Weighted Anisotropic Morrey Spaces Estimates for Anisotropic Maximal Operators

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Abstract The aim of this paper can give weighted anisotropic Morrey Spaces estimates for anisotropic maximal functions.

Keywords: Weighted anisotropic Morrey Space, anisotropic maximal function, A_p weights

1 Introduction

To characterize the regularity of solutions to some partial differential equations(PDEs), Morrey [1] first introduced classical Morrey spaces $M_{p,\lambda}$ which naturally are generalizations of Lebesgue spaces. We also refer to [2],[3] for the latest research on the theory of Morrey spaces associated with harmonic analysis. Next, we recall the definition of weighted Lebesgue spaces. By a "weight" we will mean a non-negative function w that is positive measure a.e. on $\mathbb{R}^n$.

Definition 1. (**Weighted Lebesgue space**) Let $1 \leq p \leq \infty$ and w be a weight function; $w(x) \geq 0$ and $w \in L^{loc}(\mathbb{R}^n)$, we shall define weighted Lebesgue spaces as

$$L_p(w) \equiv L_p(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_{p,w}} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty \right\}, \qquad 1 \leq p < \infty.$$

$$L_{\infty,w} \equiv L_\infty(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_{\infty,w}} = \operatorname*{esssup}_{x \in \mathbb{R}^n} |f(x)| w(x) < \infty \right\}.$$

Then, Komori and Shirai [4] introduced a version of the weighted Morrey space $L_{p,\kappa}(w)$, which is a natural generalization of the weighted Lebesgue space $L_p(w)$, and investigated the boundedness of classical operators in harmonic analysis (see [4] for details). Motivated by [4], in this paper we shall introduce the weighted anisotropic Morrey Spaces and investigate the boundedness of the anisotropic maximal functions on this space.

2 Definition and Notations

Throughout this paper all notation is standard or will be defined as needed.

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space of points $x = (x_1, ..., x_n)$ with norm $|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$, $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r. Given a cube Q and $\lambda > 0$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q. A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight function w and a measurable set E, we define $w(E) = \int_E w(x) dx$, the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E. Given a weight function w, we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any cube Q, we have $w(2Q) \leq Dw(Q)$. When w satisfies this condition, we denote $w \in \Delta_2$, for short.

Let $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$ and $\mathbb{Z}$ be the set of integer numbers. Let also $a = (a_1, ..., a_n)$ be a fixed vector from $\mathbb{R}^n$ with $a_i > 0$, $i = 1, \ldots, n$. Consider a real $n \times n$ matrix A with eigenvalues λ_j, $\mathrm{Re}\lambda_j = a_j > 0$ and let $Q = trA$ be its trace. The matrix A determines a one-parameter group $A_t = \exp(A \ln t)$, $t > 0$ of nonsingular transformations of $\mathbb{R}^n$. Denote by diag $\{a_1, ..., a_n\}$ the matrix with numbers $a_1, ..., a_n$ on the main diagonal and zero off -diagonal elements and let $a_{\max} = \max\limits_{1 \le i \le n} a_i$. Associated with the group A_t is the A_t-homogeneous metric $\rho : \mathbb{R}_0^n \to \mathbb{R}_+$, $\rho(A_t x) = t\rho(x)$ which is smooth on $\mathbb{R}_0^n$.

For $x \in \mathbb{R}_0^n$, let $[x]_a$ be a positive solution to the equation $\sum\limits_{i=1}^{n} x_i^2 [x]_a^{-2a_i} = 1$ and $|x|_a = \max\limits_{1 \le i \le n} |x_i|^{\frac{1}{a_i}}$. Note that $\rho(x)$ is equivalent to $|x|_a$, i.e.,

$$c_1 |x|_a \le \rho(x) \le c_2 |x|_a.$$

For $x \in \mathbb{R}^n$ and $r > 0$ we define the one-parametric parallelepiped

$$\begin{aligned} E(x,t) &= \{y \in \mathbb{R}^n : |x - y|_a \le t\} \\ &= \{y \in \mathbb{R}^n : |y_i - x_i| \le t^{a_i}, \ i = 1, \ldots, n\} \end{aligned}$$

and by $E = E(\alpha)$ we denote the set of all $E(x,t)$ with $x \in \mathbb{R}^n$, $t > 0$. If $a_1 = \cdots = a_n$, then $E(x,t)$ is a cube.

All parallelepipeds are assumed to have their sides parallel to the coordinate axes. $E = E(x_0, r)$ denotes the parallelepiped centered at x_0 with side length $r^{\alpha_1}, \ldots, r^{\alpha_n}$ consequently. Given a parallelepiped E and $\lambda > 0$, $\lambda^a E$ denotes the parallelepiped with the same center as E whose side length is $(\lambda r)^{a_1}, \ldots, (\lambda r)^{a_n}$ consequently.

Unless otherwise indicated, the letter C is used for various constants, and may change from one occurrence to another. First we introduce a weighted anisotropic Morrey space.

Definition 2. (**Weighted anisotropic Morrey spaces**) Let $1 \le p < \infty$, $0 \le \kappa < 1$ and w be a weight. Then a weighted anisotropic Morrey space is defined by

$$L_{p,\kappa,a}(w) := \left\{ f \in L^{loc}(w) : \|f\|_{L_{p,\kappa,a}(w)} < \infty \right\},$$

where

$$\|f\|_{L_{p,\kappa,a}(w)} = \sup_{E} \left(\frac{1}{w(E)^\kappa} \int_E |f(x)|^p w(x)dx \right)^{\frac{1}{p}}$$

and the supremum is taken over all parallelepipeds E on $\mathbb{R}^n$. In the case of $a = (1, \ldots, 1)$, we get weighted Morrey spaces $L_{p,\kappa}(w) = L_{p,\kappa,1}(w)$.

Remark 1. Alternatively, we could define the weighted Morrey spaces with anisotropic balls instead of parallelepipeds. Hence we shall use these two definitions of weighted anisotropic Morrey spaces appropriate to calculation. Also, we could define the weighted Morrey spaces with cubes instead of parallelepipeds. Moreover, when $a_1 = \cdots = a_n = 1$, then $\rho(x) = |x|$ and $\alpha = n$. In this case, weighted anisotropic Morrey spaces is reduced to the weighted Morrey space:

$$\|f\|_{L_{p,\kappa,a}(w)} = \|f\|_{L_{p,\kappa}(w)}.$$

Thus from this viewpoint, Theorem 4 and Theorem 4 below improve the corresponding results in [3],[4]. In other words, in this paper our goal is to extend results in [3],[4].

Remark 2. (1) If $w \equiv 1$ and $\kappa = \lambda/n$ with $0 < \lambda < n$, then $L_{p,\kappa}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey spaces. If $w \equiv 1$ and $\kappa = \frac{\lambda}{|a|}$ with $0 < \lambda < |a|$, $|a| = a_1 + \cdots + a_n$, then $L_{p,\kappa,a}(w) = L_{p,\lambda,a}(w)$ the anisotropic Morrey spaces. (2) Let $w \in \Delta_2$. If $\kappa = 0$, then $L_{p,0}(w) = L_p(w)$ is the weighted Lebesgue spaces. If $\kappa = 1$, then $L_{p,1}(w) = L_\infty(w)$ by the Lebesgue differentiation theorem with respect to w (see

[5]). (3) In the one-dimensional case, let a weight $w(x) = |x|^\alpha$ for some $-\frac{1}{2} < \alpha < 0$ and a function $f(x) = \chi_{(0,1)} |x|^{-\frac{1}{2}}$. Then $f \in L^{1, \frac{\alpha + \frac{1}{2}}{\alpha + 1}}(w) \setminus L^{2(\alpha + 1)}(w)$.

Let $f \in L^{loc}(\mathbb{R}^n)$. The anisotropic maximal function Mf and the sharp maximal function $f^{\#}$ are defined by

$$Mf(x) = \sup_{t > 0} |E(x,t)|^{-1} \int_{E(x,t)} |f(y)| dy$$

and

$$f^{\#}(x) = \sup_{t > 0} |E(x,t)|^{-1} \int_{E(x,t)} |f(y) - f_{E(x,t)}| dy,$$

where $f_{E(x,t)} = |E(x,t)|^{-1} \int_{E(x,t)} |f(y)| dy$.

If $a_1 = \cdots = a_n$, then $E(x,t)$ is a cube and Mf becomes the usual Hardy-Littlewood maximal function. For $r > 0$, we denote $M_r f(x)$ by $(M|f|^r(x))^{\frac{1}{r}}$.

Let w be a weight. M_w denotes the anisotropic maximal operator with respect to the measure $w(x) dx$ defined by

$$M_w f(x) = \sup_E \frac{1}{w(E)} \int_E |f(y)| w(y) \, dy \qquad (1)$$

Here and after, A_p denotes the Muckenhoupt classes (see [3]). In other words, let $1 < p < \infty$. One says that $w(x) \in A_p(\mathbb{R}^n)$, if the following condition holds:

$$[w]_{A_p} := \sup_E [w]_{A_p(E)}$$

$$= \sup_E \left(\frac{1}{|E|} \int_E w(x) dx \right) \left(\frac{1}{|E|} \int_E w(x)^{1-p'} dx \right)^{p-1} < \infty, \qquad (2)$$

where the supremum is taken with respect to any parallelepiped E and $p' = \frac{p}{p-1}$. The condition (2) is called the A_p-condition, and the weights which satisfy it are called A_p-weights. The property of the A_p-weights implies that generally speaking, we should check whether a weight w satisfies an A_p-condition or not. The expression $[w]_{A_p}$ is also called characteristic constant of w. For the case $p = 1$, $w \in A_1(\mathbb{R}^n)$ if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \inf_{x \in B} w(x),$$

and the infimum of these constant C is called the A_1 constant of w and denoted by $[w]_{A_1}$. Since the A_p classes are increasing with respect to p, the A_∞ class of weights is defined in a natural way by $A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n)$ and the A_∞ constant of $w(x) \in A_\infty(\mathbb{R}^n)$ is the smallest of the infimum of the A_p constant such that $w(x) \in A_p(\mathbb{R}^n)$, $[w]_{A_\infty} = \inf_{1 \leq p < \infty} [w]_{A_p}$ and $[w]_{A_\infty} \leq [w]_{A_p}$.

It is easy to verify that, $\rho(x)^\alpha \in A_p$ if and only if $-|a| < \alpha < |a|(p-1)$ for $1 < p < \infty$ and $\rho(x)^\alpha \in A_1$ if and only if $-|a| < \alpha \leq 0$.

3 Lemmas and Well Known Results

In this section, we shall prove some lemmas and describe the well-known result about the weighted L_p spaces.

Theorem 1. ([[6], Theorem 2.6, p. 146]) If $1 < p < \infty$ and $w \in \Delta_2$, then the operator M_w is bounded on $L_p(w)$.

The next lemma plays an important role in our proofs of theorems. We say that w satisfies the reverse doubling condition if w has the property (3) of the following lemma.

Lemma 1. ([4])If $w \in \Delta_2$, then there exists a constant $D_1 > 1$ such that

$$w(2E) \geq D_1 w(E). \tag{3}$$

Proof. Fix a parallelepiped $E = E(x_0, r)$. Then we can choose a parallelepiped $R \subset 2E$ with side length $\frac{r}{2}$ which is disjoint from the parallelepiped E. Hence

$$w(E) + w(R) \leq w(2E).$$

On the other hand, since $E \subset 5R$ we have $w(E) \leq w(5R) \leq D^3 w(R)$, where D is a doubling constant. Therefore we have

$$w(E) + \frac{w(E)}{D^3} \leq w(2E).$$

Lemma 2. ([7]) The following statements hold:
(1) If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$ we have

$$w(\lambda E) \leq \lambda^{np}[w]_{A_p} w(E).$$

(2) Let $w \in A_p$ for some $1 \leq p < \infty$. Then we have

$$Mf(x) \leq [w]_{A_p}^{\frac{1}{p}} \left(M_w\left(|f|^p\right)(x) \right)^{\frac{1}{p}}.$$

Proof. (1) Let $w \in A_p$ for some $1 \leq p < \infty$ and $\lambda > 1$. Then

$$\frac{w(\lambda E)}{w(E)} = \left(\frac{|\lambda E|}{|E|} \right)^p \frac{[w]_{A_p}(\lambda E)}{[w]_{A_p}(E)} \frac{\left(\int_E w(x)^{1-p'} dx \right)^{p-1}}{\left(\int_{\lambda E} w(x)^{1-p'} dx \right)^{p-1}}$$

$$\leq \lambda^{np}[w]_{A_p}.$$

(2) Let $w \in A_p$ for some $1 \leq p < \infty$. Applying the Hölder's inequality, we get

$$Mf(x) = \sup_E \frac{1}{|E|} \int_E |f(x)| dx$$

$$\leq \sup_E \frac{1}{|E|} \left(\int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left(\int_E w(x)^{1-p'} dx \right)^{\frac{1}{p'}}$$

$$= \sup_E \left(\frac{1}{w(E)} \int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left(\int_E w(x)^{1-p'} dx \right)^{\frac{1}{p'}} \left(\frac{w(E)}{|E|} \right)^{\frac{1}{p}}$$

$$= \left(\sup_E \frac{1}{w(E)} \int_E |f(x)|^p w(x) dx \left(\frac{1}{|E|} \int_E w(x) dx \right) \left(\int_E w(x)^{1-p'} dx \right)^{p-1} \right)^{\frac{1}{p}}$$

$$\leq [w]_{A_p}^{\frac{1}{p}} \left(M_w\left(|f|^p\right)(x) \right)^{\frac{1}{p}}.$$

4   Anisotropic Maximal Function

In this section, we shall state the boundedness of the anisotropic maximal operators on weighted anisotropic Morrey Spaces.

Theorem 2. (Our main result) If $1 < p < \infty$, $0 < \kappa < 1$ and $w \in \Delta_2$, then the operator M_w is bounded on $L_{p,\kappa,a}(w)$.

Proof. Fix a parallelepiped $E \subset \mathbb{R}^n$. We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{3E}$. Since M_w is a sublinear operator, we have

$$\left(\int_E M_w f(x)^p w(x) dx \right)^{\frac{1}{p}} \leq \left(\int_E M_w f_1(x)^p w(x) dx \right)^{\frac{1}{p}} + \left(\int_E M_w f_2(x)^p w(x) dx \right)^{\frac{1}{p}}$$
$$= I + II.$$

For the term I, since M_w is bounded on $L_p(w)$ (see Theorem 3), we obtain

$$I \leq \left(\int_E M_w f_1(x)^p w(x) dx \right)^{\frac{1}{p}} \leq C \left(\int_{3E} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}$$
$$\leq C \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa}{p}}.$$

Next we estimate the term II. By simple geometric observation, we have for any $x \in E$, note that for all $\tilde{E}$ such that $x \in \tilde{E}$, $\tilde{E} \cap (3E)^c \neq \emptyset$ there exists R such that $E \subset 3R$ and $\frac{1}{3}R \subset \tilde{E} \subset R$.

Then $w(\tilde{E}) \geq w\left(\frac{1}{3}R\right) \geq \frac{1}{D_1} w(R)$ and

$$M_w f_2(x) \leq D_1 \sup_{R: E \subset 3R} \frac{1}{w(R)} \int_R |f(y)| w(y) \, dy.$$

Note that

$$\frac{1}{w(R)} \int_R |f(y)| w(y) \, dy \leq \frac{1}{w(R)} \left(\int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \left(\int_R w(y) dy \right)^{\frac{1}{p'}}$$
$$\leq \frac{1}{w(R)^{1-\frac{1}{p'}}} \left(\int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} = \frac{1}{w(R)^{\frac{1}{p}}} \left(\frac{1}{w(R)^\kappa} \int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} w(R)^{\frac{\kappa}{p}}$$
$$\leq \left(\frac{1}{w(R)^\kappa} \int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} w(R)^{\frac{\kappa-1}{p}} \leq C \|f\|_{L_{p,\kappa,a}(w)} w(R)^{\frac{\kappa-1}{p}},$$

if $E \subset 3R$. So we obtain

$$II = \left(\int_E M_w f_2(x)^p w(x) dx \right)^{\frac{1}{p}} \leq \left(\int_E \left[\sup_{x \in R} \frac{1}{w(R)} \int_R |f(y)| w(y) \, dy \right]^p w(x) dx \right)^{\frac{1}{p}}$$
$$\leq \left(\int_E \left[c \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa-1}{p}} \right]^p w(x) dx \right)^{\frac{1}{p}} \leq c \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa-1}{p}} \left(\int_E w(x) dx \right)^{\frac{1}{p}}$$
$$= c \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa}{p}}.$$

Therefore

$$II \leq C \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa}{p}}.$$

This completes the proof.

Corollary 1. ([4],Theorem 3.1) If $1 < p < \infty$, $0 < \kappa < 1$ and and $w \in \Delta_2$, then the operator M_w is bounded on $L_{p,\kappa}(w)$.

Theorem 3. (Our main result) If $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$, then the anisotropic maximal operator M is bounded on $L_{p,\kappa,a}(w)$.

If $p = 1$, $0 < \kappa < 1$ and $w \in A_1$, for all $t > 0$ and any parallelepiped E,

$$w\left(x \in E : Mf(x) > t\right) \leq \frac{C}{t} \|f\|_{L_{1,\kappa,a}(w)} w(E)^{\kappa}.$$

Proof. Let $1 < p < \infty$. By the reverse Hölder inequality (see [8]), there exists $1 < r < p$ such that $w \in A_r$. Hence it follows from Lemma 3 (2) and Theorem 4 that

$$\left(\frac{1}{w(E)^{\kappa}}\int_E Mf(x)^p w(x)dx\right)^{\frac{1}{p}} \leq C\left(\frac{1}{w(E)^{\kappa}}\int_E M_w\left(|f|^r\right)(x)^{\frac{p}{r}} w(x)dx\right)^{\frac{1}{p}}$$

$$\leq C\|M_w\left(|f|^r\right)\|_{L_{\frac{p}{r},\kappa,a}(w)}^{\frac{1}{r}} \leq C\||f|^r\|_{L_{\frac{p}{r},\kappa,a}(w)}^{\frac{1}{r}}$$

$$= c\left[\sup_E \left(\frac{1}{w(E)^{\kappa}}\int_E |f(x)|^{r\left(\frac{p}{r}\right)} w(x)dx\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} = C\|f\|_{L_{p,\kappa,a}(w)}.$$

When $p = 1$, we use the Fefferman-Stein maximal inequality

$$\int_{\{x : Mf(x)>t\}} \varphi(x)\,dx \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(x)| (M\varphi)(x)\,dx$$

for any functions f and $\varphi \geq 0$ (see [9],[6]).

Fix a parallelepiped $E = E(x_0, r)$. Put $\varphi(x) = w(x)\chi_E(x)$. Then we have

$$\int_{\{x : Mf(x)>t\}} \chi_E(x) w(x)\,dx \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(x)| M(w\chi_E)(x)\,dx$$

$$= \frac{C}{t}\left(\int_{3E} + \int_{(3E)^c}\right) = \frac{C}{t}\{I + II\}$$

for all $t > 0$.

We now estimate the term I. Since $w \in A_1$, it follows that

$$M(w\chi_E)(x) \leq M(w)(x) \leq Cw(x).$$

So it follows that

$$I \leq Cw(3E)^{\kappa} \|f\|_{L_{1,\kappa,a}(w)} \leq Cw(E)^{\kappa} \|f\|_{L_{1,\kappa,a}(w)}.$$

To estimate the term II, we consider the form

$$\frac{1}{|R|}\int_{R\cap E} w(y)\,dy$$

for $x \in (3E)^c \cap R$ and $R \cap E \neq \emptyset$. By simple geometric observation, we have

$$\frac{1}{|R|}\int_{R\cap E} w(y)\,dy \leq C_n\left(\frac{1}{|x-x_0|_a^{|a|}}\int_E w(y)\,dy\right) \leq C_n |x-x_0|_a^{-|a|} w(E).$$

Therefore we obtain

$$M\left(w\chi_E\right)(x) \le C_n \left|x - x_0\right|_a^{-|a|} w\left(E\right).$$

Since $w \in A_1$, we have $w \in \Delta_2$ by Lemma 3 (1). Using Lemma 3, we have $w\left(3E\right) \ge w\left(2E\right) \ge D_1 w\left(E\right)$ with $D_1 > 1$. Thus we can estimate the term II as follows:

$$
\begin{aligned}
II &\le Cw\left(E\right) \int\limits_{(3E)^c} \frac{|f(x)|}{|x - x_0|_a^{|a|}} dx = C\sum_{j=1}^{\infty} \int\limits_{3^{j+1}E \backslash (3^j E)} \frac{|f(x)|}{|x - x_0|_a^{|a|}} dx \\
&\le Cw\left(E\right) \sum_{j=1}^{\infty} \frac{1}{|3^j E|} \int\limits_{3^{j+1}E} |f(x)| \, dx \\
&\le Cw\left(E\right) \sum_{j=1}^{\infty} \frac{1}{|3^j E|} \left(\operatorname*{esssup}_{x\in 3^{j+1}E} \frac{1}{w(x)}\right) \int\limits_{3^{j+1}E} |f(x)| \, w(x) \, dx \\
&= Cw\left(E\right) \sum_{j=1}^{\infty} \frac{1}{|3^j E|} \frac{\left|3^{j+1}E\right|}{w\left(3^{j+1}E\right)} \int\limits_{3^{j+1}E} |f(x)| \, w(x) \, dx \\
&= Cw\left(E\right)^{\kappa} \sum_{j=1}^{\infty} \frac{w\left(E\right)^{1-\kappa}}{w\left(3^{j+1}E\right)^{1-\kappa}} \frac{1}{w\left(3^{j+1}E\right)^{\kappa}} \int\limits_{3^{j+1}E} |f(x)| \, w(x) \, dx \\
&\le Cw\left(E\right)^{\kappa} \|f\|_{L_{1,\kappa,a}(w)} \sum_{j=1}^{\infty} \frac{w\left(E\right)^{1-\kappa}}{w\left(3^{j+1}E\right)^{1-\kappa}} \le Cw\left(E\right)^{\kappa} \|f\|_{L_{1,\kappa,a}(w)}.
\end{aligned}
$$

The last series converges since the reverse doubling constant is larger than one (see Lemma 3). This completes the proof.

Corollary 2. ([4],Theorem 3.2) If $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$, then the Hardy-Littlewood maximal operator M is bounded on $L_{p,\kappa}(w)$. If $p = 1$, $0 < \kappa < 1$ and $w \in A_1$, for all $t > 0$ and any cube Q,

$$w\left(\{x \in Q : Mf(x) > t\}\right) \le \frac{C}{t} \|f\|_{L_{1,\kappa}(w)} w\left(Q\right)^{\kappa}.$$

Corollary 3. ([3],Theorem 2) Let $1 < p < \infty$, $0 \le \kappa < 1$ and $w \in A_p$. If A is a (not necessarily quasi-homogenous) pseudo-differential operator of order 0, then the operator A is bounded on $L_{p,\kappa}(w)$.

Corollary 4. ([3], Corollary 1) Let $m \in \mathbb{N}$, $1 \le j \le n$, $1 < p < \infty$, $0 \le \kappa < 1$, $w \in A_p$ and $a(x,\xi) = \frac{\xi_j^m}{(1+|\xi|^2)^{\frac{m}{2}}}$. Then

$$\left\|F^{-1}(\xi_j^m(1 + |\xi|^2)^{-\frac{m}{2}} Ff)\right\|_{L_{p,\kappa}(w)} \le C \|f\|_{L_{p,\kappa}(w)}.$$

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