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Reliability of Modified Exponential Distribution

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Abstract This paper deals with the estimation of the stress-strength model, when stress and strength are independent and follow modified exponential distribution. The maximum likelihood estimator and the uniformly minimum variance unbiased estimator are obtained for the stress-strength model. Based on the exact and the asymptotic distributions of the maximum likelihood estimator, an exact and an asymptotic confidence intervals of the reliability has been obtained. Bayes estimates of the reliability and the associated credible intervals are also derived under the assumptions of independent conjugate gamma and non-informative priors. An extensive computer simulation is used to compare the performance of the proposed estimators. Finally, data analysis is considered.

Keywords Stress-Strength model, modified exponential distribution, maximum likelihood estimator, minimum variance unbruised estimator, Bayes estimator
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1. Introduction

The reliability of systems or components can be defined by the stress-strength model as $R = P(X < Y)$, where X and Y represent the stress and the strength random variables, respectively. In various practical problems, R is of great interest, since it provides a general measure of the difference between two populations. For instance, R may be used in treatment comparisons .Thus, stress-strength model has many applications in engineering concepts, deterioration of rocket motors, fatigue of ceramic components and fatigue of aircraft structure are some of its applications. A great deal of the literature has been published for evaluating the reliability R, its computation and its estimation under many statistical parametric and non-parametric assumptions on the model. See, for example: Downton [7], Beg and Singh [4], Constantine et al. [5], Ivshin and Lumelskii [10], Maiti [15], Mokhlis [16], Kundu and Gupta [13], Rao [18] and Al-Mutairi et al. [1].

The modified exponential distribution (MED) with two parameters is mentioned in Elbatal and Aryal [8] as a special case of the transmuted family, see also Das [6] and Khan [11]. The cumulative distribution function (cdf) of MED is defined for T>0 as:

$$F(t;\alpha,\lambda) = 1 - e^{-(\alpha+\lambda)t}, \qquad \alpha,\lambda > 0 \tag{1.1}$$

Therefore, the corresponding probability distribution function (pdf) is

$$f(t;\alpha,\lambda) = (\alpha+\lambda)\, e^{-(\alpha+\lambda)t} \tag{1.2}$$

where α and λ are scale parameters. Notice that either $\alpha = 0$ or $\lambda = 0$, leads to the usual negative exponential distribution (ED).

The main purpose of this paper is to develop the inference on $R = P(X < Y)$, where X and Y are independent modified exponential distribution with different scale parameters (α_1, λ_1) and (α_2, λ_2), respectively. The paper is organized as follows: in Section 2, the stress-strength model, R, is derived for the modified exponential distributions. In Section 3, different estimators of R are discussed, namely, maximum likelihood estimator

(MLE), uniformly minimum variance unbiased estimator (UMVUE) and Bayesian estimators corresponding to two different priors which are conjugate and non-informative priors. In Section 4, exact and asymptotic confidence intervals (ACI) for the stress-strength model are constructed. In addition, Bayesian credible intervals with respect to conjugate and non-informative priors are derived. In Section 5, a simulation study is performed to compare the different estimators (MLE, UMVUE and Bayes) of R. Finally, the procedures are illustrated by analyzing a real data set in Section 6..

2. Stress-Strength Model

If X and Y are independent where $X \sim ME(\alpha_1, \lambda_1)$ and $Y \sim ME(\alpha_2, \lambda_2)$. Let $\theta = (\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ be a vector of unknown parameters, then the stress-strength model, R, can be derived as

$$R(\theta) = P(X < Y) = \frac{\alpha_1 + \lambda_1}{\alpha_1 + \lambda_1 + \alpha_2 + \lambda_2} \tag{2.1}$$

Notice that equation (2.1) can be rewritten as

$$R = \frac{\alpha_1 + \lambda_1}{\alpha_1 + \lambda_1 + \alpha_2 + \lambda_2} \Rightarrow \frac{R}{1-R} = \frac{\alpha_1 + \lambda_1}{\alpha_2 + \lambda_2} \tag{2.2}$$

3. Point Estimation of *R*

Here, we derive the current the MLE, UMVUE and Bayes estimators of stress-strength model for the MED.

3.1. Maximum Likelihood Estimator of *R*

Suppose $X = (X_1, X_2, \ldots, X_{n_1})$ and $Y = (Y_1, Y_2, \ldots, Y_{n_2})$ are independent random samples where $X \sim ME(\alpha_1, \lambda_1)$ and $Y \sim ME(\alpha_2, \lambda_2)$. Then, the likelihood function becomes

$$L(\theta) = (\alpha_1 + \lambda_1)^{n_1} (\alpha_2 + \lambda_2)^{n_2} e^{-\alpha_1 \sum_{i=1}^{n_1} x_i - \lambda_1 \sum_{i=1}^{n_1} x_i - \alpha_2 \sum_{j=1}^{n_2} y_j - \lambda_2 \sum_{j=1}^{n_2} y_j} \tag{3.1}$$

Then, the MLEs of the parameters are obtained by maximizing the log-likelihood function with respect to the parameters as following

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{n_1}{(\alpha_1 + \lambda_1)} - \sum_{i=1}^{n_1} x_i = 0 \tag{3.2}$$

$$\frac{\partial \ln L}{\partial \alpha_2} = \frac{n_2}{(\alpha_2 + \lambda_2)} - \sum_{j=1}^{n_2} y_i = 0 \tag{3.4}$$

$$\frac{\partial \ln L}{\partial \lambda_1} = \frac{n_1}{(\alpha_1 + \lambda_1)} - \sum_{i=1}^{n_1} x_i = 0 \tag{3.3}$$

$$\frac{\partial \ln L}{\partial \lambda_2} = \frac{n_2}{(\alpha_2 + \lambda_2)} - \sum_{j=1}^{n_2} y_i = 0 \tag{3.5}$$

Then, the MLE of R is given be

$$\hat{R}_1 = \frac{\hat{\alpha}_1 + \hat{\lambda}_1}{\hat{\alpha}_1 + \hat{\lambda}_1 + \hat{\alpha}_2 + \hat{\lambda}_2} = \frac{\overline{Y}}{\overline{X} + \overline{Y}} \tag{3.6}$$

Notice that $\hat{R}_1$ can be expressed by

$$\frac{1 - \hat{R}_1}{\hat{R}_1} = \frac{1}{\hat{R}_1} - 1 = \frac{\overline{X}}{\overline{Y}} \tag{3.7}$$

Next we prove the following interesting results.

Theorem 1: If $X = (X_1, X_2, \ldots, X_{n_1})$ and $Y = (Y_1, Y_2, \ldots, Y_{n_2})$ are independent random samples where $X \sim ME(\alpha_1, \lambda_1)$ and $Y \sim ME(\alpha_2, \lambda_2)$. Then,

1. $T_X = n_1 \overline{X} \sim Gamma(n_1, \alpha_1 + \lambda_1)$ and $T_Y = n_2 \overline{Y} \sim Gamma(n_2, \alpha_2 + \lambda_2)$ (3.8)
2. $2n_1(\alpha_1 + \lambda_1)\overline{X} \sim \chi^2_{(2n_1))}$ and $2n_2(\alpha_2 + \lambda_2)\overline{Y} \sim \chi^2_{(2n_2))}$. (3.9)

3. $F_1 = \dfrac{(\alpha_1+\lambda_1)\overline{X}}{(\alpha_2+\lambda_2)\overline{Y}} \sim F(2n_1, 2n_2).$ (3.10)

Proof:

The moment generating function of $X \sim ME(\alpha_1, \lambda_1)$ is given by

$$M_X(t) = \int_0^\infty e^{tx}(\alpha_1+\lambda_1)e^{-\alpha_1 x - \lambda_1 x}dx = \left(1 - \frac{t}{\alpha_1+\lambda_1}\right)^{-1}$$

Therefore,

$$M_{\sum_{i=1}^{n_1} X_i}(t) = (M_X(t))^{n_1} = \left(1 - \frac{t}{\alpha_1+\lambda_1}\right)^{-n_1}$$

Thus, $T_X = n_1\overline{X} \sim Gamma(n_1, \alpha_1+\lambda_1)$ which yields to $2n_1(\alpha_1+\lambda_1)\overline{X} \sim \chi^2_{(2n_1))}$. Similarly, if $Y \sim ME(\alpha_2, \lambda_2)$. Then, $T_Y = n_2\overline{Y} \sim Gamma(n_2, \alpha_2+\lambda_2)$ and $2n_2(\alpha_2+\lambda_2)\overline{Y} \sim \chi^2_{(2n_2))}$. Now, if we define F_1 as following

$$F_1 = \frac{(\alpha_1+\lambda_1)\overline{X}}{(\alpha_2+\lambda_2)\overline{Y}}$$

which is distributed as F_1 with $(2n_1, 2n_2)$ degrees of freedom.

3.2. Uniformly Minimum Variance Unbiased Estimator of *R*

Let $X = (X_1, X_2, ..., X_{n_1})$ and $Y = (Y_1, Y_2, ..., Y_{n_2})$ be two independent random samples where $X \sim ME(\alpha_1, \lambda_1)$ and $Y \sim ME(\alpha_2, \lambda_2)$. Then, the likelihood function becomes

$$L = (\alpha_1+\lambda_1)^{n_1}(\alpha_2+\lambda_2)^{n_2} e^{-(\alpha_1+\lambda_1)\sum_{i=1}^{n_1} x_i - (\alpha_2+\lambda_2)\sum_{j=1}^{n_2} y_j} \tag{3.11}$$

By Factorization Theorem (see Kotz et al. [12]), we get

1. $T_X = \sum_{i=1}^{n_1} x_i$ and $T_Y = \sum_{i=1}^{n_2} y_i$ are sufficient statistics for *X* and *Y*.
2. The indicator $T = I(X_1 < Y_1)$ is an unbiased estimator of *R*.

Using Rao-Blackwell and Lehman-Sheffes' Theorems, see Mood et al. [17], the UMVUE, $\hat{R}_2$, of *R* is given by

$$\hat{R}_2 = E(T \mid T_X, T_Y) = \int_{x_1}\int_{y_1} t\ f(x_1, y_1 \mid t_x, t_y)dx_1 dy_1$$

where $f(x_1, y_1 \mid t_x, t_y)$ is the conditional pdf of X_1, Y_1 given that T_X, T_Y.

Notice that X_1 and Y_1, are independent modified exponential random variables with parameters (α_1, λ_1) and (α_2, λ_2), respectively. Recall that from Theorem 1, T_X and T_Y are independent gamma random variables with parameters $(n_1, \alpha_1+\lambda_1)$ and $(n_2, \alpha_2+\lambda_2)$, respectively. Therefore, $T_X - X_1$ and $T_Y - Y_1$ are independent gamma random variables with parameters $(n_1 - 1, \alpha_1+\lambda_1)$ and $(n_2 - 1, \alpha_2+\lambda_2)$, respectively. Moreover $T_X - X_1$ and X_1 are independent, as well as $T_Y - Y_1$ and Y_1 are also independent. We obtain that

$$\hat{R}_2 = \frac{(n_1-1)(n_2-1)}{T_X^{n_1-1}\, T_Y^{n_2-1}} \int_{x_1}\int_{y_1} t\ (T_x - x_1)^{n_1-2}(T_Y - y_1)^{n_2-2}\, dy_1 dx_1$$

Then, the $\hat{R}_2$ is derived as follows

$$\hat{R}_2 = \begin{cases} \dfrac{(n_1-1)(n_2-1)}{T_X^{n_1-1}\, T_Y^{n_2-1}} \displaystyle\int_0^{T_X}\int_{x_1}^{T_Y} (T_X - x_1)^{n_1-2}(T_Y - y_1)^{n_2-2}\, dy_1 dx_1\ , & if\ \ T_X < T_Y \\ \dfrac{(n_1-1)(n_2-1)}{T_X^{n_1-1}\, T_Y^{n_2-1}} \displaystyle\int_0^{T_Y}\int_0^{y_1} (T_X - x_1)^{n_1-2}(T_Y - y_1)^{n_2-2}\, dx_1 dy_1\ , & if\ \ T_Y \le T_X \end{cases}$$

Using formula (3.11) in Gradshteyn and Ryzhik [9], the $\hat{R}_2$ can be written as

$$\hat{R}_2 = \begin{cases} {}_2F_1\left(1,-(n_2-1),n_1;\dfrac{T_X}{T_Y}\right)\ , & if\ \ T_X < T_Y \\ 1-{}_2F_1\left(1,-(n_1-1),n_2;\dfrac{T_Y}{T_X}\right)\ , & if\ \ T_Y \le T_X \end{cases} \tag{3.12}$$

where ${}_2F_1(\alpha,\beta,\gamma;z)$ is Gauss hypergeometric function given by

$${}_2F_1(\alpha,\beta,\gamma;z) = 1 + \frac{\alpha\beta}{\gamma.1}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1).1.2}z^2 + \ldots \tag{3.13}$$

Thus, the UMVUE of R can be rewritten as

$$\hat{R}_2 = \begin{cases} \displaystyle\sum_{i=0}^{n_1-1}(-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1-i)\Gamma(n_2+i)}\left(\frac{T_X}{T_Y}\right)^i\ , & if\ \ T_X < T_Y \\ 1-\displaystyle\sum_{i=0}^{n_2-1}(-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1+i)\Gamma(n_2-i)}\left(\frac{T_Y}{T_X}\right)^i\ , & if\ \ T_Y \le T_X \end{cases} \tag{3.14}$$

3.3. Bayesian Estimators of *R*

Here, the Bayes estimators of R with respect to conjugate and non-informative prior distributions are obtained. We show that the Bayes estimator of R with respect to non-informative prior distribution is superior to that with respect to conjugate distribution.

3.3.1. Bayes Estimator with Conjugate Prior of *R*

Let $X = (X_1, X_2, \ldots, X_{n_1})$ and $Y = (Y_1, Y_2, \ldots, Y_{n_2})$ be two independent random samples drawn from the modified exponential distributions with parameters (α_1, λ_1) and (α_2, λ_2), respectively. Assuming that $\theta = (\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ are independent, having conjugate gamma prior distributions, with parameters $(\beta_1,\delta_1),(\beta_2,\delta_2),(\beta_3,\delta_3)$ and (β_4,δ_4), respectively. Then,

$$\pi_1(\theta \mid X,Y) e^{-\alpha_1\left(\delta_1+\sum_{i=1}^{n_1}x_i\right)-\lambda_1\left(\delta_2+\sum_{i=1}^{n_1}x_i\right)-\alpha_2\left(\delta_3+\sum_{j=1}^{n_2}y_j\right)-\lambda_2\left(\delta_4+\sum_{j=1}^{n_2}y_j\right)}$$

Thus, the posterior distributions of $\alpha_1, \lambda_1, \alpha_2$ and λ_2 are gamma with parameters $(\gamma_1,\eta_1),(\gamma_2,\eta_2),(\gamma_3,\eta_3)$ and (γ_4,η_4), respectively, where

$$\gamma_1 = n_1 + \beta_1 - k_1,\quad \eta_1 = \delta_1 + \sum_{i=1}^{n_1} x_i\,,\quad \gamma_2 = \beta_2 + k_1\,,\quad \eta_2 = \delta_2 + \sum_{i=1}^{n_1} x_i$$

$$\gamma_3 = n_2 + \beta_3 - k_2\,,\quad \eta_3 = \delta_3 + \sum_{j=1}^{n_2} y_j\,,\ \gamma_4 = \beta_4 + k_2\,,\quad \eta_4 = \delta_4 + \sum_{j=1}^{n_2} y_j \tag{3.15}$$

when $k_1 = 0,.., n_1,\ \ k_2 = 0,..., n_2$. Thus,

$$\pi_1(\alpha_1, \lambda_1, \alpha_2, \lambda_2 \mid X, Y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \frac{\eta_1^{\gamma_1}\eta_2^{\gamma_2}\eta_3^{\gamma_3}\eta_4^{\gamma_4}}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)\Gamma(\gamma_4)}\alpha_1^{\gamma_1-1}\lambda_1^{\gamma_2-1}\alpha_2^{\gamma_3-1}\lambda_2^{\gamma_4-1}\, e^{-\alpha_1\eta_1-\lambda_1\eta_2-\alpha_2\eta_3-\lambda_2\eta_4},\ \ \alpha_1, \lambda_1, \alpha_2, \lambda_2 > 0$$

The Bayes estimator of R with respect to the mean squared error loss function, $\hat{R}_3$, is

$$\hat{R}_3 = E(R \mid X, Y)$$

After making suitable transformations and simplifications and using Formula 3.197.3 in Gradshteyn and Ryzhik [9], we get, if $\eta_1 < \eta_3$

$$\hat{R}_3 = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \frac{1}{B(\gamma_1+\gamma_2, \gamma_3+\gamma_4)}\left(\frac{\eta_1}{\eta_3}\right)^{\gamma_1+\gamma_2}\int_0^1 R^{\gamma_1+\gamma_2}(1-R)^{\gamma_3+\gamma_4-1}\left(1-\left(1-\frac{\eta_1}{\eta_3}\right)R\right)^{-(\gamma_1+\gamma_2+\gamma_3+\gamma_4)} dR \tag{3.16}$$

$$= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\right)\left(\frac{\eta_1}{\eta_3}\right)^{\gamma_1+\gamma_2} {}_2F_1\left(\gamma_1+\gamma_2+\gamma_3+\gamma_4, \gamma_1+\gamma_2+1, \gamma_1+\gamma_2+\gamma_3+\gamma_4+1;\ 1-\frac{\eta_1}{\eta_3}\right) \tag{3.17}$$

Equation (3.16) can also be written as

$$\hat{R}_3 = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \frac{1}{B(\gamma_1+\gamma_2, \gamma_3+\gamma_4)}\left(\frac{\eta_3}{\eta_1}\right)^{\gamma_3+\gamma_4}\int_0^1 R^{\gamma_1+\gamma_2}(1-R)^{\gamma_3+\gamma_4-1}\left(1-\left(1-\frac{\eta_3}{\eta_1}\right)(1-R)\right)^{-(\gamma_1+\gamma_2+\gamma_3+\gamma_4)} dR$$

Using the transformation $R^* = 1 - R$, we obtain, if $\eta_3 < \eta_1$

$$\hat{R}_3 = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \frac{1}{B(\gamma_1+\gamma_2, \gamma_3+\gamma_4)}\left(\frac{\eta_3}{\eta_1}\right)^{\gamma_3+\gamma_4}\int_0^1 (R^*)^{\gamma_3+\gamma_4-1}(1-R^*)^{\gamma_1+\gamma_2}\left(1-\left(1-\frac{\eta_3}{\eta_1}\right)R^*\right)^{-(\gamma_1+\gamma_2+\gamma_3+\gamma_4)} dR^*$$

$$= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\right)\left(\frac{\eta_3}{\eta_1}\right)^{\gamma_3+\gamma_4} {}_2F_1\left(\gamma_1+\gamma_2+\gamma_3+\gamma_4, \gamma_3+\gamma_4, \gamma_1+\gamma_2+\gamma_3+\gamma_4+1;\ 1-\frac{\eta_3}{\eta_1}\right) \tag{3.18}$$

If $\eta_1 = \eta_3$, we get

$$\hat{R}_3 = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \frac{1}{B(\gamma_1+\gamma_2, \gamma_3+\gamma_4)}\int_0^1 R^{\gamma_1+\gamma_2}(1-R)^{\gamma_3+\gamma_4-1} dR = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\frac{\gamma_1+\gamma_2}{\gamma_1+\gamma_2+\gamma_3+\gamma_4} \tag{3.19}$$

Combining (3.17), (3.18) and (3.19), we obtain

$$\hat{R}_3 = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{cases}\dfrac{\gamma_1+\gamma_2}{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\left(\dfrac{\eta_1}{\eta_3}\right)^{\gamma_1+\gamma_2} {}_2F_1\left(\gamma_1+\gamma_2+\gamma_3+\gamma_4, \gamma_1+\gamma_2+1, \gamma_1+\gamma_2+\gamma_3+\gamma_4+1;\ 1-\dfrac{\eta_1}{\eta_3}\right), & if\ \eta_1 < \eta_3 \\[2ex] \dfrac{\gamma_1+\gamma_2}{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\left(\dfrac{\eta_3}{\eta_1}\right)^{\gamma_3+\gamma_4} {}_2F_1\left(\gamma_1+\gamma_2+\gamma_3+\gamma_4, \gamma_3+\gamma_4, \gamma_1+\gamma_2+\gamma_3+\gamma_4+1;\ 1-\dfrac{\eta_3}{\eta_1}\right), & if\ \eta_3 \le \eta_1\end{cases} \tag{3.20}$$

Based on Lindley [14] and Awad and Gharraf [2], we could use the means of an empirical Bayes procedure to estimate the parameters of the prior distributions of (α_1, λ_1) and (α_2, λ_2) in (3.20) as follows:

Let $X=\left(X_1,X_2,...,X_{n_1}\right)$ and $Y=\left(Y_1,Y_2,...,Y_{n_2}\right)$ be two independent random samples drawn from modified exponential distributions with parameters (α_1,λ_1) and (α_2,λ_2), respectively. Then, the likelihood function of each sample will be, respectively,

$$L(X\mid\alpha_1,\lambda_1)\propto\sum_{k_1=1}^{n_1}\alpha_1^{n_1-k_1}\lambda_1^{k_1}e^{-\alpha_1\sum_{i=1}^{n_1}x_i-\lambda_1\sum_{i=1}^{n_1}x_i}\quad\text{and}\quad L(Y\mid\alpha_2,\lambda_2)\propto\sum_{k_2=1}^{n_2}\alpha_2^{n_2-k_2}\lambda_2^{k_2}e^{-\alpha_2\sum_{j=1}^{n_2}y_j-\lambda_2\sum_{j=1}^{n_2}y_j}$$

Notice that $L(X\mid\alpha_1,\lambda_1)$ is a function of α_1 and λ_1 which are gamma densities with parameters $\left(n_1-k_1+1,\sum_{i=1}^{n_1}x_i\right)$ and $\left(k_1+1,\sum_{i=1}^{n_1}x_i\right)$. Thus, it is proposed to estimate the prior parameters (β_1,δ_1) and (β_2,δ_2) from the samples by $\left(n_1-k_1+1,\sum_{i=1}^{n_1}x_i\right)$ and $\left(k_1+1,\sum_{i=1}^{n_1}x_i\right)$, respectively. Similarly, α_2 and λ_2 could be estimated from the samples by $\left(n_2-k_2+1,\sum_{j=1}^{n_2}y_j\right)$ and $\left(k_2+1,\sum_{j=1}^{n_2}y_j\right)$. Therefore, from (3.15), we get

$$\hat{\gamma}_1=2n_1-2k_1+1,\ \hat{\eta}_1=2\sum_{i=1}^{n_1}x_i,\ \hat{\gamma}_2=2k_1+1,\ \hat{\eta}_2=2\sum_{i=1}^{n_1}x_i,\ k_1=0,1,...,n_1$$

$$\hat{\gamma}_3=2n_2-2k_2+1,\ \hat{\eta}_3=2\sum_{j=1}^{n_2}y_j,\ \hat{\gamma}_4=2k_2+1,\ \hat{\eta}_4=2\sum_{j=1}^{n_2}y_j,\ k_2=0,1,...,n_2 \tag{3.21}$$

The Bayes estimator of R in (3.20) could be given as

$$\hat{R}_3=\begin{cases}\dfrac{n_1+1}{n_1+n_2+2}\left(\dfrac{T_X}{T_Y}\right)^{2n_1+2}{}_2F_1\left(2(n_1+n_2+2),2n_1+3,2n_1+2n_2+5;\ 1-\dfrac{T_X}{T_Y}\right), & if\ T_X<T_Y\\[2ex] \dfrac{n_1+1}{n_1+n_2+2}\left(\dfrac{T_Y}{T_X}\right)^{2n_2+2}{}_2F_1\left(2(n_1+n_2+2),2(n_2+1),2n_1+2n_2+5;\ 1-\dfrac{T_Y}{T_X}\right), & if\ T_Y\le T_X\end{cases} \tag{3.22}$$

where $T_X=n_1\overline{X}$ and $T_Y=n_2\overline{Y}$.

3.3.2. Bayes Estimator with Non-Informative Prior of *R*

Suppose $X=\left(X_1,X_2,...,X_{n_1}\right)$ and $Y=\left(Y_1,Y_2,...,Y_{n_2}\right)$ be two independent random samples drawn from MED with parameters (α_1,λ_1) and (α_2,λ_2), respectively. Then, the Jeffrey's priors of the parameters $\alpha_1,\lambda_1,\alpha_2$ and λ_2 are obtained, respectively, as follows:

$$\pi_{21}(\alpha_1)\propto\frac{1}{\alpha_1+\lambda_1}\ ,\ \pi_{22}(\lambda_1)\propto\frac{1}{\alpha_1+\lambda_1}\ ,\ \pi_{23}(\alpha_2)\propto\frac{1}{\alpha_2+\lambda_2}\ ,\ \text{and}\ \pi_{24}(\lambda_2)\propto\frac{1}{\alpha_2+\lambda_2}.$$

The posterior joint distribution of independent (α_1,λ_1) and (α_2,λ_2) will be

$$\pi_2(\alpha_1,\lambda_1,\alpha_2,\lambda_2\mid X,Y)\propto\sum_{k_1=1}^{n_1-2}\sum_{k_2=1}^{n_2-2}\alpha_1^{n_1-k_1-2}\lambda_1^{k_1}\alpha_2^{n_2-k_2-2}\lambda_2^{k_2}e^{-(\alpha_1+\lambda_1)\sum_{i=1}^{n_1}x_i-(\alpha_2+\lambda_2)\sum_{j=1}^{n_2}y_j}$$

Thus, the posterior distributions of $\alpha_1, \lambda_1, \alpha_2$ and λ_2 are gamma with parameters $(\gamma_1, \eta_1), (\gamma_2, \eta_2), (\gamma_3, \eta_3)$ and (γ_4, η_4), respectively, where

$$\gamma_1 = n_1 - k_1 - 1, \quad \eta_1 = \sum_{i=1}^{n_1} x_i, \quad \gamma_2 = k_1 + 1, \quad \eta_2 = \sum_{i=1}^{n_1} x_i$$

$$\gamma_3 = n_2 - k_2 - 1, \quad \eta_3 = \sum_{j=1}^{n_2} y_j, \quad \gamma_4 = k_2 + 1, \quad \eta_4 = \sum_{j=1}^{n_2} y_j \tag{3.23}$$

Then, the Bayes estimators of R corresponding to the Jeffrey's priors is obtained as follows:

$$\hat{R}_4 = \begin{cases} \dfrac{1}{B(\gamma_1+\gamma_2, \gamma_3+\gamma_4)} \left(\dfrac{\eta_1}{\eta_3}\right)^{\gamma_1+\gamma_2} \displaystyle\int_0^1 R^{\gamma_1+\gamma_2}(1-R)^{\gamma_3+\gamma_4-1}\left(1-\left(1-\dfrac{\eta_1}{\eta_3}\right)R\right)^{-(\gamma_1+\gamma_2+\gamma_3+\gamma_4)} dR\,, & if\ \eta_1 < \eta_3 \\ \dfrac{1}{B(\gamma_1+\gamma_2, \gamma_3+\gamma_4)} \left(\dfrac{\eta_3}{\eta_1}\right)^{\gamma_3+\gamma_4} \displaystyle\int_0^1 R^{\gamma_1+\gamma_2}(1-R)^{\gamma_3+\gamma_4-1}\left(1-\left(1-\dfrac{\eta_3}{\eta_1}\right)(1-R)\right)^{-(\gamma_1+\gamma_2+\gamma_3+\gamma_4)} dR\,, & if\ \eta_3 \le \eta_1 \end{cases}$$

Using the Formula 3.197.3 in Gradshteyn and Ryzhik [9], we get the Bayes estimator of R with respect to Jeffrey's priors as follows

$$\hat{R}_4 = \begin{cases} \dfrac{n_1}{n_1+n_2}\left(\dfrac{T_X}{T_Y}\right)^{n_1} {}_2F_1\left(n_1+n_2, n_1+1, n_1+n_2+1;\ 1-\dfrac{T_X}{T_Y}\right), & if\ T_X < T_Y \\ \dfrac{n_1}{n_1+n_2}\left(\dfrac{T_Y}{T_X}\right)^{n_2} {}_2F_1\left(n_1+n_2, n_2, n_1+n_2+1;\ 1-\dfrac{T_Y}{T_X}\right), & if\ T_Y \le T_X \end{cases} \tag{3.24}$$

Notice that, in this section, we have no need to estimate the prior parameters such as in the previous section. This approach also overcomes the sensitivity of $\hat{R}_3$ to the parameters of the prior distributions.

4. Interval Estimation of *R*

Two different confidence intervals of R are derived in this section, based on the exact and asymptotic distributions of the maximum likelihood estimator of R.

4.1. Exact Confidence Interval for *R*

Let $X_1, X_2, ..., X_{n_1}$ and $Y_1, Y_2, ..., Y_{n_2}$ be two independent random samples drawn from modified exponential with parameters (α_1, λ_1) and (α_2, λ_2), respectively.

Using (3.7) and (3.10), we get

$$\hat{R}_1 = \left(1 + \frac{\bar{X}}{\bar{Y}}\right)^{-1} = \left(1 + \frac{(\alpha_2+\lambda_2)}{(\alpha_1+\lambda_1)} F_1\right)^{-1}$$

where $F_1 = \dfrac{(\alpha_1+\lambda_1)\bar{X}}{(\alpha_2+\lambda_2)\bar{Y}}$ is an F distributed random variable with $(2n_1, 2n_2)$ degrees of freedom. From (2.4) and (3.7), F_1 can be written as

$$F_1 = \frac{\bar{X}}{\bar{Y}} \frac{R}{1-R}. \tag{4.1}$$

which is used as a pivotal quantity. Hence, the $(1-\varsigma)100\%$ confidence interval for R is obtained as

$$CI_1 = \left(\frac{F_{1-\frac{\zeta}{2},(2n_1,2n_2)}}{F_{1-\frac{\zeta}{2},(2n_1,2n_2)} + \frac{\bar{X}}{\bar{Y}}}, \frac{F_{\frac{\zeta}{2},(2n_1,2n_2)}}{F_{\frac{\zeta}{2},(2n_1,2n_2)} + \frac{\bar{X}}{\bar{Y}}} \right) \tag{4.2}$$

where $F_a(b,c)$ is the $(1-a)$th quantile of an *F* distributed random variable with (b,c) degrees of freedom.

4.2. Asymptotic Confidence Interval for *R*

Since, the MLEs $\hat{\theta} = (\hat{\alpha}_1, \hat{\lambda}_1, \hat{\alpha}_2, \hat{\lambda}_2)$ have approximately normally distribution and according to Kotz et al. [12], the MLE $\hat{R}_1$ is approximately normally distributed given by

$$\hat{R}_1 \sim N(R, \mathrm{A}'\Sigma^{-1}\mathrm{A}) \tag{4.3}$$

where $A = \left[\frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \lambda_1}, \frac{\partial R}{\partial \alpha_2}, \frac{\partial R}{\partial \lambda_2} \right]'$ and Σ is the variance-covariance matrix. Therefore, the asymptotic $(1-\zeta)100\%$ CI for *R* can be obtained as

$$CI_2 = \left(\frac{\bar{Y}}{\bar{X}+\bar{Y}} - Z_{1-\frac{\zeta}{2}} \sqrt{\mathrm{A}'\Sigma^{-1}\mathrm{A}}, \frac{\bar{Y}}{\bar{X}+\bar{Y}} + Z_{1-\frac{\zeta}{2}} \sqrt{\mathrm{A}'\Sigma^{-1}\mathrm{A}} \right) \tag{4.4}$$

4.3. Bayesian Credible Intervals

4.3.1 Bayes Estimator with Conjugate Prior of *R*

We conclude from Section 3.3.1 that the posterior distributions of α_1 and λ_1 corresponding to gamma priors are gamma with parameters $(2n_1 - 2k_1 + 1, 2n_1\bar{X})$ and $(2k_1 + 1, 2n_1\bar{X})$, respectively. Similarly, the posterior distributions of α_2 and λ_2 corresponding to gamma priors are gamma with parameters $(2n_2 - 2k_2 + 1, 2n_2\bar{Y})$ and $(2k_2 + 1, 2n_2\bar{Y})$, respectively. Thus,

$$4n_1\bar{X}(\alpha_1 + \lambda_1) \sim \chi^2_{4(n_1+1)} \text{ and } 4n_2\bar{Y}(\alpha_2 + \lambda_2) \sim \chi^2_{4(n_2+1)}$$

Define, F_2 as

$$F_2 = \frac{n_1(n_2+1)(\alpha_1+\lambda_1)\bar{X}}{n_2(n_1+1)(\alpha_2+\lambda_2)\bar{Y}}$$

Since $\frac{(\alpha_1+\lambda_1)}{(\alpha_2+\lambda_2)} = \frac{R}{(1-R)}$, then F_2 can be rewritten as

$$F_2 = \frac{n_1(n_2+1)\bar{X}}{n_2(n_1+1)\bar{Y}} \frac{R}{(1-R)} \tag{4.5}$$

which is distributed as $F_{(4(n_1+1),4(n_2+1))}$. Using F_2 in (4.5) as a pivotal quantity, we get $(1-\zeta)100\%$ Bayes credible interval for *R* as follows

$$CI_3 = \left(\frac{F_{1-\frac{\zeta}{2},(4(n_1+1),4(n_2+1))}}{F_{1-\frac{\zeta}{2},(4(n_1+1),4(n_2+1))} + \frac{n_1(n_2+1)\bar{X}}{n_2(n_1+1)\bar{Y}}}, \frac{F_{\frac{\zeta}{2},(4(n_1+1),4(n_2+1))}}{F_{\frac{\zeta}{2},(4(n_1+1),4(n_2+1))} + \frac{n_1(n_2+1)\bar{X}}{n_2(n_1+1)\bar{Y}}} \right) \quad (4.6)$$

4.3.2. Bayesian Interval with Non-Informative Prior for *R*

We have seen in Section 3.3.2 that, assuming independence and non-informative prior distributions for $\alpha_1, \lambda_1, \alpha_2$ and λ_2, the posterior distributions of $\alpha_1, \lambda_1, \alpha_2$ and λ_2 are gamma with parameters $\left(n_1 - k_1 - 1, \sum_{i=1}^{n_1} X_i\right), \left(k_1 + 1, \sum_{i=1}^{n_1} X_i\right), \left(n_2 - k_2 - 1, \sum_{j=1}^{n_2} Y_j\right)$, and $\left(k_2 + 1, \sum_{j=1}^{n_2} Y_j\right)$, respectively. Thus,

$$2n_1(\alpha_1 + \lambda_1)\bar{X} \sim \chi^2_{2n_1} \text{ and } 2n_2(\alpha_2 + \lambda_2)\bar{Y} \sim \chi^2_{2n_2}$$

Therefore, if we define F_3 as

$$F_3 = \frac{(\alpha_2 + \lambda_2)\bar{Y}}{(\alpha_1 + \lambda_1)\bar{X}} \sim F_{(2n_2, 2n_1)}$$

and since $\frac{(\alpha_2 + \lambda_2)}{(\alpha_1 + \lambda_1)} = \left(\frac{1}{R} - 1\right)$, then F_3 can be rewritten as

$$F_3 = \frac{\bar{Y}}{\bar{X}}\left(\frac{1}{R} - 1\right) \quad (4.7)$$

which is distributed as $F_{(2n_2, 2n_1)}$. Using F_3 in (4.7) as a pivotal quantity, we get $(1-\zeta)100\%$ Bayes credible interval for *R* as follows:

$$CI_4 = \left(\frac{1}{1 + F_{\frac{\zeta}{2},(2n_2,2n_1)} \frac{\bar{X}}{\bar{Y}}} < R < \frac{1}{1 + F_{1-\frac{\zeta}{2},(2n_2,2n_1)} \frac{\bar{X}}{\bar{Y}}} \right) \quad (4.8)$$

which is the same as the exact confidence interval CI_1.

5. Simulation

In this section, we introduce a simulation study to compare between the estimators of the stress-strength model for the modified exponential distribution, namely $\hat{R}_1$, $\hat{R}_2$, $\hat{R}_3$ and $\hat{R}_4$, and the corresponding confidence intervals using two different methods. The mean squared errors (MSE) of the estimators of *R* and the average lengths (AL) of the intervals of these estimators are discussed in this comparison. The cases when $R = 0.5$, $R = 0.6$, $R = 0.7$, $R = 0.8$, $R = 0.9$ and $R = 0.97$ are studied. Without loss of generality, we take the case when $\alpha_1 = \lambda_1 = 4$ and different values of $\alpha_2 = \lambda_2 = 0.1, 0.4, 1, 1.5, 2.5, 3.5$ with the sample sizes $n_1 = n_2 = 5, 10, 20, 30, 50$. The results of this simulation is denoted in the Appendix (Tables 6 & 7).

5.1. Mean Squared Errors Estimators

From Table 1, we conclude that there are five different cases, which are given as following:

1- $MSE(\hat{R}_4) < MSE(\hat{R}_3) < MSE(\hat{R}_1) < MSE(\hat{R}_2)$:

when $n_1 = n_2 = 5$ for $\alpha_2 = \lambda_2 = 2.5, 3.5$

$n_1 = n_2 = 10$ for $\alpha_2 = \lambda_2 = 1.5, 2.5, 3.5$

$n_1 = n_2 = 20$ for $\alpha_2 = \lambda_2 = 2.5$.

2- $MSE(\hat{R}_1) < MSE(\hat{R}_4) < MSE(\hat{R}_3) < MSE(\hat{R}_2)$:

when $n_1 = n_2 = 5$ for $\alpha_2 = \lambda_2 = 1.5$

$n_1 = n_2 = 20$ for $\alpha_2 = \lambda_2 = 1.5, 3.5$

$n_1 = n_2 = 3, 50$ for $\alpha_2 = \lambda_2 = 1.5, 2.5, 3.5$.

3- $MSE(\hat{R}_1) < MSE(\hat{R}_2) < MSE(\hat{R}_3) < MSE(\hat{R}_4)$:

when $n_1 = n_2 = 5, 20$ for $\alpha_2 = \lambda_2 = 1$

$n_1 = n_2 = 3, 50$ for $\alpha_2 = \lambda_2 = 0.1, 0.4, 1$.

4- $MSE(\hat{R}_2) < MSE(\hat{R}_1) < MSE(\hat{R}_3) < MSE(\hat{R}_4)$:

when $n_1 = n_2 = 5, 20$ for $\alpha_2 = \lambda_2 = 0.1, 0.4$.

5- $MSE(\hat{R}_2) < MSE(\hat{R}_3) < MSE(\hat{R}_4) < MSE(\hat{R}_1)$:

when $n_1 = n_2 = 10$ for $\alpha_2 = \lambda_2 = 0.1, 0.4, 1$.

In another way, we can compare between the estimators of *R* by the value of *R* as follows:

1- If $R = 0.5$ and $R = 0.6$:

when $(n_1 = n_2) < 30$, $MSE(\hat{R}_4)$ is the smallest, otherwise $MSE(\hat{R}_1)$.

2- If $R = 0.7$ and $R = 0.8$:

when $n_1 = n_2 = 10$, $MSE(\hat{R}_4)$ is the smallest, otherwise $MSE(\hat{R}_1)$.

3- If $R = 0.7$ and $R = 0.8$:

when $(n_1 = n_2) < 30$, $MSE(\hat{R}_2)$ is the smallest, otherwise $MSE(\hat{R}_1)$.

Therefore, we can conclude that the MLE of *R*, $\hat{R}_1$, is the best estimate of *R* when the sample size is greater than 30 for different values of *R* and also when the sample size is less than 30 for $0.6 < R < 0.9$. On the other hand, if the sample size is less than 30, the BE with Jeffrey's prior, $\hat{R}_4$, is a better estimator of *R* than the other estimators for $R \le 0.6$. In contrast, $\hat{R}_2$, the UMVUE of *R*, is the best estimator if $R \le 0.6$ when $(n_1 = n_2) < 30$.

We also observe that the mean squared errors of the four estimates decrease as the sample sizes and/or *R* increase. When *R* is large, *R* = 0.9, 0.97, the differences between the mean squared errors of the four estimates are very small.

5.2. Using Average Length of the Intervals

The exact, asymptotic confidence intervals and the confidence intervals of the posterior prior distribution for the Bayes estimators of *R*, namely CI_1, CI_2, CI_3 and CI_4, respectively.

As with the earlier numerical illustration, taking $\zeta = 0.05$, the average length of each of the intervals is obtained and calculated for 1000 simulation runs for various sample sizes ($n_1 = n_2 = 5, 10, ,20, 30, 50$) and for the cases when R = 0.5, R = 0.6, R = 0.7, R = 0.8, R = 0.9, and R = 0.97.

From Table 2, we notice that the average lengths of CI_3 are the shortest average lengths. Moreover, in general, if the sample sizes larger than 20, the average lengths of CI_2 smaller than the average lengths of CI_1 and vice versa. However, the differences in average lengths are small. The average lengths of all intervals decrease as $n_1 = n_2$ increases. Therefore, we conclude that the Bayes estimator of R with conjugate prior has the smallest confidence intervals.

Table 1: R Estimators for MED when $\alpha_1 = \lambda_1 = 4$.

n_1=n_2	$\alpha_2 = \lambda_2$	R	$\hat{R}_1$	MSE	$\hat{R}_2$	MSE	$\hat{R}_3$	MSE	$\hat{R}_4$	MSE
5			0.525511	0.022102	0.522178	0.027702	0.519529	0.021597	0.518679	0.019816
10			0.52919	0.012049	0.537108	0.013727	0.534711	0.011986	0.533926	0.011446
20	3.5	0.53	0.530934	0.005897	0.532657	0.006573	0.531518	0.006126	0.531143	0.005983
30			0.535688	0.003971	0.533445	0.004378	0.532653	0.004176	0.532391	0.00411
50			0.533739	0.002161	0.532825	0.002452	0.532347	0.002383	0.532189	0.00236
5			0.600885	0.020884	0.604676	0.025679	0.592568	0.020779	0.588646	0.019366
10			0.607606	0.011293	0.618733	0.012363	0.611328	0.010996	0.608886	0.01058
20	2.5	0.62	0.611239	0.003787	0.614669	0.003969	0.610846	0.003855	0.609582	0.00382
30			0.61638	0.003078	0.615435	0.003232	0.61282	0.003169	0.611953	0.003149
50			0.615092	0.002275	0.614917	0.002423	0.613324	0.002395	0.612795	0.002387
5			0.706363	0.016641	0.71817	0.019098	0.695839	0.017252	0.688286	0.016806
10			0.716092	0.008683	0.729848	0.008709	0.717479	0.008259	0.713277	0.008156
20	1.5	0.73	0.721603	0.003987	0.726592	0.004226	0.720157	0.004144	0.717988	0.004132
30			0.72668	0.002548	0.727256	0.002784	0.722893	0.002748	0.721427	0.002743
50			0.726248	0.00139	0.726898	0.001571	0.724244	0.001561	0.723355	0.001561
5			0.777895	0.012335	0.792568	0.013154	0.767002	0.013322	0.757897	0.013566
10			0.7883	0.006181	0.800214	0.00583	0.786687	0.005932	0.781939	0.006029
20	1	0.8	0.794223	0.002705	0.799411	0.002772	0.792435	0.002828	0.790037	0.002866
30			0.798762	0.001684	0.799958	0.001819	0.795259	0.001844	0.793659	0.001861
50			0.79876	0.000915	0.799705	0.001028	0.796859	0.001039	0.795897	0.001046
5			0.892682	0.004416	0.9051	0.004055	0.884248	0.005316	0.875716	0.005991
10			0.900659	0.00201	0.910073	0.001546	0.899949	0.001838	0.896093	0.001993
20	0.4	0.91	0.905116	0.000784	0.908759	0.000751	0.90366	0.000834	0.901829	0.000875
30			0.907848	0.000466	0.909049	0.00049	0.90566	0.000525	0.90447	0.000543
50			0.908125	0.00025	0.908944	0.000277	0.906909	0.00029	0.906209	0.000296
5			0.969719	0.00048	0.974387	0.000404	0.966366	0.000688	0.962476	0.00087
10			0.972674	0.000199	0.975891	0.00013	0.97231	0.000177	0.970832	0.000205
20	0.1	0.98	0.974278	0.000069	0.97551	0.000063	0.973752	0.000075	0.973096	0.000081
30			0.975142	0.000039	0.975594	0.000041	0.974438	0.000046	0.974022	0.000048
50			0.975273	0.000021	0.975568	0.000023	0.974879	0.000025	0.974639	0.000026

Table 2: AL of the Intervals when $\alpha_1 = \lambda_1 = 4$ and $\zeta = 0.05$.

$\alpha_2 = \lambda_2$	R	$n_1 = n_2$	$\mathrm{AL}(CI_1, CI_4)$	$\mathrm{AL}(CI_2)$	$\mathrm{AL}(CI_3)$
		5	0.537909	0.468571	0.380794
		10	0.404345	0.342223	0.286508
3.5	0.53	20	0.296776	0.238015	0.210207
		30	0.245103	0.194291	0.17353
		50	0.19213	0.145736	0.135972
		5	0.523399	0.450825	0.368792
2.5	0.62	10	0.389885	0.38087	0.275382
		20	0.284497	0.227294	0.201129

		30	0.233854	0.184612	0.165332
		50	0.183178	0.138668	0.129523
1.5	0.73	5	0.471778	0.390091	0.326928
		10	0.341361	0.280573	0.238548
		20	0.244293	0.192408	0.171635
		30	0.198654	0.15498	0.139823
		50	0.154879	0.116471	0.109209
1	0.8	5	0.412298	0.325089	0.280201
		10	0.288547	0.230848	0.199277
		20	0.202058	0.156684	0.141024
		30	0.162639	0.125389	0.113948
		50	0.126099	0.094182	0.088664
0.4	0.91	5	0.259462	0.180519	0.167312
		10	0.165979	0.124374	0.111405
		20	0.110143	0.082478	0.075741
		30	0.086749	0.065241	0.060179
		50	0.066412	0.048896	0.046418
0.1	0.98	5	0.092336	0.05545	0.055772
		10	0.052844	0.037118	0.034465
		20	0.033168	0.024098	0.022515
		30	0.025642	0.018902	0.017643
		50	0.019418	0.014135	0.013507

6. Data Analysis

In this section, we discuss the problem of fitting the MED to well-known data sets and compare its goodness-of-fit with ED using the Kolmogorov-Smirnov (K-S) statistic and the likelihood ratio test.

The present sets of data were reported by Badar and Priest [3] and represent the strength easured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20, and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150, and 300 mm. For illustrative purpose in this section, we consider the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge length, with sample sizes $n_1 = 69$ and $n_2 = 63$, respectively. This data is presented in Table 3. We analyze the data by subtracting 1.0 and 1.8 from the first and second data sets, respectively. These transformed data sets were analyzed by Raqab and Kundu (2005).

Table 3: Carbon-Fiber Data Sets (Badar and Priest [3]).

Data Set I: Gauge lengths of 10 mm.					**Data Set II: Gauge lengths of 10 mm.**				
1.312	1.314	1.479	1.552	1.7	1.901	2.132	2.203	2.228	2.257
1.803	1.861	1.865	1.944	1.958	2.35	2.361	2.396	2.397	2.445
1.966	1.997	2.006	2.021	2.027	2.454	2.474	2.518	2.522	2.525
2.055	2.063	2.098	2.14	2.179	2.532	2.575	2.614	2.616	2.618
2.224	2.24	2.253	2.27	2.272	2.624	2.659	2.675	2.738	2.74
2.274	2.301	2.301	2.359	2.382	2.856	2.917	2.928	2.937	2.937
2.382	2.426	2.434	2.435	2.478	2.977	2.996	3.03	3.125	3.139
2.49	2.511	2.514	2.535	2.554	3.145	3.22	3.223	3.235	3.243
2.566	2.57	2.586	2.629	2.633	3.264	3.272	3.294	3.332	3.346
2.642	2.648	2.684	2.697	2.726	3.377	3.408	3.435	3.493	3.501
2.77	2.773	2.8	2.809	2.818	3.537	3.554	3.562	3.628	3.852
2.821	2.848	2.88	2.954	3.012	3.871	3.886	3.971	4.024	4.027
3.067	3.084	3.09	3.096	3.128	4.225	4.395	5.02		
3.233	3.433	3.585	3.585						

Table 4 gives MLEs and MMEs of parameters of the MED. The 95% asymptotic confidence intervals (ACI) for the MLEs and the interval lengths for the two data sets are calculated in Table 7. 3The log-likelihood values, the Kolmogorov-Smirnov statistics based on the MLEs and the corresponding *p*-values for the modified exponential distribution and the exponential distribution are represented in Table 6. Notice that the log-likelihood values are the same for the ED and the MED. On the other hand, the MED has a smaller K-S statistic than the ED in the second data set. It is observed that the fitting results for the MED and the ED are almost the same.
The four estimators of reliability parameter $R = P(X < Y)$, when $X \sim MED(\alpha_1, \lambda_1)$ and $Y \sim MED(\alpha_2, \lambda_2)$ are estimated in Table 7 with the corresponding confidence interval (CI) and interval length (IL). Noticed that the average lengths of $\hat{R}_3$ are the shortest average lengths, which is the same result of the simulation.

Table 4: Parameter Estimations for the MED and ED

Data	**Estimator**	**MED(α, λ)**		**ED(β)**
		α	λ	β
Set I	MLE	0.244511	0.444511	0.689022
	MME	0.361464	0.561464	0.851795
Set II	MLE	0.347045	0.447045	0.794091
	MME	0.404399	0.604399	1.017675

Table 5: ACI and IL for MLEs of MED and ED

Data Set		**MED(α, λ)**		**ED(β)**
		α	λ	β
Set I	ACI	(0.08193, 0.40708)	(0.28193, 0.60708)	(0.52644, 0.85159)
	IL	0.32515	0.32515	0.325152
Set II	ACI	(0.15095, 0.54313)	(0.25095, 0.64313)	(0.59800, 0.99017)
	IL	0.39217	0.39217	0.392173

Table 6: Log-likelihood and K-S statistic for MED and ED

Data	**Model**	**Log-Likelihood**	**K-S Statistic**	***p*-value**
Set I	MED	-94.7013	0.36224	6.96×10^{-11}
	ED	-94.7013	0.36224	1.57×10^{-8}
Set II	MED	-77.5251	0.23606	0.00011
	ED	-77.5251	0.27450	0.00011

Table 7: *R* Estimators, the CI and IL for MED

	$\hat{R}_1$	$\hat{R}_2$	$\hat{R}_3$	$\hat{R}_4$
R Estimators	0.464578	0.464477	0.464455	0.465014
CI	(0.3814, 0.5504)	(0.2946, 0.6344)	(0.4057, 0.5247)	(0.3814, 0.5504)
IL	0.169037	0.339825	0.11894	0.169037

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