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The conservative averaging method development in the petroleum science

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Abstract In this paper we look over history of conservative averaging method in last 50 - 100 years. Some well-known mathematicians in classical papers investigated problems with non-classical boundary conditions, which have derivatives in boundary conditions equal or higher as derivatives in the main partial differential equation. Forty or fifty years ago it was important to construct mathematical models for intensification of the crude oil or gas output. We have experience in around 30 years in these fields. Most of the papers were published in Russian language. We have made short overlook of some most important papers. In these papers were investigating problems with non-classical boundary conditions, mathematical basis of method of conservative averaging was constructed. For the situation with layered media we have introduced integral spline, which fulfils the energy or mass conservation in new simplified (less dimensional) formulation of the problem. We introduce new hyperbolic approximation for conservative averaging method. In the paper is given a new representation for classic cubic spline. Our new formula allows calculating spline values with *O(n)* operations.

Keywords Conservative averaging, Energy conservation, Petroleum models, Integral parabolic spline, Cubic spline

1. Introduction

We start with history in the last 100 years [1]-[11]. Good short history about thermal conductivity in the 18^{th} and 19^{th} century and Fourier research is given in the paper [12].

Real processes take place in natural or technical systems with complicated structure. Very often such systems consist of separate layers with different thickness and different physical properties. It means that on the surfaces between two adjacent layers we have jump in coefficients of differential equations mathematically describing correspondent physical process. A. Buikis has developed and mathematically justified special method – conservative averaging method (CAM) [15] – [55], [60] – [63]. The main idea of CAM is that the new problem formulation in main domain has fulfilled all energy peculiarities, and fulfils conservation laws.

For the multi-layered strata we have introduced a special new type of spline [39] - [41]. We used this new type spline for multi-layer system: [21] - [24]. We use CAM for solving such problems for wide class of direct [15], [16], [67] – [82], [85] and inverse [17] and [25] - [55], [60] – [63] problems for partial differential equations with discontinuous coefficients.

I give modified description of integral parabolic spline and employ this spline for some groundwater (or other fluids) flows and pollution problems in layered stratum. Proposed method differs from methods traditionally used by mathematical modelling of groundwater pollution [8] - [14], [56] – [58] or other transport processes in natural or artificial porous media. This method as outcome gives little bit more complicated mathematical model, but it allows describing broader spectrum of physical phenomena and wider variety of geometrical and physical parameters. By the way, we give a new representation for classical cubic spline [42]. The new form of cubic spline is given in papers [42], [43]. We finish paper with generalization of Green function method for non-canonical domain [79], [85], [86].

2. Short Hystory of Conservative Averaging Method

2.1. Classical Paper

The history of conservative averaging method started with paper [1] of Kneser. The Journal *Rendiconti del circolo matematico di Palermo was* very important at beginning of 20th century. In the Editors were: E. Bertini, E. Borel, C. Caratheodory, U. Dini, R. Forsyth, I. Fredholm, J. Hadamard, D. Hilbert, F. Klein, T. Levi –Civita, A. Liapounoff, G. Mittag – Leffler, F.Osgood, E. Picard, C. Segre, W. Stekloff, Ch. De la Vallee Poussin and e. a.

In year 1914 Kneser Adolf in this reputation journal publishes paper [1] about loaded integral equations with concentrate mass:

$$\int \varphi_m(x)\varphi_n(x)\,dx + \int_{(i)} M(a)\varphi_m(a)\varphi_n(a)\,da = 0$$

In the second integral the function $M(a)$ is positive. He called such integral equations with this equality as loaded orthogonal. In his paper Kneser A. investigate boundary problem:

$$\begin{aligned} &\frac{\partial^2 U}{\partial t^2} = a^2\frac{\partial^2 U}{\partial x^2}, 0<x<l, U(0,t)=0, \\ &x=0: U=0, x=l: M\frac{\partial^2 U}{\partial t^2}+c^2u=-T\frac{\partial U}{\partial x}. \end{aligned} \qquad (1)$$

Important is moment that in the right boundary condition we have the second derivative relatively to time argument as in the main equation.

Samarskii Alexander in his paper [2] looked on slab, which was heated from left end with the furnace. Samarskii gives such mathematical model. Hi introduces two different temperatures, between whom a temperature jump is possible:

$$U(x,t)=\begin{cases} U(t), at\, x=0,\ furnace\, temp., \\ u(x,t), at\, 0<x\le l, slab\, temp. \end{cases} \qquad (2)$$

This allow to use

$$\begin{aligned} &\frac{\partial^2 U}{\partial x^2}=\frac{1}{a^2}\frac{\partial U}{\partial t}+\beta^2 U(x,t), 0<x<l, \\ &U(x,0)=0,\ 0\le x\le l, \\ &k\frac{\partial U}{\partial x}(l,t)=0, \\ &k\frac{\partial U}{\partial x}(0+0,t)=h\left[U(0+0,t)-U(0,t)\right]. \end{aligned} \qquad (3)$$

Heat exchange between furnace and slab is given by following formula (4):

$$\begin{aligned} &Q_0-\alpha_2 U(0,t)-C_0\frac{dU(0,t)}{dt}= \\ &-k\frac{\partial U}{\partial x}(0+0,t). \end{aligned} \qquad (4)$$

The definition (2) allows formulating ordinary derivative relative to t in (4). In formulas (3), (4) we have in the point $x=0$ two temperatures: $U(0+0,t), U(0,t).$ Accordingly, we have two conditions – the last one in (3) and (4).

Next interesting paper is in year 1950 written by Tikhonov A. N. [3]. In papers of Kneser and Samarskii in boundary condition (BC) the derivative relative to t is in the main equation and in BC (1), (4). In paper the derivative in BC is higher than in main differential equation (heat equation):

$$\frac{\partial U}{\partial t}=a^2\frac{\partial^2 U}{\partial x^2},$$

$$\sum_{k=0}^{m}\alpha_k\frac{\partial^k u(0,t)}{\partial x^k}=f(t),\alpha_m\neq 0,\alpha_0\neq 0,m>1. \quad (5)$$

In paper [3] the existence and uniqueness of problem is investigated.

The books [4], [5] look at different mathematical models and authors also consider the condition of non-ideal thermal contact between two different materials. Authors states that "temperature has discontinuity when passing through the boundary of non-ideal contact, with the height of the step being proportional to the heat flow, i.e.

$$\left[T\right]=\frac{k}{\alpha}\frac{\partial T}{\partial n},\left(x,y,z\right)\in S,$$

where the coefficient of contact heat transfer α is associated with the contact conditions" [5]. In our papers [16], [17] was found, that coefficient α can be expressed trough physical and geometrical properties of the interlayer:

$$\alpha=\frac{k_0}{\delta}.$$

Here k_0 is heat conductivity coefficient of interlayer and δ is interlayer thickness.

2.2. Praxis in the Petroleum Output

In time after Second World War was very important to raise oil output, especially by thermal methods. The books [8] - [10], [56] - [58] and the paper [6] are good examples. The pumping hot or cold water in the petroleum layer was investigated in [6]. Lauwerier assumed that in oil layer heat conductivity in orthogonal direction ς is equivalence to infinity, but in the horizontal direction ξ in the oil layer and in neighborhood material heat conductivity is zero. The first assumption means that in oil layer the temperature in vertical direction is constant. This supposition allows us to get from second partial differential equation a new BC with highest derivative in time. The mathematical model looks like this:

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial \varsigma^2},\xi>0,\varsigma>0,\tau>0,$$

$$\frac{\partial u}{\partial \tau}=\frac{\partial u}{\partial \varsigma}-\frac{\partial u}{\partial \xi},\varsigma=0,\tau>0,$$

$$u\big|_{\xi=\varsigma=0}=1,u\big|_{x=0}=0.$$

In this case the non-dimensional solution is in the form:

$$u(\xi,\varsigma,\tau)=erfc\frac{\xi+\varsigma}{\sqrt{\tau-\xi}}\eta\left(\tau-\xi\right). \quad (6)$$

In our paper [35] were looked at Lauwerier solution with different, third type boundary condition. It gives continuity of solution in the corner point and can be represented as Lauwerier solution plus function which is going to zero as $\frac{1}{\sqrt{\tau}}$ for each fixed point.

In the paper [7] authors looked at tertiary method of oil production: the simplest possible initial value problem of heat conduction with a free boundary moving within the concentrate capacity. In the porous layer a non-isothermal incompressible fluid flow is accompanied by a phase change: the melting of paraffin sediments in oil saturated media by means of hot liquid injection. The mathematical model is generalizing of the Lauwerier's scheme with heat conduction in the oil layer. The mathematical model is following:

$$k_0 u_{0zz} = u_{0t}, -\infty < x < \infty, z > 0, t > 0,$$
$$k_i u_{ixx} + k_0 / h\, u_{0z}\big|_{z=0} - c_i v u_{ix} = c_i u_{it},$$
$$-\infty < x < \infty, z = 0,$$
$$u_1 : -\infty < x < s(t), u_2 : s(t) < x < \infty, t > 0,$$
$$\dot{s}(t) = u_{2x}\left[s(t),t\right] - u_{1x}\left[s(t),t\right], t > 0.$$

The main equation must be added boundary and initial condition, which we omit. Important is to see in this model the second equation, which is boundary conditions, with higher derivative according argument t .

2.3 Temperature fields in petroleum layers

All our papers were written in Russian in the 1960-1980-ies and were not known in English speaking countries. Here we give short overlies for part of these investigations.

The Latvian University Computing Center from Moscow at the end of 1950ties had received computer BESM-2 (large electronic big mainframe computer). One of the most important problems was the extraction of oil in multi-layer systems, if the in hot water injection is used. A typical oilfield was in Westernn Kazakhstan – Usen deposit. There was quite a lot of paraffin in oil layer, which sets hard around 40 ° C. These many layers in the system could lead to the fact that lower permeably layers are blocked [51], [52], [13], [14]. The second important property was high viscosity of petroleum in comparison with water viscosity. The Latvian University Computing Center in second part of 60-ies had research agreement with All-union of Petroleum and Gas Investigation Institute about thermal methods in the multi-layered petroleum stratum of petroleum investigation in the West Kazakhstan Usen deposit. A. Buikis was leading researcher in this agreement. Main results were published in the articles [51] - [54]. A. Buikis in these papers solved the Buckley-Leverett equation [56] - [58] for immiscible two phase fluids flow in the multi-layered system. The Buckley-Leverett equation with artificial viscosity has the form:

$$\frac{\partial \sigma}{\partial t} + \frac{v}{m}\frac{\varphi(\sigma, \mu(T))}{\partial x} = a^2 \frac{\partial^2 \sigma}{\partial x^2}. \quad (7)$$

Here $\sigma(x,t)$ is the saturation of the two phase fluid, the φ is Buckley function, the μ is viscosity of water and oil ratio and $T(x,t)$ is temperature field.

The Buckley-Leverett equation was approximated by difference scheme:

$$\frac{2\sigma_i^{k+1} - \sigma_{i+1}^k - \sigma_{i-1}^k}{2\Delta t} + \frac{v}{m}\frac{\varphi_{i+1}^k - \varphi_{i-1}^k}{2\Delta x} =$$
$$a^2 \frac{\sigma_{i+1}^{k+1} - 2\sigma_i^{k+1} + \sigma_{i-1}^{k+1}}{(\Delta x)^2}, \frac{\Delta x}{\Delta t} = \tilde{v}_s,$$
$$\sigma_i^k = \sigma(x_i, t_k), \varphi_i^k = \varphi\left(\sigma_i^k, \mu\left(T(x_i, t_k)\right)\right).$$

Further $\tilde{v}_s = \frac{v}{m}\frac{\partial}{\partial \sigma}\varphi(\tilde{\sigma}_s, \mu(T_0))$, $\tilde{\sigma}_s$ is saturation jump and T_0 is initial temperature field. The disappeared diffusivity was calculated from inequality:

$$a^2 = \Delta x \tilde{v}_s \frac{\sqrt{1+\tilde{v}_s^2}}{\sqrt{1+\tilde{v}_s^2} - \tilde{v}_s}.$$

In the papers was evaluated the procedures and calculation results for non-isothermal two phase flow for cold and hot water injection. This immiscible two phase fluids flow in the multi-layered system for cases with given filtration velocities and for given pressure field between injection and production holes. On the basis of this investigation atom power station was built to produce hot water. Some years later two dimensional for immiscible two phase fluids flow when one of fluids - petroleum has non-Newtonian properties [54] were numerically calculated. The temperature field in the multilayered system was calculated by Lauwerier solution, modified by "additive formula" [14]. Temperature in separate layer was calculated as sum of other layers. If the sum was greater as maximal temperature the "additive formula" gives the maximal temperature. We have calculated the error which was about 30%. After we think about this problem and later we introduce the integral splines, see section 4.

In this time it was popular to think about underground nuclear explosions to intensify petroleum and gas outcome [59]. We investigate two dimensional gas movements in the papers [60]-[62]. The mathematical models were constructed in polar system of co-ordinates with very big non-homogeneousness: permeability changed 10^3 times. In all our papers some averaging procedures were made.

In the paper [25] was investigated heat transfer by conduction and convection in porous media (layer), if we distinguish the temperature of the fluid $u\left(x_2,t\right)$ and the temperature of the matrix $v(x_1,x_2,t)$:

- The temperature of the layer itself, averaged by the density of the layer (co-ordinate x_1) with the method of conservative averaging, approximated with a constant;
- Heat transfer in layer and surrounding media in the direction of convection (co-ordinate x_2) is not taken into account;
- The layer itself is considered as two temperatures in medium: porous matrix and the liquid that is flowing threw it.

In nature two mathematical models are investigated

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x_1^2}, x_1, x_2, t>0,$$

$$\begin{cases}\dfrac{\partial v}{\partial t}=\dfrac{\partial v}{\partial x_1}+\alpha\left(u-v\right),\\ \qquad\qquad\qquad x_1=0, x_2, t>0,\\ \dfrac{\partial u}{\partial t}=-\dfrac{w}{m}\dfrac{\partial u}{\partial x_2}+\alpha_0\left(v-u\right),\end{cases}\qquad(8)$$

$$v\big|_{t=0}=v^0(x_1,x_2), u\big|_{t=0}=u^0(x_2),$$

$$u\big|_{x_1=x_2=0}=\mu\left(t\right), \left|u\right|\Big|_{x_1+x_2\to\infty}\to const.$$

In mathematical model the averaging of the temperature regarding to averaged thermal qualities are carried out (index "0" refers to liquid, index "1" refers to porous matrix).

$$\tilde{u}(x_2,t)=\frac{mc_0\,u\left(x_2,t\right)+\left(1-m\right)c_1\,v(0,x_2,t)}{mc_0+\left(1-m\right)c_1}.$$

If we investigate the layer as homogeneous media (with averaged thermal qualities), then this kind of problem is also called the Lauwerier scheme $u_L\left(x_2,t\right)$ (see formula (6)). In the same layer the temperature is defined with equation

$$\frac{\partial v}{\partial t}=\beta\frac{\partial v}{\partial x_1}-\tilde{w}\frac{\partial v}{\partial x_2}.$$

The values of coefficients $\beta,\tilde{w}$ depend on the physical and geometrical qualities of porous media and the quantity of filtration velocity. In this work is shown (with the help of analytical formulae's and method of finite difference) that both approaches give fundamentally different results.

One of the authors of the articles [49], [50] dealt with the problem of the heat water-oil fluid and the layer porous matrix when temperatures are different. Mathematical model of the problem with formulation in the conditions of concentrated heat capacity, and its solution is obtained with the Laplace integral transform method. The solution of the problem shows that the problem of the homogeneous layer is the same as the problem of cracked porous layer. The temperature fronts are in completely different positions, and it was substantial to the influence of temperature to the field of porous oil.

In paper [28] is considered the heat transfer in the way of heat conduction and convection in multi layered porous system, if temperature of separate layers is averaged by the density of corresponding layer (assumptions are similar as in previous paper – without the second component). In the paper finite difference scheme with energy conservation is offered. Further stability theorem of the scheme in C metrics is formulated and an economic, factorization type algorithm for solution for three point schemes (on coordinate perpendicular to depth of layers) is offered [26], [27]:

$$\frac{\partial u}{\partial t} = a_k^2 \frac{\partial^2 u}{\partial x_1^2}, x_1^{(k-1)} < x_1 < x_1^{(k)}, x_2, t > 0, k = \overline{1, K+1};$$

$$\frac{\partial u}{\partial t} = \frac{\beta_k^+}{2} \frac{\partial u}{\partial x_1}\bigg|_{x_1 = x_1^{(k)}+0} - \frac{\beta_k^-}{2} \frac{\partial u}{\partial x_1}\bigg|_{x_1 = x_1^{(k)}-0} - w_k \frac{\partial u}{\partial x_2},$$

$$x_1 = x_1^{(k)}, x_2, t > 0, k = \overline{1, K};$$

$u|_{t=0} = u^0(x_1, x_2), u|_{x = x_0^{(k)}} = \mu_k(t), |u|_{x_1^2 + x_2 \to \infty} < \infty.$ Boundary and initial conditions are evident for the difference schemes. We give the most important difference equations:

$$y_t = a_k^2 \Lambda_{11} y^{\left(\sigma_1^{(k)}\right)}, N_1^{(k-1)} < i < N_1^{(k)}, k = \overline{1, K+1};$$

$$\left(1 + \frac{\beta_k^+ h_1^{(k+1)}}{4a_{k+1}^2} + \frac{\beta_k^- h_1^{(k)}}{4a_k^2}\right) y_t = \frac{\beta_k^+}{2} \Lambda_1^+ y^{\left(\sigma_1^{(k+1)}\right)} - \quad (9)$$

$$-\frac{\beta_k^-}{2} \Lambda_1^- y^{\left(\sigma_1^{(k)}\right)} - w_k \Lambda_2^- y^{\left(\sigma_1^{(k)}\right)}, i = N_k^{(k)}, k = \overline{1, K}.$$

Theorem 1. If for $0 \le \sigma_1^{(k)} < 1,\ 0 \le \sigma_2^{(k)} < 1$, conditions are fulfilled:

$$\tau \le \left(\min_{1 \le k \le K+1} \tau_1^{(k)}, \min_{1 \le k \le K} \tau_2^{(k)} \right).$$

Where

$$\tau_1^{(k)} = \frac{\left(h_1^{(k)}\right)^2}{2a_k^2\left(1 - \sigma_1^{(k)}\right)}, \tau_2^{(k)} =$$

$$\frac{1 + \dfrac{\beta_k^+ h_1^{(k+1)}}{4a_{k+1}^2} + \dfrac{\beta_k^- h_1^{(k)}}{4a_k^2}}{\dfrac{\beta_k^+\left(1 - \sigma_1^{(k+1)}\right)}{2h_1^{(k+1)}} + \dfrac{\beta_k^-\left(1 - \sigma_1^{(k)}\right)}{2h_1^{(k)}} + \dfrac{w_k\left(1 - \sigma_2^{(k)}\right)}{h_2}},$$

then difference scheme (8) is stable in C norm. The difference scheme with $\sigma_1^{(k)} = \sigma_2^{(k)} = 1$ is absolutely stable.

In paper [27] is considered the heat transfer in the way of heat conduction and convection in two dimensional multilayered porous – fractured systems, if temperature that separates layers is averaged conservatively by the width of corresponding layer (consumptions as in papers [28]):

$$\frac{\partial T}{\partial t} = a^2\left(x_1\right)\frac{\partial^2 T}{\partial x_1^2},\ x_1 \neq x_1^{(k)}, x_2, t > 0;$$
$$\frac{\partial T}{\partial t} = \frac{\beta_k^+}{2}\frac{\partial T}{\partial x_1}\bigg|_{x_1 = x_1^{(k)}+0} - \frac{\beta_k^-}{2}\frac{\partial T}{\partial x_1}\bigg|_{x_1 = x_1^{(k)}-0} -$$
$$-w_k\frac{\partial T}{\partial x_2} + \alpha_k\left(\Theta_k - T\right), x_1 = x_1^{(k)}, x_2, t > 0;$$
$$\frac{\partial \Theta_k}{\partial t} = -w_k^0\frac{\partial \Theta_k}{\partial x_2} + \alpha_k^0\left(T - \Theta_k\right),$$
$$T\big|_{t=0} = T^0(x_1, x_2),\ \Theta_k\big|_{t=0} = \Theta_k^0(x_2),$$
$$T\big|_{x=x_1^{(k)}, x_2=0} = T_k^1(t),\ \Theta_k\big|_{x_2=0} = \Theta_k^1\left(t\right).$$

In paper there is shown the scheme of finite differences with scales. In the new moment of time the equation of liquid is switched off and a classical type algorithm is offered [32], [33].

In paper [34] two dimensional heat transfer in the way of heat conducting and convection in multilayered porous system, witch contains sub-layers between any of two basic layers is inspected.

Averaging via corresponding depth of layer is carried out, and the analogue of Lauwerier model for multi-layered system is obtained:

$$\frac{\partial T}{\partial t} = a^2\left(x_1\right)\frac{\partial^2 T}{\partial x_1^2},\ x_1 \neq x_1^{(k)}, x_2, t > 0;$$
$$\frac{\partial T}{\partial t} = \beta_k^+\frac{\partial T}{\partial x_1}\bigg|_{x_1 = x_1^{(k)}+0} - \beta_k^-\frac{\partial T}{\partial x_1}\bigg|_{x_1 = x_1^{(k)}-0} -$$
$$-w_k\frac{\partial T}{\partial x_2},\ x_1 = x_1^{(k)}, x_2, t > 0;$$
$$T\big|_{t=0} = T^0(x_1, x_2),\ T\big|_{x=x_1^{(k)}, x_2=0} = T_k^1(t).$$

For this kind of equation system a difference scheme with weights is offered. The stability term in steady metrics (analogue of theorem 1) for difference scheme and algorithm for the solution of difference scheme is formulated. Also the statement for multi – fractured system (the extension of short material [26]) is examined as well as the exclusion of difference equation for liquid from the system of equations. At the end of the paper calculations are given, which show that the assumption about the linearity of temperature in intermediate layer made in monographers' [18], [20], cannot fulfill if the depth of layer is thick enough.

In paper [26] is considered the heat transfer in the way of heat conduction and convection in two dimensional multilayered porous or porous – fractured systems, if temperatures that separate layers are averaged conservatively by the depth of corresponding layer. In the end of the paper there are given concrete calculations, which show that the assumption about the linearity of the temperature in sub-layers can not fulfill: the solution can be with the inner minimum.

In paper [31] with the help of the method of conservative averaging the problem of convection – diffusion of ground – waters is considered. For the obtained non-classical problem a finite difference scheme with scales is constructed, the sufficient parameter of stability in steady metrics is obtained, as well as considerations about the reduction of numerical diffusion in calculations are expressed. The results are in graphics, they show the placement of convective front in the end of calculations: after 27.5 years.

In author's paper [31] mathematical model for two layered system with diffusion crosswise the elongated direction of the layer, when the coefficients of both layers varies greatly, is inspected. With the help of averaging process, the differential equation of second layer is excluded, and it transfers to non – classical border condition. Vast correlation of diffusion coefficients is considered. Full statement is given; other solutions: constant in the layer with major, IPS there in and both layers IPS. For the new statement the difference scheme is offered, with the help of maximum principle a criteria of stability for difference scheme is extracted and a version of calculations is offered. So in paper [31], the transfer of substance transverse (by the depth of layer), diffusion and convection by the longitudinal line of the layer in two dimensional multi layered, porous, isolated media is considered.

Two dimensional difference schemes are offered. Afterwards, the statement with conservative averaging by layer, in which the coefficient of diffusion is bigger, is examined; as the result we obtain problem for one layer with non-classical border condition and corresponding difference scheme, as in [26]. Further approximation with polynomial of second degree is considered and for this model a difference scheme is offered. Finally, a version with approximation with second degree polynomial (IPS) in both layers is examined, and also in this case a difference scheme is offered. At the end of the paper the results of calculations with different parameters (when the coefficients of diffusion differ 20 times and 5 times) are compared.

In papers [30]-[33] is given a mathematical model of transfer for two dimensional multi layered system with sub-layers, when diffusion takes place crosswise the elongated direction of the layer, when the diffusion coefficients and speed of convection of both layers varies. In sub-layers linear approximation is used. With the help of generalized IPS in irregular mode system of one dimensional equation is obtained, and there is shown that the obtained system differs from classical scheme [11], which does not allow the depth of sub-layers to be equal to zero. Two layer systems with one interlayer were described by system:

$$H_i k_i \left(\frac{\partial^2 \Phi_i}{\partial x^2} + \frac{\partial^2 \Phi_i}{\partial y^2} \right) + \alpha_{01} \left(\Phi_j - \Phi_i \right) = -H_i f_i, i = 0,1;\ j = 1-i. \qquad (10)$$

Index $i = 0,1$ is for layers, but $i = 1/2$ - for interlayer (aquitard).

For classical scheme $\alpha_{01} = k_{1/2} H_{1/2}^{-1}$, our statement gives [44]:

$$\tilde{\alpha}_{01} = \left[\frac{H_0}{3k_0} + \frac{H_{1/2}}{k_{1/2}} + \frac{H_1}{3k_1} \right]^{-1}. \qquad (11)$$

It is easy to see, that system (9) are degenerated for $H_{1/2} \to +0$, but with coefficient $\tilde{\alpha}_{01}$ the system of differential equations is correct.

3. Conservative Averaging Method with Hyperbolic Function

In all situations there were two or more sub-domains. Now we describe case, in which only one domain exists. In this case the character scale in one direction is smaller than in other direction. In our publications [15] – [24], [21] – [24] were only the first case was examined: with two or more sub-domains.

Let it be given a continuous function $U(x), x \in [0,l]$ and positive constant k. This function is unknown, but we have information about averaged value:

$$u = l^{-1} \int_0^l U(x) dx. \qquad (12)$$

Additionally is known boundary conditions in points $x = 0, x = l$.

We will approximate the function $U(x)$ in the form (parameter α is free constant):

$$U(x) = u + +ml\frac{\sinh\left[\alpha\left(x-\frac{l}{2}\right)\right]}{2\sinh\left(\alpha\frac{l}{2}\right)} +$$

$$eG\left[\frac{\sinh^2\left[\alpha\left(x-\frac{l}{2}\right)\right]}{\sinh^2\left(\alpha\frac{l}{2}\right)} - A_0\right],$$

$$A_0 = \frac{\frac{\sinh(\alpha l)}{\alpha l - 1}}{\cosh(\alpha l) - 1}, G = \frac{l}{k}.$$

This form fulfills the equality (11). Two independent coefficients m, e we can determine from boundary conditions.

We start with first type BC:

$$U(0) = \Phi_0, U(l) = \Phi_1. \qquad (13)$$

From BC we obtain:

$$e = \frac{\Phi_0 + \Phi_1 - u}{2G(1-A_0)}, m = \frac{\Phi_0 + \Phi_1}{2} - \frac{2\Phi_0}{l} - u\left(1 - \frac{2}{l}\right).$$

For the second type BC:

$$U'(0) = \Phi_0, U'(l) = \Phi_1.$$

Similarly we obtain:

$$m = \frac{\Phi_0 + \Phi_1}{\alpha l}\tanh\left(\frac{\alpha l}{2}\right), e = \frac{\Phi_0 + \Phi_1}{\alpha G}\tanh\left(\frac{\alpha l}{2}\right).$$

Third type BC:

$$-U'(0) + \lambda_0 U(0) = \Phi_0,\ U'(l) + \lambda_1 U(l) = \Phi_1.$$

4. Integral Spline

In this section we shortly describe integral parabolic spline from papers [39] - [39], [7] – [10]. Spline with integral values in the recent years is often in scientific publications [63]. This spline is important because it helps solving mathematical models with mechanical or physical content [65] as this was for integral parabolic or rational spline [41]. The article [32] deals with the problem for layered media that can be solved with averaged integral parabolic or averaged integral rational spline.

Second part of this section is non-traditional representations of classic cubic spline [42], [43] and monograph [65].

4.1. Integral parabolic spline

Let it be given a continuous, piecewise-smooth function $U(x), x \in [a,b]$. Further, let it be given, that the first derivative $U'(x)$ has a finite jump in the inner points x_i :

$$k_{i-1}U'(x_i - 0) = k_i U'(x_i + 0),\ i = 1,\ldots,N. \quad (14)$$

Here $k_i, i = 0,\ldots,N$ are known (given) strongly positive coefficients. We additionally have following continuity equalities in the same points:

$$U(x_i - 0) = U(x_i + 0), i = 1,\ldots,N. \qquad (15)$$

Let's additionally be given the average integral values u_i of the function $U(x)$ over the all sub-segments $[x_i, x_{i+1}],\ i = 0,...,N, x_0 = a, x_{N+1} = b$:

$$H_i = x_{i+1} - x_i, i = 0,...,N. \qquad (16)$$

The goal of the interpolation problem is to (approximately) reconstruct the function $U(x)$, being based on conditions (14)-(16) and following general boundary conditions (BC) on the interval end points $x = a$ and $x = b$:

$$\begin{aligned} &-\nu_0 k_0 U'(a) + \lambda_0 U(a) = \Phi_0, \\ &\nu_1 k_N U'(b) + \lambda_1 U(b) = \Phi_1. \end{aligned} \qquad (17, 18)$$

Such form of BC is typical for the ordinary or partial differential equations. In papers [7]-[10] it was proved, that this interpolation problem can be solved by second order polynomial spline of following form:

$$S(x) = u_i + m_i(x - \bar{x}_i) + e_i\left[\frac{(x-\bar{x}_i)^2}{k_i H_i} - \frac{G_i}{12}\right], \quad (19)$$

$$\bar{x}_i = \left(x_i + x_{i+1}\right)/2, G_i = H_i / k_i > 0.$$

This form of spline <u>exactly fulfills</u> the integral equalities (3) for all real values of unknown coefficients m_i, e_i. For the determination of $2(N+1)$ free coefficients; we have exactly the same number of equations (13), (14), (16) and (17). In papers [37], [38], [21]-[24] it was shown, that all coefficients m_i can be represented through coefficients e_i and we obtain the system of linear algebraic equations for $i = \overline{1, N-1}$.

We propose in thesis a different <u>normalized</u> form for the calculation of the spline coefficients e_i :

$$\begin{aligned} &a_i e_{i-1} + (1 + a_i + b_i) e_i + b_i e_{i+1} = \\ &f_i^- u_{i-1} - f_i u_i + f_i^+ u_{i+1}, i = 1,..., N-1. \end{aligned} \qquad (20)$$

Additionally we use other from [21]-[23] form for the first and last equations of the system of linear algebraic equations:

$$\begin{aligned} &(1 + a_0 + b_0) e_0 + b_0 e_1 = f_0^- u_{-1} - f_0 u_0 + f_0^+ u_1, \\ &a_N e_{N-1} + (1 + a_N + b_N) e_N = \\ &f_N^- u_{N-1} - f_N u_N + f_N^+ u_{N+1}. \end{aligned} \qquad (21)$$

Here $f_i = f_i^- + f_i^+$ and

$$\begin{aligned} &a_i = G_{i-1} / \left(G_i + G_{i-1}\right), b_i = G_{i+1} / \left(G_i + G_{i+1}\right), \\ &f_i^- = 3 / \left(G_i + G_{i-1}\right), f_i^+ = 3 / \left(G_i + G_{i+1}\right). \end{aligned}$$

Instead of (21) here we also propose other form of the explicit representation for coefficients e_i. This representation shows in explicit form the influence of the BC type and its right hand side on the spline:

$$\begin{aligned} &e_i = \gamma_i^{(0)} f_0^- u_{-1} + \gamma_i^{(1)} f_N^+ u_{N+1} + \sum_{j=0}^{N} \beta_{ij} u_j, \\ &i = \overline{0, N}. \end{aligned} \qquad (22)$$

The coefficients in the representation (22) are determinates from following systems of linear algebraic equations:

a) the system for $\gamma_i^{(0)}$:

$$\begin{aligned}
&(1+a_0+b_0)\gamma_0^{(0)}+b_0\gamma_1^{(0)}=1,\\
&a_i\gamma_{i-1}^{(0)}+(1+a_i+b_i)\gamma_i^{(0)}+b_i\gamma_{i+1}^{(0)}=0,\\
&i=1,\dots,N-1,\\
&a_N\gamma_{N-1}^{(0)}+(1+a_N+b_N)\gamma_N^{(0)}=0;
\end{aligned}\tag{23}$$

b) the system for $\gamma_i^{(1)}$:

$$\begin{aligned}
&(1+a_0+b_0)\gamma_0^{(1)}+b_0\gamma_1^{(1)}=0,\\
&a_i\gamma_{i-1}^{(1)}+(1+a_i+b_i)\gamma_i^{(1)}+b_i\gamma_{i+1}^{(1)}=0,\\
&i=1,\dots,N-1,\\
&a_N\gamma_{N-1}^{(1)}+(1+a_N+b_N)\gamma_N^{(1)}=1.
\end{aligned}\tag{24}$$

b) and $N+1$ systems ($j=0,\dots,N$) for β_{ij}:

$$\begin{aligned}
&(1+a_0+b_0)\beta_{0,j}+b_0\beta_{1,j}=0,\\
&a_i\beta_{i-1,j}+(1+a_i+b_i)\beta_{ij}+b_i\beta_{i+1,j}=\\
&f_j^-\delta_{i-1,j}-f_j\delta_{i,j}+f_j^+\delta_{i+1,j}, i=1,\dots,N-1,\\
&a_N\beta_{N-1,j}+(1+a_N+b_N)\beta_{N,j}=0.
\end{aligned}\tag{25}$$

We would like to draw reader's attention to several important aspects of this new type of integral spline. Firstly, this spline interpolates exactly the average integral value (16) of the function $U(x)$. Secondly, it fulfills exactly both conjugations conditions (17), (18). Thirdly, the new type of representation (22) has interesting and very important property in application to differential equations. As reader can see, the components of the vector $\gamma^{(k)}=(\gamma_i^{(k)})_{i=0}^N, k=\{0,1\}$ and of the matrix $\beta=(\beta_{ij})_{i,j=0}^N$ depend on the location of grid points x_i, coefficients k_i and type of BC (17), (18), but they are independent from averaged integral values a u_i and right hand sides' values Φ_0,Φ_1 of BC. This property implies that for fixed grid points and coefficients k_i we need to calculate the components of the two vectors $\gamma^{(k)}$ and the matrix β only once. After this calculation, for the construction of the integral parabolic spline we need only to compute the finite sum (22). This representation (22) is very important by utilizing this spline for the differential equations with discontinuous coefficients. The approximation error of the IPS can be estimated as follows.

Theorem 2. If the continuous, piecewise smooth function $U(x), x\in[a,b]$ fulfills the conditions (14)-(17):

$$U(x_i-0)=U(x_i+0), k_{i-1}U'(x_i-0)=k_iU'(x_i+0),$$

$$i=1,\dots,N, u_i=H_i^{-1}\int_{x_i}^{x_{i+1}}U(x)dx,$$

then the interpolation with the IPS can be estimated with following inequality:

$$\begin{aligned}
&\left|U^{(p)}(x)-S^{(p)}(x)\right|\le C_p\alpha_N\left\|\tilde\Delta_N\right\|^{2-p},\\
&p=0,1,2; \tilde\Delta_N=\max_i G_i, \alpha_N=\omega\left(U'',\left\|\tilde\Delta_N\right\|\right).
\end{aligned}\tag{26}$$

Here ω is continuity modulus of corresponding function on the grid:

$$\omega(U,\delta)=\max_{\substack{|h|\le\delta\\ x,x+h\in[a,b]}}\left|U(x+h)-U(x)\right|.$$

This type of splines was used in different papers [21]-[34] of filtration processes in the multi-layered systems. The paper [63] is devoted to the field transistor technology. The mathematical model with integral parabolic spline allows reducing the two-dimensional problem to the system of one-dimensional system of partial differential equations.

4.2. New representation for classic cubic spline

The classical interpolation problem for function $u(x)$ is formulated as follows. Let it be given the values $u_i = u(x_i), i = 0,...,N$ on the grid $\Delta := \{a = x_0 < x_1 < ... < x_N = b\}$, $H_i = x_{i+1} - x_i$. Interpolation by classical cubic spline $S_3(x) \in C^2[a,b]$ for sub segment $x \in [x_i, x_{i+1}]$ regarding the normalized argument $t = (x - x_i)/H_i$ reduces to determination of the cubic spline of second derivatives $M_i = S_3''(x_i)$ [64]. The second derivative of the cubic spline is linear function:

$$S_3''(x) = \frac{1}{H_i}\left[M_i(x_{i+1} - x) + M_{i+1}(x - x_i)\right],$$
$$i = \overline{0, N-1}.$$

The integrating gives [42]:

$$S_3'(x) = \frac{1}{2H_i}\left[-M_i(x_{i+1} - x)^2 + M_{i+1}(x - x_i)^2\right] + C_1$$

The integrating ones more give the representation for the spline with two constants:

$$S_3(x) = \frac{1}{2H_i}\left[M_i(x_{i+1} - x)^3 + M_{i+1}(x - x_i)^3\right]$$
$$+ C_1 x + C_2$$

Both two constants can be easy calculates in the points $M_i = S_3''(x_i)$ and $x = x_{i+1}$:

$$H_i^2/6M_i + C_1 x_i + C_0 = u_i,$$
$$H_i^2/6M_{i+1} + C_1 x_{i+1} + C_0 = u_{i+1.}$$

Subtracting from the second equality the first one we obtain:

$$C_1 = \frac{u_{i+1} - u_i}{H_i} - \frac{H_i}{6}(M_{i+1} - M_i).$$

With this value of constant we can rewrite the first derivative in the form:

$$S_3'(x) = \frac{1}{2H_i}\left[-M_i(x_{i+1} - x)^2 + M_{i+1}(x - x_i)^2\right]$$
$$-\frac{H_i}{6}(M_{i+1} - M_i) + \frac{u_{i+1} - u_i}{H_i}.$$

The continuity for the first derivative in the point x_i:

$$S_3'(x_i + 0) = -\frac{M_i H_i}{2} - \frac{H_i}{6}(M_{i+1} - M_i) + \frac{u_{i+1} - u_i}{H_i},$$
$$S_3'(x_i - 0) = \frac{M_i H_{i-1}}{2} - \frac{H_{i-1}}{6}(M_i - M_{i-1}) + \frac{u_i - u_{i-1}}{H_{i-1}}.$$

The equality of both equations gives:

$$\frac{H_{i-1}}{6}M_{i-1}+\frac{H_{i-1}+H_i}{3}M_i+\frac{H_i}{2}M_{i+1}= \frac{u_{i-1}-u_i}{H_{i-1}}+\frac{u_{i+1}-u_i}{H_i}. \quad (27)$$

We can rewrite last system of linear equations in the following form [42]:

$$a_iM_{i-1}+2M_i+b_iM_{i+1}= f_i^-u_{i-1}-f_iu_i+f_i^+u_{i+1}, i=1,...,N-1.$$

The 3-diagonal system of linear algebraic equations for calculation of the coefficients M_i is identical with the system (20). The formulae for the coefficients for cubic spline are as follows:

$$a_i=H_{i-1}/\left(H_{i-1}+H_i\right), b_i=H_i/\left(H_{i-1}+H_i\right),$$
$$f_i^-=6a_i/H_{i-1}^2,\ f_i^+=6b_i/H_i^2.$$

We can see that we have following equality:

$$a_i+b_i=1.$$

It means that system of linear equations (27) is equivalent with system for integral parabolic spline (20). We confine ourselves to two of most frequently used types of BC:

1) Is given first derivative on both end points:

$$S_3'(a)=u_0', S_3'(b)=u_N'; \quad (29')$$

2) In both end points are given second derivative:

$$S_3''(a)=u_0'', S_3''(b)=u_N''. \quad (29'')$$

We write them in the common form:

$$2M_0+b_0M_1=f_0^+(u_1-u_0)+F_0,$$
$$a_NM_{N-1}+2M_N=f_N^-(u_{N-1}-u_N)+F_N.$$

Here coefficients of the equations (29’), (29’’) have following expressions:

1) $$b_0=1, F_0=-6u_0'/H_0, a_N=1, \quad F_N=6u_N'/H_{N-1}; \quad (30)$$

2) $$b_0=f_0^+=0, F_0=2u_0'', \quad a_N=f_N^-=0, F_N=2u_N''. \quad (31)$$

For the coefficients M_i of the cubic spline instead of representation (22) for integral parabolic spline we have following expression:

$$M_i=\gamma_i^{(0)}F_0+\gamma_i^{(1)}F_N+\sum_{j=0}^{N}\beta_{ij}u_j,\ i=\overline{0,N}. \quad (32)$$

For the coefficients we have following systems of the linear algebraic equations. Firstly for the two coefficients $\gamma_i^{(0)},\gamma_i^{(1)}$:

$$2\gamma_0^{(0)}+b_0\gamma_1^{(0)}=1,$$
$$a_i\gamma_{i-1}^{(0)}+2\gamma_i^{(0)}+b_i\gamma_{i+1}^{(0)}=0, i=1,...,N-1,$$
$$a_N\gamma_{N-1}^{(0)}+2\gamma_N^{(0)}=0.$$

$$2\gamma_0^{(1)} + b_0\gamma_1^{(1)} = 0,$$
$$a_i\gamma_{i-1}^{(1)} + 2\gamma_i^{(1)} + b_i\gamma_{i+1}^{(1)} = 0, i = 1,..., N-1,$$
$$a_N\gamma_{N-1}^{(1)} + 2\gamma_N^{(1)} = 1.$$

Finally for the $N+1$ systems for coefficients β_{ij} :

$$2\beta_{0,j} + b_0\beta_{1,j} = 0,$$
$$a_i\beta_{i-1,j} + 2\beta_{ij} + b_i\beta_{i+1,j} =$$
$$f_j^-\delta_{i-1,j} - f_j\delta_{i,j} + f_j^+\delta_{i+1,j}, i = 1,..., N-1,$$
$$a_N\beta_{N-1,j} + 2\beta_{N,j} = 0.$$

This gives interesting new explicit formula for classical cubic spline [42], [43]:

$$S_3(t) = u_i(1-t) + u_{i+1}t - \frac{H_i^2}{6}t(1-t)\times\Big[3(\gamma_i^{(0)}F_0 +$$
$$\gamma_i^{(1)}F_N) + \sum_{j=0}^{N}[(2-t)\beta_{ij} + (1+t)\beta_{i+1,j}]u_j\Big].$$

Cubic spline explicit formula on the according argument x is following [42]:

$$S_3(x) = \frac{H_i - x + x_i}{H_i}u_i + \frac{x - x_i}{H_i}u_{i+1} -$$
$$\frac{(x-x_i)(H_i - x + x_i)}{6_i}\Big[3(\gamma_i^{(0)}F_0 + \gamma_i^{(1)}F_N) +$$
$$\sum_{j=0}^{N}[\left(\frac{1}{a_{i+1}} - \frac{x}{H_i}\right)\beta_{ij} + (\frac{1}{a_{i+1}} + \frac{x}{H_i})\beta_{i+1,j}]u_j\Big].$$

4.3. On recurrence equation-based solution for the cubic spline

In the paper [66] was written cubic spline in the form that is similar to (28):

$$H_{i-1}M_{i-1} + 2\left(H_{i-1} + H_i\right)M_i + H_iM_{i+1} = \frac{3}{H_{i-1}}u_{i-1}$$
$$-\left(\frac{3}{H_{i-1}} + \frac{3}{H_i}\right)u_i + \frac{3}{H_i}u_{i+1}, i = 1,..., N-2.$$

The author Peter Revesz did not use the possibility to write boundary conditions in the form (30). He looked for special cases $M_0 = M_N = 0$ or $H_i = H$.

But formula (34) is more efficiency as algorithm from paper [64] or books [65], [66].

5. The development of conservative averaging method in the last years

In the last years we together with colleagues have developed conservative averaging method. We approximated spline with exponential (hyperbolic) function [47]. This form of approximation allowed us good approximation of non-monotone solution [45] – [47].

In the last 20 years we use conservative averaging method for system with fins or complicated systems [67] – [82].

Green functions have long been used to create integral representations for boundary value problems [83]. The method is usually Green function method for canonical domains. We had to use this method for the non-canonical adjoining domains. We generalize the Green function method by conjugation conditions for non-canonical domains. This generalization is being used for elliptic, parabolic and hyperbolic types of partial differential equations. We demonstrate the idea of this method for two hollow cylinders [84], [85]. For the first cylinder we have partial differential equation:

$$\frac{1}{a^2}\frac{\partial U}{\partial t}=\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}+\frac{\partial^2 U}{\partial z^2}\right]+$$

$$F(r,\varphi,z,t), r\in[R_0,R], \varphi\in[0,2\pi], z\in[0,l].$$

For the second cylinder which can be of different material:

$$\frac{1}{a_0^2}\frac{\partial U_0}{\partial t}=\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_0}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U_0}{\partial \varphi^2}+\frac{\partial^2 U_0}{\partial z^2}\right],$$

$$r\in[R_0,R_1], \varphi\in[0,2\pi], z\in[l,l_0].$$

We formulate the conjugations conditions as ideal thermal contact for $r\in[R_0,R_1]$:

$$U\big|_{z=l-0}=U_0\big|_{z=l+0}, k\frac{\partial U}{\partial z}\bigg|_{z=l-0}=k_0\frac{\partial U_0}{\partial z}\bigg|_{z=l+0}.$$

The boundary conditions we assume as third type. The initial conditions for the both cylinders are assumed in following form:

$$U\big|_{t=0}=U_0(r,\varphi,z), U_0\big|_{t=0}=U_{00}(r,\varphi,z).$$

The solution for three-dimensional problem for the wall is in following form with Green function [85]:

$$U(r,\varphi,z,t)=H(r,\varphi,z,t)+a^2\int_0^t d\tau\int_0^{2\pi}d\varsigma$$

$$\int_{R_0}^{R_1}F_0(\xi,\varsigma,\tau)G(r,\varphi,z,\xi,\varsigma,l,t-\tau)d\xi$$

$$+\int_{R_0}^{R_1}\xi d\xi\int_0^{2\pi}d\varsigma\times \tag{33}$$

$$\int_l^{l_1}U_0(\xi,\eta,\varsigma)G(r,\varphi,z,\xi,\eta,\varsigma,t)d\eta.$$

The function $H(r,\varphi,z,t)$ is dependent from boundary conditions. The combination of conjugations conditions gives such third type boundary condition:

$$\left(\frac{\partial U}{\partial z}+k_1 U\right)\bigg|_{z=l-0}=F_0(r,\varphi,t)= \tag{34}$$

$$\left(\frac{k_0}{k}\frac{\partial U_0}{\partial z}+k_1 U_0\right)\bigg|_{z=l+0}.$$

The solution in three-dimensional problem for the fin is in following form [85]:

$$U_0(r,\varphi,z,t)=H_0(r,\varphi,z,t)+\int_{R_0}^{R_1}\xi d\xi\int_0^{2\pi}d\eta\times$$

$$\int_l^{l_1}U_{00}(\xi,\eta,\varsigma)G_0(r,\varphi,z,\xi,\eta,\varsigma,t)d\varsigma-a^2\int_0^t d\tau\times$$

$$\int_0^{2\pi} d\eta \int_{R_0}^{R_1} F(\xi,\eta,\tau)\xi G_0(r,\varphi,z,\xi,\eta,l,t-\tau)d\xi.$$

The solution in three-dimensional problem for the fin is in following form: $U_0(r,\varphi,z,t) =$

$$-\bar{H}_0(r,\varphi,z,t) - a^2 \int_0^t d\tau \int_0^{2\pi} d\eta \times \qquad (35)$$

$$\int_{R_0}^{R_1} F(r,\varphi,t) G_0(r,\varphi,z,\xi,\eta,l,t-\tau)d\xi,$$

$$\bar{H}_0(r,\varphi,z,t) = H_0(r,\varphi,z,t) + \int_{R_0}^{R_1} \xi d\xi \int_0^{2\pi} d\eta$$

$$\times \int_l^{l_1} U_{00}(\xi,\eta,\varsigma) G_0(r,\varphi,z,\xi,\eta,\varsigma,t)d\varsigma.$$

The combination of conjugations conditions gives such third type boundary condition:

$$\left.\left(\frac{\partial U_0}{\partial z} - U_0\right)\right|_{z=l} = F(r,\varphi,t) = \left.\left(\frac{k}{k_0}\frac{\partial U}{\partial z} - U\right)\right|_{z=l-0}.$$

Unfortunately, the representation (33) is unusable as solution for the fin because of the unknown function $F(r,\varphi,t)$, i.e. temperature in the wall $U(r,\varphi,z=l,t)$ and derivative of temperature. From equation (35) we can write the combination $F_0(r,\varphi,t)$.

$$F_0(r,\varphi,t) = -\hat{\bar{H}}_0(r,\varphi,t) - a^2 \int_0^t d\tau \int_0^{2\pi} d\eta \times$$

$$\int_{R_0}^{R_1} F(\xi,\eta,\tau) \hat{G}_0(r,\varphi,\xi,\eta,\varsigma,t-\tau)d\xi, \hat{\bar{H}}_0(r,\varphi,t)$$

$$= \left.\left(\frac{k_0}{k}\frac{\partial}{\partial z}\bar{H}_0(r,\varphi,z,t) - \bar{H}_0(r,\varphi,z,t)\right)\right|_{z=l},$$

Similarly we do with equations (23) and (18):

$$\hat{G}_0(r,\varphi,\xi,\eta,\varsigma,t) = \left[\frac{k_0}{k}\frac{\partial}{\partial z} G_0(r,\varphi,z,\xi,\eta,\varsigma,t)\right.$$

$$\left. - G_0(r,\varphi,z,\xi,\eta,\varsigma,t)\right]\Big|_{z=l}$$

$$U(r,\varphi,z,t) = \bar{H}(r,\varphi,z,t) + a^2 \int_0^t d\tau \times$$

$$\int_0^{2\pi} d\varsigma \int_{R_0}^{R_1} F_0(\xi,\varsigma,\tau) G(r,\varphi,z,\xi,\varsigma,l,t-\tau)d\xi,$$

$$\bar{H}\left(r,\varphi,z,t\right)=H\left(r,\varphi,z,t\right)+\int_{R_0}^{R_1}\xi d\xi\int_0^{2\pi}d\varsigma\times$$

$$\int_l^{l_1}U_0(\xi,\eta,\varsigma)G(r,\varphi,z,\xi,\eta,\varsigma,t)d\eta.$$

From equation (23) we can write the combination $F(r,\varphi,t)$:

$$F(r,\varphi,t)=\hat{\bar{H}}\left(r,\varphi,t\right)+a^2\int_0^t d\tau\times$$

$$\int_0^{2\pi}d\eta\int_{R_0}^{R_1}F_0\left(\xi,\eta,\tau\right)\hat{G}(r,\varphi,l,\xi,\eta,l,t-\tau)d\xi,$$

$$\hat{\bar{H}}\left(r,\varphi,t\right)=\left[\frac{k}{k_0}\frac{\partial}{\partial z}\bar{H}\left(r,\varphi,t\right)-\bar{H}\left(r,\varphi,t\right)\right]\Bigg|_{z=l},$$

$$\hat{G}(r,\varphi,l,\xi,\varsigma,l,t)=$$

$$\left[\frac{k}{k_0}\frac{\partial}{\partial z}G(r,\varphi,z,\xi,\varsigma,l,t)-G(r,\varphi,z,\xi,\varsigma,l,t)\right]\Bigg|_{z=l}.$$

As the last step we substitute the combination $F_0(\xi,\eta,\tau)$ from (33):

$$F_0(\xi,\eta,\tau)=-\hat{\bar{H}}_0\left(\xi,\eta,\tau\right)-a^2\int_0^{\tau}d\tau_1\int_0^{2\pi}d\eta_1\times$$

$$\int_{R_0}^{R_1}F\left(\xi_1,\eta,\tau_1\right)\hat{G}_0(\xi,\eta,\xi,\eta_1,\varsigma,\tau-\tau_1)d\xi_1,$$

$$F(r,\varphi,t)=\hat{\bar{H}}\left(r,\varphi,t\right)+a^2\int_0^t d\tau\times$$

$$\int_0^{2\pi}d\eta\int_{R_0}^{R_1}F_0\left(\xi,\eta,\tau\right)\hat{G}(r,\varphi,l,\xi,\eta,l,t-\tau)d\xi.$$

in the equation (29):

Finally we obtain following non-homogeneous Fredholm integral equation of 2nd kind [87] :

$$F(r,\varphi,t)=\Phi\left(r,\varphi,t\right)+a^2\int_0^t d\tau\int_0^{\tau}d\tau_1\int_0^{2\pi}d\eta_1\times$$

$$\int_0^{2\pi}d\eta\int_{R_0}^{R_1}F\left(\xi_1,\eta,\tau_1\right)\Gamma(r,\varphi,l,\xi,\eta,l,t-\tau)d\xi,$$

$$\Gamma(r,\varphi,l,\xi,\eta,l,t)=\int_{R_0}^{R_1}\hat{G}(r,\varphi,l,\xi,\eta,l,t)d\xi\times$$

$$\int_{R_0}^{R_1}\hat{G}_0(\xi,\eta,\xi_1,\eta_1,\varsigma,\tau-\tau_1)d\xi_1.$$

Knowing the function $F(r,\varphi,t)$ we can calculate the solution for the fin from (34) and know the function $F_0(\xi,\eta,\tau)$ we have solution (33) for the wall.

6. Conclusions

We have shortly reconstructed history of conservative capacity boundary conditions in the last 100 years. We have given short history of oil output investigation. In paper we propose new hyperbolic approximation for conservative averaging method. This approximation is not in the form of polynomial. The new representation for classical cubic spline is given.

In this paper and papers [41] – [43] we construct conservative averaging method. This method interpolates exactly the average integral value of the function. Secondly, it fulfills exactly both conjugations or (and) boundary conditions.

In other words: The main idea for the method of conservative averaging is to fulfill in the simplified problem formulation the conservation of energy or mass. All of main differential equations fulfill energy or mass conservation. The conjugations conditions fulfill energy exchange between two neighboring sub-domains or boundary conditions fulfill the energy conservation between object and surrounding media.

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