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Criteria for Nonsingular *H*-tensors

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Abstract. Tensor is a high-level extension of the matrix, *H*-tensor is a special tensor and it is a new developed concept in tensor analysis. In this paper, we introduce some definitions and theorems firstly, then establish some implementable criteria in identifying nonsingular *H*-tensor, and at last give two numerical examples to prove the criteria are reliable.

Keywords: *H*-tensor, generalized diagonally dominant, *M*-tensor.

1 Introduction

Tensor analysis and computing has received much attention of researchers in recent decade since tensors have wide applications in signal and image processing, continuum physics, higher-order statistics [1].

Generally, tensor is a higher-order extension of matrix. A high order tensor is a multi-way array whose entries are addressed via multiple indices in the following form:

$$\mathrm{A} = (a_{i_1 i_2 \cdots i_m}), i_j = 1, 2, \cdots, n_j, j = 1, 2, \cdots, m,$$

where $a_{i_1 i_2 \cdots i_m}$ are real numbers. If $n_1 = n_2 = \cdots = n_m = n$, then A is called a square tensor, otherwise it is called a rectangular tensor.

For tensor A and matrix X, their product on mode-k [1] is defined as

$$(\mathrm{A} \times_k X)_{i_1 i_2 \cdots j_k \cdots i_m} = \sum_{i_k=1}^{n} \mathrm{A}_{i_1 i_2 \cdots i_k \cdots i_m} X_{i_k j_k}$$

which denotes that

$$\mathrm{A}X^{m-1} = \mathrm{A} \times_2 X \times_3 X \cdots \times_m X .$$

For tensor A and vector $x \in R^n$, $\mathrm{A}X^{m-1}$ is a vector in R^n with entries

$$(\mathrm{A}X^{m-1})_i = \sum_{i_2, i_3, \cdots, i_m = 1}^{n, n, \cdots n} a_{i i_2 i_3 \cdots i_m} X_{i_2} X_{i_3} \cdots X_{i_m}, i = 1, 2, \cdots, n.$$

and $\mathrm{A}X^m$ is a scalar with

$$\mathrm{A}X^m = \sum_{i_1, i_2, \cdots, i_m = 1}^{n, n, \cdots n} a_{i_1 i_2 \cdots i_m} X_{i_1} X_{i_2} \cdots X_{i_m} .$$

The paper uses I to denote m-th order n-dimensional identity tensor with entries

$$I_{i_1 \cdots i_m} = \begin{cases} 1 & i_1 = \cdots = i_m, \\ 0 & otherwise. \end{cases}$$

and define the following notation

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1 & i_1 = \cdots = i_m, \\ 0 & otherwise. \end{cases}$$

In paper [2] and [4], the authors gave some properties and applications of *M*-tensors. In paper [3], the authors gave some properties of *H*-tensors. *H*-tensor plays an important role in identifying positive definiteness of even-order real symmetric tensors and it contains *M*-tensor as special cases. This paper

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establishes some new implementable criteria in identifying nonsingular *H*-tensors and gives two numerical examples.

2 *H*-tensors and Their Properties

The paper first presents some definitions developed in tensor analysis and then introduces some kinds of specially structured tensors. For a real *m*-order *n*-dimensional tensor $\mathcal{A}$ and a scalar $\lambda \in C$, if there exists nonzero vector $X \in C^n$ such that

$$\mathcal{A}X^{m-1} = \lambda X^{[m-1]}.$$

where $X^{[m-1]} \in C^n$ with $(X^{[m-1]})_i \in X_i^{m-1}, i = 1, 2, \cdots n.$ then λ is said to be an eigenvalue of tensor $\mathcal{A}$ and X an eigenvector associated with eigenvalue λ. In particular, if X is real, then λ is also real, and $(\lambda; X)$ is said to be an *H*-eigenpair of tensor $\mathcal{A}$. The largest modulus of eigenvalue of tensor $\mathcal{A}$ is called the spectral radius of tensor $\mathcal{A}$ and denotes it by $\rho(\mathcal{A})$. Motivated by the characteristics of nonsingular matrices and say a square tensor is nonsingular if its all eigenvalues are nonzero.

Definition 2.1[2] Tensor $\mathcal{A}$ is said to be a *Z*-tensor if it can be written as $\mathcal{A} = cI - \mathcal{B}$, where $c > 0$ and $\mathcal{B}$ is a nonnegative tensor. Furthermore, if $c \geq \rho(\mathcal{B})$, then $\mathcal{A}$ is said to be an *M*-tensor, and if $c > \rho(\mathcal{B})$, and then $\mathcal{A}$ is said to be a nonsingular *M*-tensor. It is easy to see that all the off diagonal entries of a *Z*-tensor are non-positive.

Proposition 2.1[2] Let $\mathcal{A}$ be a *Z*-tensor. Then it is a nonsingular *M*-tensor if and only if one of the following conditions holds.

(1) The real part of any eigenvalue of tensor $\mathcal{A}$ is positive;

(2) There exists positive vector $X \in R^n$ such that $\mathcal{A}X^{m-1} > 0$.

Definition 2.2[2] For *m*-order *n*-dimensional tensor $\mathcal{A}$, its comparison tensor denoted by $M_{\mathcal{A}}$, is defined as

$$M_{\mathcal{A}} = \begin{cases} \left|a_{i_1 i_2 \cdots i_m}\right| & \text{if } i_1 = i_2 = \cdots = i_m, \\ -\left|a_{i_1 i_2 \cdots i_m}\right| & \text{otherwise.} \end{cases}$$

Definition 2.3[2] If comparison tensor $M_{\mathcal{A}}$ of tensor $\mathcal{A}$ is an *M*-tensor, then tensor $\mathcal{A}$ is called an *H*-tensor, and if comparison tensor $M_{\mathcal{A}}$ is a nonsingular *M*-tensor, then tensor $\mathcal{A}$ is called a nonsingular *H*-tensor.

Definition 2.4[3] Tensor $\mathcal{A}$ is called diagonally dominant if

$$\left|a_{ii\cdots i}\right| \geq \sum_{i_2 \cdots i_m \neq ii \cdots i} \left|a_{i i_2 i_3 \cdots i_m}\right|, \forall i = 1, 2, \cdots, n, \tag{2.1}$$

and tensor $\mathcal{A}$ is called strictly diagonally dominant if all the inequalities hold with strict inequality.

Theorem 2.1[3] If square tensor $\mathcal{A}$ is strictly diagonally dominant or it is irreducible and diagonally dominant with at least one strict inequality holding in (2.1), then it is a nonsingular *H*-tensor.

Definition 2.5[4] Tensor $\mathcal{A}$ is said to be generalized strictly diagonally dominant if there exists positive diagonal matrix D such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant.

Proposition 2.2[2] Tensor $\mathcal{A}$ is a nonsingular *H*-tensor if and only if $\mathcal{A}$ is generalized strictly diagonally dominant.

Corollary 2.1[2] For square tensor $\mathcal{A}$, if there exists a positively diagonal matrix $D \in R^{n \times n}$ such that $\mathcal{A}D^{m-1}$ is a nonsingular *H*-tensor, then $\mathcal{A}$ is a nonsingular *H*-tensor.

Proposition 2.3[2] If tensor $\mathcal{A}$ is irreducible and diagonally dominant with at least one strict inequality holding in (2.1), then it is generalized diagonally dominant.

3 Criteria for Nonsingular *H*-tensors

Now, the study turns to considering one kind of tensor diagonal product dominance. Let S be a subset of N and $\bar{S} = N \setminus S$. Then the following multiple index is defined

$$\Lambda = \left\{ i_2 i_3 \cdots i_m \mid i_k \in S \ for\ any\ k = 2,3,\cdots,m \right\},$$
$$\overline{\Lambda} = \left\{ i_2 i_3 \cdots i_m \mid i_k \in \overline{S} \ for\ any\ k = 2,3,\cdots,m \right\}.$$

Based on the above sets denote that

$$R_i^{\Lambda}(\mathrm{A}) = \sum_{\substack{i_2 i_3 \cdots i_m \neq ii\cdots i \\ i_2 i_3 \cdots i_m \in \Lambda}} \left| a_{i i_2 \cdots i_m} \right|, \qquad R_i^{\overline{\Lambda}}(\mathrm{A}) = \sum_{\substack{i_2 i_3 \cdots i_m \neq ii\cdots i \\ i_2 i_3 \cdots i_m \in \overline{\Lambda}}} \left| a_{i i_2 \cdots i_m} \right|.$$

Then the paper can have the following conclusion.

Theorem 3.1 For tensor $\mathrm{A} = (a_{i_1 i_2 \cdots i_m})$, if there exists a partition $(S, \overline{S})$ of the index set N such that

$$\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A}) > 0, p \in S; \ \left| a_{qq\cdots q} \right| - R_q^{\overline{\Lambda}}(\mathrm{A}) > 0, q \in \overline{S}.$$
$$(\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A}))^{\alpha} (\left| a_{qq\cdots q} \right| - R_q^{\overline{\Lambda}}(\mathrm{A}))^{1-\alpha} > R_p^{\overline{\Lambda}}(\mathrm{A})^{\alpha} (R_q^{\Lambda}(\mathrm{A}))^{1-\alpha} \tag{3.1}$$

and if $\alpha \in \left[0, \frac{1}{2}\right]$, then A is a nonsingular H-tensor.

Proof: From

$$(\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A}))^{\alpha} (\left| a_{qq\cdots q} \right| - R_q^{\overline{\Lambda}}(\mathrm{A}))^{1-\alpha} > R_p^{\overline{\Lambda}}(\mathrm{A})^{\alpha} (R_q^{\Lambda}(\mathrm{A}))^{1-\alpha},$$

then

$$\left(\frac{\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\overline{\Lambda}}(\mathrm{A})}\right)^{\alpha} > \left(\frac{R_q^{\Lambda}(\mathrm{A})}{\left| a_{qq\cdots q} \right| - R_q^{\overline{\Lambda}}(\mathrm{A})}\right)^{1-\alpha}.$$

From the first inequality of (3.1), one has

$$\frac{\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\overline{\Lambda}}(\mathrm{A})} > 1.$$

If $\alpha \in \left[0, \frac{1}{2}\right]$ holds, then

$$\left(\frac{\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\overline{\Lambda}}(\mathrm{A})}\right)^{1-\alpha} \geq \left(\frac{\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\overline{\Lambda}}(\mathrm{A})}\right)^{\alpha} > \left(\frac{R_q^{\Lambda}(\mathrm{A})}{\left| a_{qq\cdots q} \right| - R_q^{\overline{\Lambda}}(\mathrm{A})}\right)^{1-\alpha}.$$

so

$$\frac{\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\overline{\Lambda}}(\mathrm{A})} > \frac{R_q^{\Lambda}(\mathrm{A})}{\left| a_{qq\cdots q} \right| - R_q^{\overline{\Lambda}}(\mathrm{A})}.$$

Hence the paper defines the following positive diagonal matrix D with diagonal entries

$$D_{ii} = \begin{cases} 1 & if\ i \in S, \\ d & if\ i \in \overline{S}. \end{cases}$$

where $d > 1$ is such that

$$\frac{\left| a_{pp\cdots p} \right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\overline{\Lambda}}(\mathrm{A})} > d^{m-1}, p \in S; \qquad d^{m-1} > \frac{R_q^{\Lambda}(\mathrm{A})}{\left| a_{qq\cdots q} \right| - R_q^{\overline{\Lambda}}(\mathrm{A})}, q \in \overline{S}.$$

Now, consider tensor $\mathrm{B} = \mathrm{A}D^{m-1}$. It is easy to see that for any $i \in N$,

$$R_i^{\Lambda}(\mathrm{B}) = \sum_{\substack{i_2 i_3 \cdots i_m \neq ii\cdots i \\ i_2 i_3 \cdots i_m \in \Lambda}} \left| b_{i i_2 \cdots i_m} \right| = \sum_{\substack{i_2 i_3 \cdots i_m \neq ii\cdots i \\ i_2 i_3 \cdots i_m \in \Lambda}} \left| a_{i i_2 \cdots i_m} \right| = R_i^{\Lambda}(\mathrm{A}), \qquad R_i^{\overline{\Lambda}}(\mathrm{B}) = \sum_{\substack{i_2 i_3 \cdots i_m \neq ii\cdots i \\ i_2 i_3 \cdots i_m \in \overline{\Lambda}}} \left| b_{i i_2 \cdots i_m} \right| \leq d^{m-1} R_i^{\overline{\Lambda}}(\mathrm{A}).$$

and

$$b_{pp\cdots p} = a_{pp\cdots p}, \quad b_{qq\cdots q} = d^{m-1} a_{qq\cdots q}.$$

Thus for $p \in S$, if $R_p^{\overline{\Lambda}}(\mathrm{A}) > 0$, then

$$\begin{aligned} R_p(\mathrm{B}) &= R_p^{\Lambda}(\mathrm{B}) + R_p^{\bar{\Lambda}}(\mathrm{B}) \le R_p^{\Lambda}(\mathrm{A}) + d^{m-1}R_p^{\bar{\Lambda}}(\mathrm{A}) \\ &< R_p^{\Lambda}(\mathrm{A}) + \frac{a_{pp\cdots p} - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})} R_p^{\bar{\Lambda}}(\mathrm{A}) \\ &= a_{pp\cdots p} = b_{pp\cdots p} \end{aligned}$$

and if $R_p^{\bar{\Lambda}}(\mathrm{A}) = 0$, then from the first inequality of (3.1),

$$\begin{aligned} R_p(\mathrm{B}) &= R_p^{\Lambda}(\mathrm{B}) + R_p^{\bar{\Lambda}}(\mathrm{B}) \le R_p^{\Lambda}(\mathrm{A}) + d^{m-1}R_p^{\bar{\Lambda}}(\mathrm{A}) \\ &= R_p^{\Lambda}(\mathrm{A}) < a_{pp\cdots p} = b_{pp\cdots p}. \end{aligned}$$

For $q \in \bar{S}$, from the second inequality of (3.1), one has

$$\begin{aligned} b_{qq\cdots q} - R_q(\mathrm{B}) &= d^{m-1}a_{qq\cdots q} - R_q^{\Lambda}(\mathrm{B}) - R_q^{\bar{\Lambda}}(\mathrm{B}) \ge d^{m-1}a_{qq\cdots q} - R_q^{\Lambda}(\mathrm{A}) - d^{m-1}R_q^{\bar{\Lambda}}(\mathrm{A}) \\ &= d^{m-1}(a_{qq\cdots q} - R_q^{\bar{\Lambda}}(\mathrm{A})) - R_q^{\Lambda}(\mathrm{A}) \\ &> \frac{R_q^{\Lambda}(\mathrm{A})}{|a_{qq\cdots q}| - R_q^{\bar{\Lambda}}(\mathrm{A})}(a_{qq\cdots q} - R_q^{\bar{\Lambda}}(\mathrm{A})) - R_q^{\Lambda}(\mathrm{A}) = 0. \end{aligned}$$

This means that tensor $\mathrm{A}D^{m-1}$ is strictly diagonally dominant, and A is generalized strictly diagonally dominant, hence it is a nonsingular *H*-tensor by Proposition 2.2.

Theorem 3.2 For irreducible tensor $\mathrm{A} = (a_{i_1 i_2 \cdots i_m})$, if there exists a partition $(S, \bar{S})$ of the index set N such that

$$\begin{aligned} &|a_{pp\cdots p}| - R_p^{\Lambda}(\mathrm{A}) \ge 0, p \in S; \quad |a_{qq\cdots q}| - R_q^{\bar{\Lambda}}(\mathrm{A}) \ge 0, q \in \bar{S}, \\ &(|a_{pp\cdots p}| - R_p^{\Lambda}(\mathrm{A}))^{\alpha}(|a_{qq\cdots q}| - R_q^{\bar{\Lambda}}(\mathrm{A}))^{1-\alpha} \ge R_p^{\bar{\Lambda}}(\mathrm{A})^{\alpha}(R_q^{\Lambda}(\mathrm{A}))^{1-\alpha}. \end{aligned} \tag{3.2}$$

and there exists index $p_0 \in S$, $q_0 \in \bar{S}$ such that

$$\begin{aligned} &|a_{p_0 p_0 \cdots p_0}| - R_{p_0}^{\Lambda}(\mathrm{A}) > 0, \\ &(|a_{p_0 p_0 \cdots p_0}| - R_{p_0}^{\Lambda}(\mathrm{A}))^{\alpha}(|a_{q_0 q_0 \cdots q_0}| - R_{q_0}^{\bar{\Lambda}}(\mathrm{A}))^{1-\alpha} > R_{p_0}^{\bar{\Lambda}}(\mathrm{A})^{\alpha}(R_{q_0}^{\Lambda}(\mathrm{A}))^{1-\alpha} \end{aligned} \tag{3.3}$$

and if $\alpha \in \left[0, \frac{1}{2}\right]$, then A is a nonsingular *H*-tensor.

Proof: From

$$(|a_{pp\cdots p}| - R_p^{\Lambda}(\mathrm{A}))^{\alpha}(|a_{qq\cdots q}| - R_q^{\bar{\Lambda}}(\mathrm{A}))^{1-\alpha} \ge R_p^{\bar{\Lambda}}(\mathrm{A})^{\alpha}(R_q^{\Lambda}(\mathrm{A}))^{1-\alpha},$$

then

$$\left(\frac{|a_{pp\cdots p}| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})}\right)^{\alpha} \ge \left(\frac{R_q^{\Lambda}(\mathrm{A})}{|a_{qq\cdots q}| - R_q^{\bar{\Lambda}}(\mathrm{A})}\right)^{1-\alpha}.$$

From the first inequality of (3.2), one has

$$\frac{|a_{pp\cdots p}| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})} \ge 1 .$$

If $\alpha \in \left[0, \frac{1}{2}\right]$ holds, then

$$\left(\frac{|a_{pp\cdots p}| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})}\right)^{1-\alpha} \ge \left(\frac{|a_{pp\cdots p}| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})}\right)^{\alpha} \ge \left(\frac{R_q^{\Lambda}(\mathrm{A})}{|a_{qq\cdots q}| - R_q^{\bar{\Lambda}}(\mathrm{A})}\right)^{1-\alpha},$$

so

$$\frac{\left|a_{pp\cdots p}\right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})} \geq \frac{R_q^{\Lambda}(\mathrm{A})}{\left|a_{qq\cdots q}\right| - R_q^{\bar{\Lambda}}(\mathrm{A})}.$$

and

$$\frac{\left|a_{p_0p_0\cdots p_0}\right| - R_{p_0}^{\Lambda}(\mathrm{A})}{R_{p_0}^{\bar{\Lambda}}(\mathrm{A})} > \frac{R_{q_0}^{\Lambda}(\mathrm{A})}{\left|a_{q_0q_0\cdots q_0}\right| - R_{q_0}^{\bar{\Lambda}}(\mathrm{A})}.$$

Hence it defines the following positive diagonal matrix D with diagonal entries

$$D_{ii} = \begin{cases} 1 & \text{if } i \in S, \\ d & \text{if } i \in \bar{S}, \end{cases}$$

where $d > 1$ is such that

$$\frac{\left|a_{pp\cdots p}\right| - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})} \geq d^{m-1}, p \in S; \qquad d^{m-1} \geq \frac{R_q^{\Lambda}(\mathrm{A})}{\left|a_{qq\cdots q}\right| - R_q^{\bar{\Lambda}}(\mathrm{A})}, q \in \bar{S}.$$

Now, consider tensor $\mathrm{B} = \mathrm{A}D^{m-1}$. The B remains irreducible as D is positively diagonal. It is easy to see that for any $i \in N$,

$$R_i^{\Lambda}(\mathrm{B}) = \sum_{\substack{i_2i_3\cdots i_m \neq ii\cdots i \\ i_2i_3\cdots i_m \in \Lambda}} \left|b_{ii_2\cdots i_m}\right| = \sum_{\substack{i_2i_3\cdots i_m \neq ii\cdots i \\ i_2i_3\cdots i_m \in \Lambda}} \left|a_{ii_2\cdots i_m}\right| = R_i^{\Lambda}(\mathrm{A}), \qquad R_i^{\bar{\Lambda}}(\mathrm{B}) = \sum_{\substack{i_2i_3\cdots i_m \neq ii\cdots i \\ i_2i_3\cdots i_m \in \bar{\Lambda}}} \left|b_{ii_2\cdots i_m}\right| \leq d^{m-1} R_i^{\bar{\Lambda}}(\mathrm{A}),$$

and

$$b_{pp\cdots p} = a_{pp\cdots p}, \quad b_{qq\cdots q} = d^{m-1} a_{qq\cdots q}.$$

Thus for $p_0 \in S$, if $R_{p_0}^{\bar{\Lambda}}(\mathrm{A}) > 0$, then

$$\begin{aligned} R_{p_0}(\mathrm{B}) &= R_{p_0}^{\Lambda}(\mathrm{B}) + R_{p_0}^{\bar{\Lambda}}(\mathrm{B}) \leq R_{p_0}^{\Lambda}(\mathrm{A}) + d^{m-1} R_{p_0}^{\bar{\Lambda}}(\mathrm{A}) \\ &< R_{p_0}^{\Lambda}(\mathrm{A}) + \frac{a_{p_0p_0\cdots p_0} - R_{p_0}^{\Lambda}(\mathrm{A})}{R_{p_0}^{\bar{\Lambda}}(\mathrm{A})} R_{p_0}^{\bar{\Lambda}}(\mathrm{A}) \\ &= a_{p_0p_0\cdots p_0} \end{aligned}$$

and if $R_{p_0}^{\bar{\Lambda}}(\mathrm{A}) = 0$, then from the first inequality of (3.3),

$$\begin{aligned} R_{p_0}(\mathrm{B}) &= R_{p_0}^{\Lambda}(\mathrm{B}) + R_{p_0}^{\bar{\Lambda}}(\mathrm{B}) \leq R_{p_0}^{\Lambda}(\mathrm{A}) + d^{m-1} R_{p_0}^{\bar{\Lambda}}(\mathrm{A}) \\ &= R_{p_0}^{\Lambda}(\mathrm{A}) < a_{p_0p_0\cdots p_0} \end{aligned}$$

for $i \in S$, if $R_p^{\bar{\Lambda}}(\mathrm{A}) > 0$, then

$$\begin{aligned} R_p(\mathrm{B}) &= R_p^{\Lambda}(\mathrm{B}) + R_p^{\bar{\Lambda}}(\mathrm{B}) \leq R_p^{\Lambda}(\mathrm{A}) + d^{m-1} R_p^{\bar{\Lambda}}(\mathrm{A}) \\ &\leq R_p^{\Lambda}(\mathrm{A}) + \frac{a_{pp\cdots p} - R_p^{\Lambda}(\mathrm{A})}{R_p^{\bar{\Lambda}}(\mathrm{A})} R_p^{\bar{\Lambda}}(\mathrm{A}) \\ &= a_{pp\cdots p} = b_{pp\cdots p} \end{aligned}$$

and if $R_p^{\bar{\Lambda}}(\mathrm{A}) = 0$ then from the first inequality of (3.2),

$$\begin{aligned} R_p(\mathrm{B}) &= R_p^{\Lambda}(\mathrm{B}) + R_p^{\bar{\Lambda}}(\mathrm{B}) \leq R_p^{\Lambda}(\mathrm{A}) + d^{m-1} R_p^{\bar{\Lambda}}(\mathrm{A}) \\ &= R_p^{\Lambda}(\mathrm{A}) \leq a_{pp\cdots p} = b_{pp\cdots p} \end{aligned}$$

For $q \in \bar{S}$, from the second inequality of (3.2), one has

$$\begin{aligned} b_{qq\cdots q} - R_q(\mathrm{B}) &= d^{m-1}a_{qq\cdots q} - R_q^{\Lambda}(\mathrm{B}) - R_q^{\bar{\Lambda}}(\mathrm{B}) \geq d^{m-1}a_{qq\cdots q} - R_q^{\Lambda}(\mathrm{A}) - d^{m-1}R_q^{\bar{\Lambda}}(\mathrm{A}) \\ &= d^{m-1}(a_{qq\cdots q} - R_q^{\bar{\Lambda}}(\mathrm{A})) - R_q^{\Lambda}(\mathrm{A}) \\ &\geq \frac{R_i^{\Lambda}(\mathrm{A})}{\left|a_{ii\cdots i}\right| - R_i^{\bar{\Lambda}}(\mathrm{A})}(a_{ii\cdots i} - R_i^{\bar{\Lambda}}(\mathrm{A})) - R_i^{\Lambda}(\mathrm{A}) = 0 \end{aligned}$$

Thus, $\mathrm{A}D^{m-1}$ is diagonally dominant with at least one strict inequality. Since $\mathrm{A}D^{m-1}$ is irreducible, and also knows that $\mathrm{A}D^{m-1}$ is generalized diagonally dominant by Proposition 2.3. So A is generalized diagonally dominant and it is a nonsingular *H*-tensor.

4 Examples

Example 1 Consider 4 order 4 dimensional tensor A with entries

$$a_{1111} = a_{2222} = a_{3333} = a_{4444} = 2,\ a_{1222} = \frac{1}{3}, a_{2111} = a_{4111} = a_{4222} = 1, a_{2444} = \frac{4}{3}.$$

and all other entries are zeros and $S = \{1,3\}$, $\bar{S} = \{2,4\}$, $p = 1$, $q = 2$, $\alpha = \frac{1}{3}$.

For this tensor $R_1(\mathrm{A}) = \frac{1}{3}, R_2(\mathrm{A}) = \frac{7}{3}, R_3(\mathrm{A}) = 0, R_4(\mathrm{A}) = 2$, then it has

$$\frac{R_q^{\Lambda}(\mathrm{A})}{\left|a_{qq\cdots q}\right| - R_q^{\bar{\Lambda}}(\mathrm{A})} = \frac{R_2^{\Lambda}(\mathrm{A})}{\left|a_{22\cdots 2}\right| - R_2^{\bar{\Lambda}}(\mathrm{A})} = \frac{1}{2 - \frac{4}{3}} = \frac{3}{2} > 1,$$

and

$$(\left|a_{pp\cdots p}\right| - R_p^{\Lambda}(\mathrm{A}))^{\alpha}(\left|a_{qq\cdots q}\right| - R_q^{\bar{\Lambda}}(\mathrm{A}))^{1-\alpha} = (\left|a_{11\cdots 1}\right| - R_1^{\Lambda}(\mathrm{A}))^{\frac{1}{3}}(\left|a_{22\cdots 2}\right| - R_2^{\bar{\Lambda}}(\mathrm{A}))^{\frac{2}{3}} = 2^{\frac{1}{3}} \times \left(\frac{2}{3}\right)^{\frac{2}{3}},$$

$$R_p^{\bar{\Lambda}}(\mathrm{A})^{\alpha}(R_q^{\Lambda}(\mathrm{A}))^{1-\alpha} = R_1^{\bar{\Lambda}}(\mathrm{A})^{\frac{1}{3}}R_2^{\Lambda}(\mathrm{A})^{\frac{2}{3}} = \left(\frac{1}{3}\right)^{\frac{1}{3}} \times 1^{\frac{2}{3}} = \left(\frac{1}{3}\right)^{\frac{1}{3}}.$$

then

$$(\left|a_{pp\cdots p}\right| - R_p^{\Lambda}(\mathrm{A}))^{\alpha}(\left|a_{qq\cdots q}\right| - R_q^{\bar{\Lambda}}(\mathrm{A}))^{1-\alpha} > R_p^{\bar{\Lambda}}(\mathrm{A})^{\alpha}(R_q^{\Lambda}(\mathrm{A}))^{1-\alpha}.$$

From Theorem 3.1, it concludes that tensor A is a nonsingular *H*-tensor.

Example 2 Consider 4 order 4 dimensional tensor A with entries

$$a_{1111} = a_{2222} = a_{3333} = a_{4444} = 4,\ a_{1222} = \frac{1}{4}, a_{2111} = a_{2444} = a_{4111} = a_{4222} = 1,$$

and all other entries are zeros and $S = \{1,3\}$, $\bar{S} = \{2,4\}$, $p = 1$, $q = 2$, $\alpha = \frac{2}{3}$.

For this tensor $R_1(\mathrm{A}) = \frac{1}{4}, R_2(\mathrm{A}) = 2, R_3(\mathrm{A}) = 0, R_4(\mathrm{A}) = 2,$ then it has

$$0 < \frac{R_q^{\Lambda}(\mathrm{A})}{\left|a_{qq\cdots q}\right| - R_q^{\bar{\Lambda}}(\mathrm{A})} = \frac{R_2^{\Lambda}(\mathrm{A})}{\left|a_{22\cdots 2}\right| - R_2^{\bar{\Lambda}}(\mathrm{A})} = \frac{1}{4-1} = \frac{1}{3} < 1,$$

and

$$(\left|a_{pp\cdots p}\right| - R_p^{\Lambda}(\mathrm{A}))^{\alpha}(\left|a_{qq\cdots q}\right| - R_q^{\bar{\Lambda}}(\mathrm{A}))^{1-\alpha} = (\left|a_{11\cdots 1}\right| - R_1^{\Lambda}(\mathrm{A}))^{\frac{2}{3}}(\left|a_{22\cdots 2}\right| - R_2^{\bar{\Lambda}}(\mathrm{A}))^{\frac{1}{3}} = 4^{\frac{2}{3}} \times 3^{\frac{1}{3}},$$

$$R_p^{\bar{\Lambda}}(\mathrm{A})^{\alpha}(R_q^{\Lambda}(\mathrm{A}))^{1-\alpha} = R_1^{\bar{\Lambda}}(\mathrm{A})^{\frac{2}{3}}(R_2^{\Lambda}(\mathrm{A}))^{\frac{1}{3}} = \left(\frac{1}{4}\right)^{\frac{2}{3}}.$$

then

$$(|a_{pp\cdots p}| - R_p^{\Lambda}(A))^{\alpha}(|a_{qq\cdots q}| - R_q^{\bar{\Lambda}}(A))^{1-\alpha} > R_p^{\bar{\Lambda}}(A)^{\alpha}(R_q^{\Lambda}(A))^{1-\alpha}.$$

From Theorem 3.1, the paper concludes that tensor A is a nonsingular *H*-tensor.

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