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***Journal of Scientific and Engineering Research*, 2017, 4(9):221-227**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

A nonconvex regularization for wavelet frame with application to Poisson noise removal

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Abstract In this paper, we propose a nonconvex variational model to Poisson noise removal. An efficient iterative algorithm based on the augmented Lagrangian technique is proposed. Numerical experiments illustrate the effectiveness of the proposed method.

Keywords Poisson noise, wavelet tight frame, nonconvex regularization

Introduction

The deblurring problem for images corrupted by Poisson noise is an important task in various applications, such as astronomical [1], medical [2] and photographic imaging [3]. The problem of restoration of Poissonian images has received considerable attention in recent years. Image restoration in such fields of applications can often be formulated as linear ill-posed problems. The goal of image deblurring and denoising is to recover approximate images of original images from blurred and noisy measurements.

Let $u \in R^{n\times n}$ be the original image, $f \in R^{n\times n}$ be the observed image, $K : R^{n\times n} \to R^{n\times n}$ is a blurring operator which model a convolution or some other linear observation mechanism, such as emission tomography. Then the degradation model can be written as

$$f = P(Ku+b), \tag{1}$$

where $P(\alpha)$ denotes a Poisson distributed random vector with mean α, and $b \geq 0$ is an array describing the expected value of the background emission.

The likelihood probability distribution of (1) can be expressed as

$$p(f|Ku) = \prod_{i,j=1}^{n} \frac{[(Ku+b)_{i,j}]^{f_{i,j}}}{f_{i,j}!} e^{-(ku+b)_{i,j}}. \tag{2}$$

By a simple calculation, the above equation is equivalent to

$$-\log[p(|Ku)] = \sum_{i,j=1}^{n} (Ku+b)_{i,j} - f_{i,j}\log(Ku+b)_{i,j}, \tag{3}$$

the unknown image can be recovered by minimizing the likelihood function (3) with respect to u.

The standard algorithm for the minimization of (3) is the Richardson-Lucy (RL) algorithm [4,5], which takes account Poisson statistics of the photon counting. However, this method is unable to prevent noise amplification sufficiently during the iterative process due to the ill-posedness of the inverse problem.

Since as a consequence of noise and ill-posedness of the inverse problem, in general the minimizers of the function $-\log[p(f|Ku)]$ are not reliable solutions of the image restoration problem and they are sparse [6-8]. A general approach to compute a useful approximation solution of (3) is to replace the system by a better-conditioned nearby system. This replacement is known as the regularization. Regularization methods formulate the image restoration problem as a minimization problem of the form

$$\min_u \sum_{i,j=1}^{n} (Ku+b)_{i,j} - f_{i,j} \log(Ku+b)_{i,j} + \alpha\Phi_{reg}(u) \quad (4)$$

where the regularization function $\Phi_{\text{reg}}(u)$ is a prior information about the object to be recovered and α is the regularization parameter.

In recent years, sparsity-based priors of images in certain domains have been widely used in many image restoration tasks, which is based on the observation that images usually have sparse representations in some transformed domains such as Fourier transforms, cosine transforms, wavelet or framelet transforms. The ability to approximate images sparsely is an important characteristic of wavelets, see [9]. There are mainly three formulations utilizing the sparseness of the wavelet frame coefficients, namely analysis based approach, synthesis based approach, and balanced approach. In our work, we will focus on the analysis approach. The analysis approach is often modeled as a regularization term in (4) as follows

$$\Phi_{\text{reg}}(u) = \|Wu\|_1, \quad (5)$$

where W is the analysis operator and Wu is the corresponding wavelet frame coefficients. We know that the wavelet frame based methods are a reasonably effective procedure for noise reduction and blur removal when the image of interest possesses a sparse wavelet representsation. However, the methods implemented by pure wavelet thresholding also revoke unpleasant artifacts around discontinuities as a result of Gibbs phenomenon.

In order to further improve the quality of the restored images, some nonconvex regularization methods are proposed. In [10], Nikolova et al. pointed out that nonconvex nonsmooth regularization has advantages over convex regularization for offering better possibilities to recover images with neat edges. However, its practical interest used to be limited due to the difficulty of the computational stage which requires a nonsmooth nonconvex minimization.

In this paper, in order to induce wavelet-domain sparsity, we apply a nonconvex penalty due to its strong sparsity-inducing properties. We use firm thresholding, a continuous piecewise-linear approximation of hard thresholding, to compute the minimization involving the non-convex penalty.

This paper is organized as follows. In Section 2, we give some preliminaries of framelets, then we present the new model for image restoration. In Section 3, we employ the ADMM to find the solution of the proposed model. Some numerical experiments are given to illustrate the performance of the proposed algorithm in Section 4.

The proposed model

In this section, we present some preliminaries of tight framelets.

Let $W = [W_0^T, W_{1,1}^T, \cdots, W_{1,J}^T, \cdots, W_{Q,J}^T]^T \in R^{(JQ+1)n^2 \times n^2}$ be a multi-level wavelet tight frame transform operator, *i.e.* $W^T W = I$, that convert an lilters that the wavelet system used. So Wf is a multi-lever wavelet tight frame transform of f.

In this paper, we proposed the following model

$$\min_u \sum_{i,j=1}^{n} (Ku+b)_{i,j} - f_{i,j} \log(Ku+b)_{i,j} + \alpha\Psi(Wf), \quad (6)$$

where $\Psi(Wf) = \sum_{i=1}^{(JQ+1)n^2} \phi((Wf)_i)$ and with

$$\phi(x) = \begin{cases} |x| - \dfrac{x^2}{2\mu}, & if\ |x| < \mu, \\ \dfrac{\mu}{2}, & if\ |x| \ge \mu. \end{cases} \quad x \in R.$$

In the function ϕ, $\mu > 0$ is a firm thresholding parameter. See [11] for more details. It is not difficult to see that ϕ is a nonconvex function. In this paper, we use the nonconvex term $\Psi(Wf)$ to improve the wavelet-domain sparsity of the restored images.

Numerical algorithm

Here we discuss the details of the algorithm to solve the hybrid model (6). The proposed algorithm is an application of the ADMM. By introducing three auxiliary variables x, y, we reformulate the model (6) as the following constrained optimization problem

$$\min_{u,x,y} \sum_{i,j=1}^{n} (x+b)_{i,j} - f_{i,j} \log(x+b)_{i,j} + \alpha\Psi(y)$$

s.t. $Ku = x, Wu = y$.

The resulting augmented Lagrangian function is

$$L(u,x,y,d_1,d_2) = \sum_{i,j=1}^{n} (x+b)_{i,j} - f_{i,j} \log(x+b)_{i,j} + \frac{\sigma_1}{2}\|Ku - x + d_1\|_2^2 + \alpha\Psi(y) + \frac{\sigma_2}{2}\|Wu - y + d_2\|_2^2.$$

Since the variables u, x and y are decoupled, this allows us to solve them more easily on their corresponding subproblems in the ADMM. We now investigate these subproblems one by one for the Poisson noise removal problem.

Firstly, we solve for the variable u in the subproblem. This subproblem corresponds to the following optimization problem

$$u^{k+1} = \arg\min_{u} \frac{\sigma_1}{2}\|Ku - x^k + d_1^k\|_2^2 + \frac{\sigma_2}{2}\|Wu - y^k + d_2^k\|_2^2.$$

The minimizer can be obtained by equivalently solving a linear system

$$(\sigma_1 K^T K + \sigma_2 W^T W)u = \sigma_1 K^T(x^k - d_1^k) + \sigma_2 W^T(y^k - d_2^k),$$

where we have used $W^T W = I$. Under the periodic boundary conditions, the matrices K have block circulant with circulant blocks (BCCB) structure, so the above linear system can be efficiently solved by using FFTs. Denoting $F(u)$ as the fast Fourier transform of u, we can write

$$u^{k+1} = F^{-1}\left(\frac{F(\sigma_1 K^T(x^k - d_1^k) + \sigma_2 W^T(y^k - d_2^k))}{F(\sigma_1 K^T K + \sigma_2 I)}\right).$$

The minimization of L with respect to x is expressed as the following simple form

$$x = \arg\min_{x}\left(\sum_{i,j=1}^{n} (x+b)_{i,j} - f_{i,j}\log(x+b)_{i,j} + \frac{\sigma_1}{2}\|Ku - x + d_1^k\|_2^2.\right.$$

The corresponding solution can be obtained

$$x^{k+1} = \frac{1}{2}[(Ku^{k+1} + b + d_1^k - \frac{1}{\sigma_1}) + \sqrt{(Ku^{k+1} + b + d_1^k - \frac{1}{\sigma_1}) + 4\frac{f^T}{\sigma_1}}] - b.$$

The minimization of L with respect to y is expressed as the following simple form

$$y = \arg\min_{y} \alpha\Psi(y) + \frac{\sigma_2}{2}\|Wu^{k+1} - y + d_2^k\|_2^2 = \arg\min_{y} \sum_{i=1}^{(JQ+1)n^2} \frac{\alpha}{\sigma_2}\phi(y_i) + \frac{1}{2}\|y - (Wu^{k+1} + d_2^k)\|_2^2.$$

The corresponding solution can be obtained

$$y_i^{k+1} = S_{firm}((Wu^{k+1} + d_2^k)_i, \frac{\alpha}{\sigma_2}, \mu),$$

where

$$S_{firm}(x,\delta,\mu) = \begin{cases} 0 & |x| < \delta \\ \frac{\mu}{\mu-\delta}(x - \delta sign(x)) & \delta \le |x| \le \mu \\ x & |x| > \mu \end{cases}.$$

On the other hand, the updating scheme of the Lagrangian multipliers can be rewritten specifically as

$$d_1^{k+1} = d_1^k + (Ku^{k+1} - x^{k+1}),$$

$$d_2^{k+1} = d_2^k + (Wu^{k+1} - y^{k+1}).$$

The resulting algorithm based ADMM is summarized as Algorithm 1.

Algorithm 1 ADMM for the image restoration problem (6)

Initialization: ε_{tol}, MaxIter, $x^0 = Ku$, $y^0 = Wu$, $\sigma_1, \sigma_2, d_i^0 = 0$ for $i = 1,2$, $k = 0$, const $= 1$.

Iteration:

While (const)

compute u: $u^{k+1} = F^{-1}(\frac{F(\sigma_1 K^T(x^k - d_1^k) + \sigma_2 W^T(y^k - d_2^k))}{F(\sigma_1 K^T K + \sigma_2 I)})$;

compute x: $x^{k+1} = \frac{1}{2}[(Ku^{k+1} + b + d_1^k - \frac{1}{\sigma_1}) + \sqrt{(Ku^{k+1} + b + d_1^k - \frac{1}{\sigma_1}) + 4\frac{f^T}{\sigma_1}}] - b$;

compute y: $y_i^{k+1} = S_{firm}((Wu^{k+1} + d_2^k)_i, \frac{\alpha}{\sigma_2}, \mu)$,

update d_1: $d_1^{k+1} = d_1^k + (Ku^{k+1} - x^{k+1})$,

update d_2: $d_2^{k+1} = d_2^k + (Wu^{k+1} - y^{k+1})$.

const= $\|u^{k+1} - u^k\|_2 / \|u^k\|_2 > \varepsilon_{tol}$ or $k <$ MaxIter;

$k = k + 1$;

Numerical experiments

In this section, we conduct several numerical experiments to illustrate the performance of the proposed hybrid model. All the experiments were performed using MATLAB 7.7.0 on a computer equipped with an Intel (R) Core (TM) 2.60 GHz processor, with 4.00 GB of RAM, and running Windows XP.

We compare our method with tight frame model (Convex) with $\Phi_{reg}(u) = \|Wu\|_1$ in (4), and we also use ADMM solve this model. The quality of the restored images is measured by Peak-signal-to-noise ratio (PSNR), and Structural similarity index (SSIM). They are defined as follows:

$$PSNR = 20\log_{10}\frac{255}{\frac{1}{mn}\|u - f^*\|_2},$$

$$SSIM = \frac{(2\mu_f^* \mu_u + C_1)(2\sigma_{f^*u} + C_2)}{(\mu_{f^*}^2 + \mu_u^2 + C_1)(\sigma_{f^*}^2 + \sigma_u^2 + C_2)},$$

where f^* is the original image, and u is the restored image. μ_{f^*} and μ_u are averages of f^* and u respectively, σ_{f^*} and σ_u are the variance of f^* and u respectively, σ_{f^*u} is the covariance of f^* and u and the positive constants C_1 and C_2 can be thought of as stabilizing constants for near-zero denominator values.

We use the stopping criterion when the maximum number of allowed outer iterations MaxIter has been carried out or the relative differences between consecutive iterates $u^1, u^2, u^3, \ldots\ldots$ satisfy

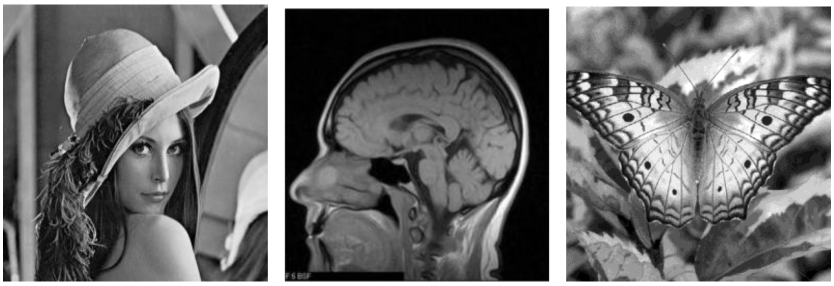
$$\frac{\left\|u^{k+1}-u^k\right\|_2}{\left\|u^{k+1}\right\|_2} < \varepsilon_{tol}.$$

In this paper, we set MaxIter=100, $\varepsilon_{tol} = 10^{-3}$ and $b = 5$ in Algorithm 1.

It is known that the quality of the restored image is highly depended on the regularization parameters. In order to have fair comparisons for the two methods, we use the best regularization parameters such that the optimal PSNR values are achieved. Regarding the penalty parameters σ's in Algorithm 1, theoretically any positive values of σ_1 and ensure the convergence of the ADMM [12]. In numerical experiments, we set $\sigma_1 = \sigma_2 = 0.005$ in the convex model and $\sigma_1 = \sigma_2 = 0.001$ in our method, i.e. the Nonconvex model.

The test images are shown in Fig. 1. In order to simulate the degraded operation in the tests, we generate the blurred and noisy images by blurring the true images with the given different point spread functions and additionally contaminate it by Poisson noise, which is implemented by applying the Matlab routine $poissrnd$.

For each image, we consider three different blurs: the out-of-focus blur proposed in [13], the Gauss blur function $psfGauss$ proposed in [14] and the linear motion blur in [15]. In this paper, we choose the out-of-focus blur with radius 9 which is generated by MATLAB function $ones(9,9)/18$, the Gauss blur with $\dim = 7$ and $s = 2$ which is generated by MATLAB function, and the linear motion blur with $r = 7$ and $\theta = 45$ which is generated by MATLAB function $fspecial('motion',7,45)$.



(a) Lena (b) Brain (c) Butterfly

Figure 1: Original images

In Table 1, the PSNR and SSIM values of three methods are presented, which shows that our method yields a better restoration result. Therefore, we conclude that the proposed method performs better than the tight frame method.

Table 1: Output of the experiments

Image	**blur**	**PSNR**		**SSIM**	
		Frame	**Proposed**	**Frame**	**Proposed**
Lena	Gauss	26.47	26.67	0.7909	0.7922
Brain	Defocus	27.78	27.85	0.7583	0.7928
Butterfly	Motion	27.98	28.11	0.9086	0.9097

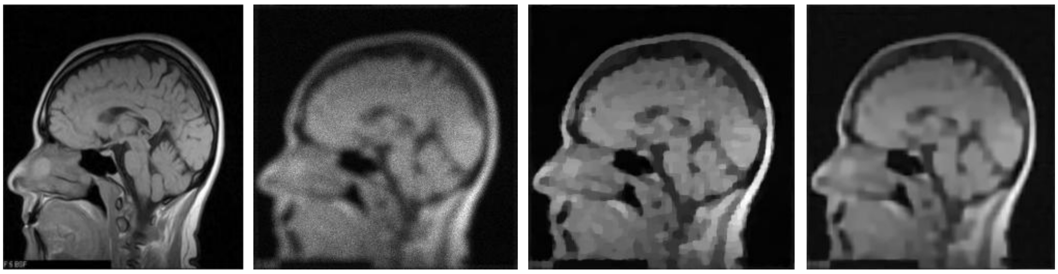
In Fig. 2, the original image, the degraded image and the restoration images of the Lena image by two methods are exhibited.

(a) Lena (b) Degraded (c) Frame (d) Proposed

Figure 2: Restoration results for the image "Lena" under the Gauss blur.

$$(\alpha = 0.005 \quad \mu = 10\alpha / \sigma_1)$$

In Fig 3, the original image, the degraded image and the restoration images of the Brain image by two methods are exhibited.



(a) Brain (b) Degraded (c) Frame (d) Proposed

Figure 3: Restoration results for the image "Brain" under the Out-of-focus blur

$$(\alpha = 0.007 \quad \mu = 12\alpha / \sigma_1)$$

In Fig 4, the original image, the degraded image and the restoration images of the Butterfly image by two methods are exhibited.

(a) Butterfly (b) Degraded (c) Frame (d) Proposed

Figure 4: Restoration results for the image "Butterfly" under the Motion blur

$$(\alpha = 0.01 \quad \mu = 10\alpha / \sigma_1)$$

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