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Research Article

Sublinear operators with rough kernel generated by fractional integrals and their commutators on generalized Morrey spaces

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Abstract In this paper some results for the boundedness of certain sublinear operators, including fractional integral operators, with rough kernels on generalized Morrey spaces are given. Moreover, the corresponding results of the commutators with rough kernels are discussed. Also, Marcinkiewicz operator which satisfies the conditions of these theorems can be considered as an example.

Keywords Sublinear operator, fractional integral operator, rough kernel, generalized Morrey space, commutator, BMO

1. Introduction and Main Results

The classical Morrey spaces $M_{p,\lambda}$ have been introduced by Morrey in [26] to study the local behavior of solutions of second order elliptic partial differential equations (PDEs). Later, there are many applications of Morrey space to the Navier-Stokes equations (see [24]), the Schrödinger equations (see [32]) and the elliptic problems with discontinuous coefficients (see [2, 29]).

We recall the definition of classical Morrey spaces $M_{p,\lambda}$ as

$$M_{p,\lambda}\left(\mathbb{R}^n\right)=\left\{ f:\|f\|_{M_{p,\lambda}\left(\mathbb{R}^n\right)}=\sup_{x\in\mathbb{R}^n,r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,r))}<\infty\right\},$$

where $f\in L_p^{loc}(\mathbb{R}^n)$, $0\le\lambda\le n$ and $1\le p<\infty$.

Note that $M_{p,0}=L_p(\mathbb{R}^n)$ and $M_{p,n}=L_\infty(\mathbb{R}^n)$. If $\lambda<0$ or $\lambda>n$, then $M_{p,\lambda}=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda}\equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f\in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}}\equiv\|f\|_{WM_{p,\lambda}(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0} r^{-\frac{\lambda}{p}}\|f\|_{WL_p(B(x,r))}<\infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned}\|f\|_{WL_p(B(x,r))} &\equiv \left\| f\chi_{B(x,r)} \right\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t \left| \{ y \in B(x,r) : |f(y)| > t \} \right|^{1/p} \\ &= \sup_{0<t\le|B(x,r)|} t^{1/p} \left(f\chi_{B(x,r)} \right)^{*}(t) < \infty,\end{aligned}$$

where g^* denotes the non-increasing rearrangement of a function g.

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also let $B(x,r)$ denotes the open ball centered at x of radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the readers to [1, 5, 31]. For further properties and applications of classical Morrey spaces, see [6, 7, 14, 17] and references therein.

After studying Morrey spaces in detail, researchers have passed to generalized Morrey spaces. Mizuhara [25] has given generalized Morrey spaces $M_{p,\varphi}$ considering $\varphi(r)$ instead of r^{λ} in the above definition of the Morrey space. Later, Guliyev [12], Guliyev et al. [13] and Karaman [22] have defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm as follows:

Definition 1 (***Generalized Morrey space***) *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

During the last decades various classical operators, such as maximal, singular and potential operators have been widely investigated in generalized Morrey spaces (see [8, 12, 13, 18, 22, 28, 34] for details).

Suppose that S^{n-1} is the unit sphere on $\mathbb{R}^n$ $(n \ge 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L_s(S^{n-1})$ with $1 < s \le \infty$ be homogeneous of degree zero. We define $s' = \frac{s}{s-1}$ for any

$s>1$. Suppose that $T_{\Omega,\alpha}$, $\alpha\in(0,n)$ represents a linear or a sublinear operator, which satisfies that for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin suppf$

$$|T_{\Omega,\alpha}f(x)|\le c_0\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy, \tag{1.1}$$

where c_0 is independent of f and x.

For a locally integrable function b on $\mathbb{R}^n$, suppose that the commutator operator $T_{\Omega,b,\alpha}$, $\alpha\in(0,n)$ represents a linear or a sublinear operator, which satisfies that for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin suppf$

$$|T_{\Omega,b,\alpha}f(x)|\le c_0\int_{\mathbb{R}^n}|b(x)-b(y)|\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy, \tag{1.2}$$

where c_0 is independent of f and x.

We point out that the condition (1.1) in the case of $\Omega\equiv 1$, $\alpha=0$ has been introduced by Soria and Weiss in [35]. The conditions (1.1) and (1.2) are satisfied by many interesting operators in harmonic analysis, such as fractional Marcinkiewicz operator, fractional maximal operator, fractional integral operator (Riesz potential) and so on (see [23], [35] for details).

In 1971, Muckenhoupt and Wheeden [27] defined the fractional integral operator with rough kernel $\overline{T}_{\Omega,\alpha}$ by

$$\overline{T}_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy \qquad 0<\alpha<n$$

and a related fractional maximal operator with rough kernel $M_{\Omega,\alpha}$ is given by

$$M_{\Omega,\alpha}f(x)=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int_{B(x,t)}|\Omega(x-y)||f(y)|dy \qquad 0<\alpha<n,$$

where $\Omega\in L_s(S^{n-1})$ with $1<s\le\infty$ is homogeneous of degree zero on $\mathbb{R}^n$ and also $\overline{T}_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ satisfy condition (1.1).

If $\alpha=0$, then $M_{\Omega,0}\equiv M_\Omega$ is the Hardy-Littlewood maximal operator with rough kernel and $\overline{T}_{\Omega,\alpha}$ also becomes a Calderón-Zygmund singular integral operator with rough kernel. It is obvious that when $\Omega\equiv 1$, $M_{1,\alpha}\equiv M_\alpha$ and $\overline{T}_{1,\alpha}\equiv\overline{T}_\alpha$ are the fractional maximal operator and the fractional integral operator, respectively.

In recent years, the mapping properties of $\overline{T}_{\Omega,\alpha}$ on some kinds of function spaces have been studied in many papers (see [4, 9, 10, 27] for details). In particular, the boundedness of $\overline{T}_{\Omega,\alpha}$ in Lebesgue spaces has been obtained.

Lemma 1 *[4, 9, 27] Let* $0<\alpha<n$, $1<p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *If* $\Omega\in L_s(S^{n-1})$, $s>\frac{n}{n-\alpha}$, *then we have*

$$\left\| \overline{T}_{\Omega,\alpha} f \right\|_{L_q} \leq C \| f \|_{L_p}.$$

Corollary 1 *Under the assumptions of Lemma 1, the operator* $M_{\Omega,\alpha}$ *is bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$. *Moreover, we have*

$$\left\| M_{\Omega,\alpha} f \right\|_{L_q} \leq C \| f \|_{L_p}.$$

Proof. Set

$$\widetilde{T}_{|\Omega|,\alpha}(|f|)(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy \qquad 0<\alpha<n,$$

where $\Omega \in L_s(S^{n-1})(s>1)$ is homogeneous of degree zero on $\mathbb{R}^n$. It is easy to see that, for $\widetilde{T}_{|\Omega|,\alpha}$, Lemma 1 is also hold. On the other hand, for any $t>0$, we have

$$\widetilde{T}_{|\Omega|,\alpha}(|f|)(x) \geq \int_{B(x,t)} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy$$

$$\geq \frac{1}{t^{n-\alpha}} \int_{B(x,t)} |\Omega(x-y)| |f(y)| dy.$$

Taking the supremum for $t>0$ on the inequality above, we get

$$M_{\Omega,\alpha} f(x) \leq C_{n,\alpha}^{-1} \widetilde{T}_{|\Omega|,\alpha}(|f|)(x) \qquad C_{n,\alpha} = |B(0,1)|^{\frac{n-\alpha}{n}}.$$

For $b \in L_1^{loc}(\mathbb{R}^n)$, the commutator $[b,\overline{T}_\alpha]$ of fractional integral operator (also known as the Riesz potential) is defined by

$$[b,\overline{T}_\alpha] f(x) = b(x)\overline{T}_\alpha f(x) - \overline{T}_\alpha(bf)(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} f(y) dy \qquad 0<\alpha<n$$

for any suitable function f.

The function b is also called the symbol function of $[b,\overline{T}_\alpha]$. The characterization of (L_p, L_q)-boundedness of the commutator $[b,\overline{T}_\alpha]$ of fractional integral operator has been given by Chanillo [3]. A well known result of Chanillo [3] states that the commutator $[b,\overline{T}_\alpha]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $1<p<q<\infty$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ if and only if $b \in BMO(\mathbb{R}^n)$. There are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutators can produce some characterizations of function spaces (see [3, 15, 16, 20, 30, 33]). The other one is that the theory of commutators plays an important role in the study of the regularity of solutions to elliptic and parabolic PDEs of the second order (see [6, 7, 34]).

Many authors are interested in the study of commutators for which the symbol functions belong to $BMO(\mathbb{R}^n)$ spaces, see [3, 13, 18, 19, 20, 22, 30] for example.

Let us recall the defination of the space of $BMO(\mathbb{R}^n)$ (bounded mean oscillation).

Definition 2 *Suppose that* $b \in L_1^{loc}(\mathbb{R}^n)$, *let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y)-b_{B(x,r)}|\,dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{loc}(\mathbb{R}^n) : \|b\|_* < \infty\}.$$

If one regards two functions whose difference is a constant as one, then the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 1 *[22]* (1) *The John-Nirenberg inequality [21]: there are constants* C_1, $C_2 > 0$, *such that for all* $b \in BMO(\mathbb{R}^n)$ *and* $\beta > 0$

$$\left|\{x \in B : |b(x)-b_B| > \beta\}\right| \leq C_1 |B| e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y)-b_{B(x,r)}|^p \,dy \right)^{\frac{1}{p}} \tag{1.3}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \leq C\|b\|_* \ln\frac{t}{r} \text{ for } 0 < 2r < t, \tag{1.4}$$

where C is independent of b, x, r and t.

Remark 2 *[22] Note that* $L_\infty(\mathbb{R}^n)$ *is contained in* $BMO(\mathbb{R}^n)$ *and we have*

$$\|b\|_{BMO} \leq 2\|b\|_\infty.$$

Moreover, BMO contains unbounded functions, in fact the function $\log|x|$ on $\mathbb{R}^n$, is in BMO but it is not bounded, so $L_\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.

Let b be a locally integrable function on $\mathbb{R}^n$, then for $0 < \alpha < n$ and f is a suitable function, we define the commutators generated by fractional integral and maximal operators with rough kernel and b as follows, respectively:

$$[b, \overline{T}_{\Omega,\alpha}] f(x) \equiv b(x)\overline{T}_{\Omega,\alpha} f(x) - \overline{T}_{\Omega,\alpha}(bf)(x) = \int_{\mathbb{R}^n} [b(x)-b(y)] \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy,$$

$$M_{\Omega,b,\alpha}(f)(x)=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int_{B(x,t)}|b(x)-b(y)||\Omega(x-y)||f(y)|dy$$

satisfy condition (1.2). The proof of boundedness of $[b,\overline{T}_{\Omega,\alpha}]$ in Lebesgue spaces can be found in [9] (by taking $w=1$ there).

Theorem 1 *[9] Suppose that* $\Omega\in L_s(S^{n-1})$, $1<s\leq\infty$, *is homogeneous of degree zero and has mean value zero on* S^{n-1}. *Let* $0<\alpha<n$, $1\leq p<\frac{n}{\alpha}$, *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ *and* $b\in BMO(\mathbb{R}^n)$. *If* $s'<p$ *or* $q<s$, *then the operator* $[b,\overline{T}_{\Omega,\alpha}]$ *is bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$.

Remark 3 *Using the method in the proof of Corollary 1 we have that*

$$M_{\Omega,b,\alpha}f(x)\leq C_{n,\alpha}^{-1}[b,\overline{T}_{|\Omega|,\alpha}](|f|)(x) \qquad C_{n,\alpha}=|B(0,1)|^{\frac{n-\alpha}{n}}. \tag{1.5}$$

By (1.5) we see that under the conditions of Theorem 1, the consequences of (L_p,L_q)-boundedness still hold for $M_{\Omega,b,\alpha}$.

Remark 4 *[33] When* Ω *satisfies the specified size conditions, the kernel of the operator* $\overline{T}_{\Omega,\alpha}$ *has no regularity, so the operator* $\overline{T}_{\Omega,\alpha}$ *is called a rough fractional integral operator. In recent years, a variety of operators related to the fractional integrals, but lacking the smoothness required in the classical theory, have been studied. These include the operator* $[b,\overline{T}_{\Omega,\alpha}]$. *For more results, we refer the reader to [3, 9, 10, 11, 15, 18, 19].*

There are many papers discussing the conditions on $\varphi(x,r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [28] by Nakai the following condition has been imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r)\leq\varphi(x,t)\leq c\varphi(x,r) \tag{1.6}$$

whenever $r\leq t\leq 2r$, where $c(\geq 1)$ does not depend on t, r and $x\in\mathbb{R}^n$, jointly with the condition:

$$\int_r^\infty t^\alpha\varphi(x,t)^q\frac{dt}{t}\leq C\varphi(x,r)^p, \tag{1.7}$$

where $C(>0)$ does not depend on r and $x\in\mathbb{R}^n$, $1\leq p<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Under the above conditions, in [28] has been obtained the boundedness of the operator $\overline{T}_\alpha$ is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$ for $p>1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi}$ for $p=1$. Later, Guliyev [12] has shown that the boundedness of the operator $\overline{T}_\alpha$ from M_{p,φ_1} to M_{q,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p=1$ by considering the following condition (1.8) instead of conditions (1.6) and (1.7)

$$\int_r^{\infty} t^{\alpha}\varphi_1(x,t)\frac{dt}{t} \le C\varphi_2(x,r), \tag{1.8}$$

where C does not depend on x and r, $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. But, Guliyev's results [12] are different from the main results in [28] as the definitions of generalized Morrey spaces of Guliyev and Nakai are different from each other. On the other hand, in [13], Guliyev et al. have introduced a weaker condition for the boundedness of certain sublinear operators, including fractional integral operators, and their commutators under generic size conditions on generalized Morrey spaces. It can be formulated to their main results as follows:

Theorem 2 *[13] Let* $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and the pair* (φ_1, φ_2) *satisfies the condition*

$$\int_r^{\infty} \frac{essinf_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}dt \le C\varphi_2(x,r), \tag{1.9}$$

where C does not depend on x and r. Let T_α be a sublinear operator satisfying condition (1.1) (by taking $\Omega \equiv 1$ there), bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p>1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p=1$. Then the operator T_α is bounded from M_{p,φ_1} to M_{q,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p=1$. Moreover, we have for $p>1$

$$\left\|T_\alpha f\right\|_{M_{q,\varphi_2}} \lesssim \left\|f\right\|_{M_{p,\varphi_1}},$$

and for $p=1$

$$\left\|T_\alpha f\right\|_{WM_{q,\varphi_2}} \lesssim \left\|f\right\|_{M_{1,\varphi_1}}.$$

Theorem 3 *[13] Let* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ *and the pair* (φ_1, φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{essinf_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}dt \le C\varphi_2(x,r)$$

where C does not depend on x and r. Let $T_{b,\alpha}$ be a sublinear operator satisfying condition (1.2) (by taking $\Omega \equiv 1$ there) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Then the operator $T_{b,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2}. Moreover, we have

$$\left\|T_{b,\alpha} f\right\|_{M_{q,\varphi_2}} \lesssim \left\|b\right\|_{*}\left\|f\right\|_{M_{p,\varphi_1}}.$$

Remark 5 *If the pair* (φ_1,φ_2) *satisfies condition (1.8), then* (φ_1,φ_2) *satisfies condition (1.9). But the opposite is not true. In general, condition (1.9) does not imply condition (1.8). For example, see Remark 5.6. in [13].*

After the establishment of the generalized Morrey boundedness of T_α under generic size conditions in Theorem 2, a natural question is: Can this result be generalized? In other words, what properties does the more general operators $T_{\Omega,\alpha}$ under generic size conditions have on the generalized Morrey space? We give an answer as follows:

Theorem 4 *(Our main result) Suppose that* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *is homogeneous of degree zero. Let* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Let* $T_{\Omega,\alpha}$ *be a sublinear operator satisfying condition (1.1), bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $p > 1$, *and bounded from* $L_1(\mathbb{R}^n)$ *to* $WL_q(\mathbb{R}^n)$ *for* $p = 1$. *Let also, for* $s' \le p$, *the pair* (φ_1,φ_2) *satisfies the condition*

$$\int_r^\infty \frac{essinf_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}dt \le C\varphi_2(x,r), \tag{1.10}$$

and for $q < s$ the pair (φ_1,φ_2) satisfies the condition

$$\int_r^\infty \frac{essinf_{t<\tau<\infty}\varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}}dt \le C\varphi_2(x,r)r^{\frac{n}{s}}, \tag{1.11}$$

where C does not depend on x and r.

Then the operator $T_{\Omega,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$. Moreover, we have for $p > 1$

$$\left\|T_{\Omega,\alpha} f\right\|_{M_{q,\varphi_2}} \lesssim \left\|f\right\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\left\|T_{\Omega,\alpha} f\right\|_{WM_{q,\varphi_2}} \lesssim \left\|f\right\|_{M_{1,\varphi_1}}.$$

Corollary 2 *Suppose that* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *is homogeneous of degree zero. Let* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *For* $s' \le p$ *the pair* (φ_1,φ_2) *satisfies condition (1.10) and for* $q < s$ *the pair* (φ_1,φ_2) *satisfies condition (1.11). Then the operators* $M_{\Omega,\alpha}$ *and* $\overline{T}_{\Omega,\alpha}$ *are bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{q,φ_2} *for* $p = 1$.

On the other hand, the result of Theorem 3 can also be generalized. We can state the BMO estimates for the commutator operators $T_{\Omega,b,\alpha}$ under generic size conditions on the generalized Morrey space as follows:

Theorem 5 *(Our main result) Suppose that* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *is homogeneous of degree zero and* $T_{\Omega,b,\alpha}$ *is a sublinear operator satisfying condition (1.2) and bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$. *Let* $1 < p < \infty$ $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Let also, for* $s' \le p$ *the pair* (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{essinf_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C\varphi_2(x,r), \tag{1.12}$$

and for $q < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{essinf_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt \le C\varphi_2(x,r) r^{\frac{n}{s}}, \tag{1.13}$$

where C does not depend on x and r.

Then, the operator $T_{\Omega,b,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2}. Moreover

$$\left\| T_{\Omega,b,\alpha} f \right\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \left\| f \right\|_{M_{p,\varphi_1}}.$$

Corollary 3 *Suppose that* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *is homogeneous of degree zero. Let* $1 < p < \infty$ $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *If for* $s' \le p$ *the pair* (φ_1, φ_2) *satisfies the condition (1.12) and for* $q < s$ *the pair* (φ_1, φ_2) *satisfies the condition (1.13). Then, the operators* $M_{\Omega,b,\alpha}$ *and* $[b, \overline{T}_{\Omega,\alpha}]$ *are bounded from* M_{p,φ_1} *to* M_{q,φ_2}.

Inspired by [18], in this paper we consider the boundedness of sublinear operators with a rough kernel generated by fractional integrals and give BMO space estimates for commutators with rough kernel on generalized Morrey spaces.

Finally, we present a relationship between essential supremum and essential infimum.

Lemma 2 *(see [41] page 143) Let* f *be a real-valued nonnegative function and measurable on* E. *Then*

$$\left(essinf_{x\in E} f(x)\right)^{-1} = esssup_{x\in E} \frac{1}{f(x)}. \tag{1.14}$$

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Proof of theorems

To prove the theorems (Theorems 4 and 5), we need the following lemmas.

Lemma 3 *[15] Suppose that* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *is homogeneous of degree zero. Let* $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Let* $T_{\Omega,\alpha}$ *be a sublinear operator satisfying condition (1.1), bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $p > 1$, *and bounded from* $L_1(\mathbb{R}^n)$ *to* $WL_q(\mathbb{R}^n)$ *for* $p = 1$.

If $p > 1$ and $s' \le p$, then the inequality

$$\|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt \tag{2.1}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

If $p > 1$ and $q < s$, then the inequality

$$\|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Moreover, for $p = 1 < q < s$ the inequality

$$\|T_{\Omega,\alpha} f\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_1(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n)$.

Lemma 4 *Suppose that* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, *is homogeneous of degree zero. Let* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, *and* $T_{\Omega,b,\alpha}$ *is a sublinear operator satisfying condition (1.2) and bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$. *Then, for* $s' \le p$ *the inequality*

$$\|T_{\Omega,b,\alpha} f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Also, for $q < s$ the inequality

$$\|T_{\Omega,b,\alpha} f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* r^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Proof. For $x \in B(x_0,t)$, notice that Ω is homogenous of degree zero and $\Omega \in L_s(S^{n-1})$, $s > 1$. Then, we obtain

$$\left(\int_{B(x_0,t)}|\Omega(x-y)|^s dy\right)^{\frac{1}{s}}=\left(\int_{B(x-x_0,t)}|\Omega(z)|^s dz\right)^{\frac{1}{s}}$$

$$\leq\left(\int_{B(0,t+|x-x_0|)}|\Omega(z)|^s dz\right)^{\frac{1}{s}}$$

$$\leq\left(\int_{B(0,2t)}|\Omega(z)|^s dz\right)^{\frac{1}{s}}$$

$$=\left(\int_0^{2t}\int_{S^{n-1}}|\Omega(z')|^s d\sigma(z')r^{n-1}dr\right)^{\frac{1}{s}}$$

$$=C\|\Omega\|_{L_s(S^{n-1})}|B(x_0,2t)|^{\frac{1}{s}}. \tag{2.2}$$

Let $1<p<\infty$, $0<\alpha<\frac{n}{p}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. For any $x_0\in\mathbb{R}^n$, set $B=B(x_0,r)$ for the ball centered at x_0 and of radius r and $2B=B(x_0,2r)$. We represent f as

$$f=f_1+f_2,\qquad f_1(y)=f(y)\chi_{2B}(y),\qquad f_2(y)=f(y)\chi_{(2B)^C}(y),\qquad r>0$$

and have

$$\|T_{\Omega,b,\alpha}f\|_{L_q(B)}\leq\|T_{\Omega,b,\alpha}f_1\|_{L_q(B)}+\|T_{\Omega,b,\alpha}f_2\|_{L_q(B)}.$$

From the boundedness of $T_{\Omega,b,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see Theorem 1) it follows that:

$$\|T_{\Omega,b,\alpha}f_1\|_{L_q(B)}\leq\|T_{\Omega,b,\alpha}f_1\|_{L_q(\mathbb{R}^n)}$$

$$\lesssim\|b\|_*\|f_1\|_{L_p(\mathbb{R}^n)}=\|b\|_*\|f\|_{L_p(2B)}.$$

It is known that $x\in B$, $y\in(2B)^C$, which implies $\frac{1}{2}|x_0-y|\leq|x-y|\leq\frac{3}{2}|x_0-y|$. Then for $x\in B$, we have

$$|T_{\Omega,b,\alpha}f_2(x)|\lesssim\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|b(y)-b(x)||f(y)|dy$$

$$\approx\int_{(2B)^C}\frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}}|b(y)-b(x)||f(y)|dy.$$

Hence we get

$$\|T_{\Omega,b,\alpha}f_2\|_{L_q(B)}\lesssim\left(\int_B\left(\int_{(2B)^C}\frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}}|b(y)-b(x)||f(y)|dy\right)^q dx\right)^{\frac{1}{q}}$$

$$\lesssim \left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}} |b(y)-b_B||f(y)|dy\right)^q dx\right)^{\frac{1}{q}}$$

$$+\left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}} |b(x)-b_B||f(y)|dy\right)^q dx\right)^{\frac{1}{q}}$$

$$= J_1 + J_2.$$

We have the following estimation of J_1. When $s' \leq p$ and $\frac{1}{\mu}+\frac{1}{p}+\frac{1}{s}=1$, by the Fubini's theorem

$$J_1 \approx r^{\frac{n}{q}} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}} |b(y)-b_B||f(y)|dy$$

$$\approx r^{\frac{n}{q}} \int_{(2B)^C} |\Omega(x-y)||b(y)-b_B||f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy$$

$$\approx r^{\frac{n}{q}} \int_{2r}^{\infty} \int_{2r\leq|x_0-y|\leq t} |\Omega(x-y)||b(y)-b_B||f(y)|dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)||b(y)-b_B||f(y)|dy \frac{dt}{t^{n+1-\alpha}} \text{ holds.}$$

Applying the Hölder's inequality and by (1.3), (1.4), (2.2) we get

$$J_1 \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)|\left|b(y)-b_{B(x_0,t)}\right||f(y)|dy \frac{dt}{t^{n+1-\alpha}}$$

$$+ r^{\frac{n}{q}} \int_{2r}^{\infty} \left|b_{B(x_0,r)}-b_{B(x_0,t)}\right| \int_{B(x_0,t)} |\Omega(x-y)||f(y)|dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|\Omega(\cdot-y)\|_{L_s(B(x_0,t))} \left\|\left(b(\cdot)-b_{B(x_0,t)}\right)\right\|_{L_\mu(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}}$$

$$+ r^{\frac{n}{q}} \int_{2r}^{\infty} \left|b_{B(x_0,r)}-b_{B(x_0,t)}\right| \|\Omega(\cdot-y)\|_{L_s(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

In order to estimate J_2 note that

$$J_2 = \left\|\left(b(\cdot)-b_{B(x_0,t)}\right)\right\|_{L_q(B(x_0,t))} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}} |f(y)|dy.$$

By (1.3), we get

$$J_2 \lesssim \|b\|_* r^{\frac{n}{q}} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}} |f(y)| dy.$$

Applying the Hölder's inequality, we get

$$\int_{(2B)^C} \frac{|f(y)||\Omega(x-y)|}{|x_0-y|^{n-\alpha}} dy$$

$$\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}. \tag{2.3}$$

Thus, by (2.2) and (2.3)

$$J_2 \lesssim \|b\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Summing up J_1 and J_2, for all $p \in (1,\infty)$ we get

$$\|T_{\Omega,b,\alpha} f_2\|_{L_q(B)} \lesssim \|b\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Finally, we have the following

$$\|T_{\Omega,b,\alpha} f\|_{L_q(B)} \lesssim \|b\|_* \|f\|_{L_p(2B)} + \|b\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

On the other hand, we have

$$\|f\|_{L_p(2B)} \approx r^{\frac{n}{q}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}}$$

$$\leq r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

By combining the above inequalities, we obtain

$$\|T_{\Omega,b,\alpha} f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

Let $1 < q < s$. Similarly to (2.2), when $y \in B(x_0,t)$, it is true that

$$\left(\int_{B(x_0,r)} |\Omega(x-y)|^s dy\right)^{\frac{1}{s}} \leq C\|\Omega\|_{L_s(S^{n-1})} \left|B\left(x_0, \frac{3}{2}t\right)\right|^{\frac{1}{s}}. \tag{2.4}$$

When $q < s$, by the Fubini's theorem and the Minkowski inequality, we get

$$J_1 \lesssim \left(\int_B \left|\int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}| |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}}\right|^q dx\right)^{\frac{1}{q}}$$

$$+\left(\int_B\left|\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\int_{B(x_0,t)}|f(y)||\Omega(x-y)|dy\frac{dt}{t^{n+1-\alpha}}\right|^q dx\right)^{\frac{1}{q}}$$

$$\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}\left|b(y)-b_{B(x_0,t)}\right|\left|f(y)\right|\left\|\Omega(\cdot-y)\right\|_{L_q(B(x_0,t))}dy\frac{dt}{t^{n+1-\alpha}}$$

$$+\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\int_{B(x_0,t)}|f(y)|\left\|\Omega(\cdot-y)\right\|_{L_q(B(x_0,t))}dy\frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim |B|^{\frac{1}{q}-\frac{1}{s}}\int_{2r}^{\infty}\int_{B(x_0,t)}\left|b(y)-b_{B(x_0,t)}\right|\left|f(y)\right|\left\|\Omega(\cdot-y)\right\|_{L_s(B(x_0,t))}dy\frac{dt}{t^{n+1-\alpha}}$$

$$+|B|^{\frac{1}{q}-\frac{1}{s}}\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\int_{B(x_0,t)}|f(y)|\left\|\Omega(\cdot-y)\right\|_{L_s(B(x_0,t))}dy\frac{dt}{t^{n+1-\alpha}}.$$

Applying the Hölder's inequality and by (1.3), (1.4), (2.4) we get

$$J_1\lesssim J_1 r^{\frac{n}{q}-\frac{n}{s}}\int_{2r}^{\infty}\left\|\left(b(\cdot)-b_{B(x_0,t)}\right)f\right\|_{L_1(B(x_0,t))}\left|B\left(x_0,\frac{3}{2}t\right)\right|^{\frac{1}{s}}\frac{dt}{t^{n+1-\alpha}}$$

$$+r^{\frac{n}{q}-\frac{n}{s}}\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\left\|f\right\|_{L_p(B(x_0,t))}\left|B\left(x_0,\frac{3}{2}t\right)\right|^{\frac{1}{s}}\frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim r^{\frac{n}{q}-\frac{n}{s}}\int_{2r}^{\infty}\left\|\left(b(\cdot)-b_{B(x_0,t)}\right)\right\|_{L_{p'}(B(x_0,t))}\left\|f\right\|_{L_p(B(x_0,t))}t^{\frac{n}{s}}\frac{dt}{t^{n+1}}$$

$$+r^{\frac{n}{q}-\frac{n}{s}}\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\left\|f\right\|_{L_p(B(x_0,t))}t^{\frac{n}{s}}\frac{dt}{t^{\frac{n}{q}+1}}$$

$$\lesssim \|f\|_* r^{\frac{n}{q}-\frac{n}{s}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{\frac{n}{s}-\frac{n}{q}-1}\left\|f\right\|_{L_p(B(x_0,t))}dt.$$

Let $\frac{1}{p}=\frac{1}{\nu}+\frac{1}{s}$, then for J_2, by the Fubini's theorem, the Minkowski inequality, the Hölder's inequality and from (2.4), we get

$$J_2\lesssim\left(\int_B\left|\int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)||b(x)-b_B||\Omega(x-y)|dy\frac{dt}{t^{n+1-\alpha}}\right|^q dx\right)^{\frac{1}{q}}$$

$$\lesssim\int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|\left\|(b(\cdot)-b_B)\Omega(\cdot-y)\right\|_{L_q(B)}dy\frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim\int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|\left\|b(\cdot)-b_B\right\|_{L_\nu(B)}\left\|\Omega(\cdot-y)\right\|_{L_s(B)}dy\frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* |B|^{\frac{1}{q}-\frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(B)} dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* r^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* r^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt.$$

By combining the above estimates, we complete the proof of Lemma 4.

The Proof of Theorem 4. Since $f \in M_{p,\varphi_1}$, by (1.14) and the non-decreasing, with respect to t, of the norm $\|f\|_{L_p(B(x_0,t))}$, we get

$$\frac{\|f\|_{L_p(B(x_0,t))}}{essinf_{0<t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq esssup_{0<t<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq esssup_{0<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq \|f\|_{M_{p,\varphi_1}}.$$

For $s' \leq p < \infty$, since (φ_1, φ_2) satisfies (1.10), we have

$$\int_r^{\infty} \|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{q}} \frac{dt}{t}$$

$$\leq \int_r^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{essinf_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \frac{essinf_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\leq C\|f\|_{M_{p,\varphi_1}} \int_r^{\infty} \frac{essinf_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\leq C\|f\|_{M_{p,\varphi_1}} \varphi_2(x_0,r).$$

Then by (2.1), we get

$$\|T_{\Omega,\alpha} f\|_{M_{q,\varphi_2}} = \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{q}} \|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))}$$

$$\leq C \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x_0,t))} t^{-\frac{n}{q}} \frac{dt}{t}$$

$$\leq C\|f\|_{M_{p,\varphi_1}}.$$

For the case of $p=1<q<s$, we can also use the same method, so we omit the details. This completes the proof of Theorem 4.

The Proof of Theorem 5. The statement of Theorem 5 follows by Lemma 4 and (1.14) in the same manner as in the proof of Theorem 4.

Now, we give the applications of Theorem 4 and Theorem 5 for the Marcinkiewicz operator.

Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n\setminus\{0\}$, that is,

$$\Omega(\mu x)=\Omega(x), \text{ for any } \mu>0, x\in\mathbb{R}^n\setminus\{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x')=0,$$

where $x'=\frac{x}{|x|}$ for any $x\neq 0$.

(c) $\Omega\in Lip_\gamma(S^{n-1})$, $0<\gamma\leq 1$, that is there exists a constant $M>0$ such that,

$$|\Omega(x')-\Omega(y')|\leq M|x'-y'|^\gamma \text{ for any } x',y'\in S^{n-1}.$$

In 1958, Stein [36] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x)=\left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x)=\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [37, 38, 39].

The Marcinkiewicz operator is defined by (see [40])

$$\mu_{\Omega,\alpha}(f)(x)=\left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x)=\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}}f(y)dy.$$

Note that $\mu_\Omega f=\mu_{\Omega,0}f$.

The sublinear commutator of the operator $\mu_{\Omega,\alpha}$ is defined by

$$[b,\mu_{\Omega,\alpha}](f)(x)=\left(\int_0^\infty |F_{\Omega,\alpha,t,b}(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,\alpha,t,b}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}}[b(x)-b(y)]f(y)dy.$$

We consider the space $H=\{h:\|h\|=(\int_0^\infty |h(t)|^2\frac{dt}{t^3})^{1/2}<\infty\}$. Then, it is clear that $\mu_{\Omega,\alpha}(f)(x)=\|F_{\Omega,\alpha,t}(x)\|$.

By the Minkowski inequality, we get

$$\mu_{\Omega,\alpha}(f)(x)\le\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}}|f(y)|\left(\int_{|x-y|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy\le C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy.$$

Thus, $\mu_{\Omega,\alpha}$ satisfies the condition (1.1). It is known that for $b\in BMO(\mathbb{R}^n)$ the operators $\mu_{\Omega,\alpha}$ and $[b,\mu_{\Omega,\alpha}]$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p>1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p=1$ (see [40]), then from Theorems 4 and 5 we get

Corollary 4 *Suppose that* $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, *is homogeneous of degree zero. Let* $0<\alpha<n$, $1\le p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Let also, for* $s'\le p$ *the pair* (φ_1,φ_2) *satisfies the condition (1.10) and for* $q<s$ *the pair* (φ_1,φ_2) *satisfies the condition (1.11) and* Ω *satisfies conditions (a)–(c). Then* $\mu_{\Omega,\alpha}$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $p>1$ *and from* M_{1,φ_1} *to* WM_{q,φ_2} *for* $p=1$.

Corollary 5 *Suppose that* $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, *is homogeneous of degree zero. Let* $1<p<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ *and* $b\in BMO(\mathbb{R}^n)$. *Let also, for* $s'\le p$ *the pair* (φ_1,φ_2) *satisfies the condition (1.12) and for* $q<s$ *the pair* (φ_1,φ_2) *satisfies the condition (1.13) and* Ω *satisfies the conditions (a)–(c). Then the operator* $[b,\mu_{\Omega,\alpha}]$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2}.

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