Positive Solutions of Some Fourth-order Two Point Boundary Value Problem with All Order Derivatives

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Abstract In this paper, by the use of a new fixed point theorem and the Boundary Value Problem's Green function. the existence of at least one positive solutions for the fourth-order two point boundary value problem with all order derivatives

$$\begin{cases} u^{(4)}(t) + u^{''}(t) = \lambda f(t, u(t), u^{'}(t), u^{''}(t), u^{'''}(t)), t \in [0,1], \\ u(0) = u(1) = u^{''}(0) = u^{''}(1) = 0. \end{cases}$$

is considered, where f is a nonnegative continuous function and $\lambda > 0, 0 < A < \pi^2$.

Keywords: Fourth-order boundary value problem, fixed point theorem in a cone, positive solution.

1 Introduction

The deformation of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by a fourth-order ordinary equation boundary value problem. Owing to its significance in physics, the existence of positive solutions for the fourth-order boundary value problem has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point index theory, the Krasnosel'skii's fixed point theorem and the method of upper and lower solutions, in reference [1-9][11]. In recent years, there has been much attention on the fourth-order differential equations with one or two parameters.

By the fixed point theorem and theory in cone [4], Bai investigated the following fourth-order two point boundary value problem

$$\begin{cases} u^{(4)}(t) - \lambda f(u(t)) = 0, t \in [0,1], \\ u(0) = u(1) = u^{''}(0) = u^{''}(1) = 0. \end{cases}$$

where λ is a normal number,$f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$

By the monotone operator theorem and the critical point theory, Li [7] proved the existence and multiplicities of positive solutions for the fourth-order two point boundary value problem

$$\begin{cases} u^{(4)}(t) - f(u(t)) = 0, t \in [0,1], \\ u(0) = u(1) = u^{''}(0) = u^{''}(1) = 0. \end{cases}$$

where $f : [0,1] \times R^1 \longrightarrow R^1$ is continuous.

All the above works were done under the assumption that the first order derivative $u^{'}$ $u^{''}$ $u^{''}$ is not involved explicitly in the nonlinear term f. We are concerned with the existence of positive solutions for the fourth-order two-point boundary value problem

$$\begin{cases} u^{(4)}(t) + u^{''}(t) = \lambda f(t, u(t), u^{'}(t), u^{''}(t), u^{'''}(t)), t \in [0,1], \\ u(0) = u(1) = u^{''}(0) = u^{''}(1) = 0. \end{cases} \tag{1.1}$$

Throughout, we assume

(H_1) $\lambda > 0, 0 < A < \pi^2$;
(H_2) $f : [0,1] \times [0,\infty) \times R \longrightarrow [0,\infty)$is continuous.

2 Preliminary

Let $Y = C[0,1]$ be the Banach space equipped with the norm $\|u\|_0 = \max_{t\in[0,1]} |u(t)|$.

Set λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 + A\lambda$, namely $\lambda_1 = 0, \lambda_2 = -A$. By (H_1), it is easy to see that $-\pi^2 < \lambda_2 < 0$.

Let $G_i(t,s)(i = 1,2)$ be the Green's function of the linear boundary value problem:$-u^{''} + \lambda_i u(t) = 0, u(0) = u(1) = 0$.Then, carefully calculation yield:

$$G_1(t,s) = \begin{cases} s(1-t), 0 \le s \le t \le 1, \\ t(1-s), 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{\sin\sqrt{A}s \sin\sqrt{A}(1-t)}{\sqrt{A}\sin\sqrt{A}}, 0 \le s \le t \le 1, \\ \dfrac{\sin\sqrt{A}t \sin\sqrt{A}(1-s)}{\sqrt{A}\sin\sqrt{A}}, 0 \le t \le s \le 1. \end{cases}$$

Lemma 2.1. *([8]) Suppose $(H_1)(H_2)$ hold. Then for any $g(t) \in C[0,1]$, BVP*

$$\begin{cases} u^{(4)}(t) + Au^{''}(t) = g(t), t \in [0,1] \\ u(0) = u(1) = u^{''}(0) = u^{''}(1) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)g(\tau)\mathrm{d}\tau\mathrm{d}s, \tag{2.2}$$

where

$$G_1(t,s) = \begin{cases} s(1-t), 0 \le s \le t \le 1, \\ t(1-s), 0 \le t \le s \le 1, \end{cases}$$

$$G_2(s,\tau) = \begin{cases} \dfrac{\sin\sqrt{A}\tau \sin\sqrt{A}(1-s)}{\sqrt{A}\sin\sqrt{A}}, 0 \le \tau \le s \le 1, \\ \dfrac{\sin\sqrt{A}s \sin\sqrt{A}(1-\tau)}{\sqrt{A}\sin\sqrt{A}}, 0 \le s \le \tau \le 1. \end{cases}$$

By $u(t)$,we get

$$u^{'}(t) = \int_t^1 \int_0^1 G_2(s,\tau)g(\tau)\mathrm{d}\tau\mathrm{d}s - \int_0^1 \int_0^1 sG_2(s,\tau)g(\tau)\mathrm{d}\tau\mathrm{d}s, \tag{2.3}$$

$$u^{''}(t) = -\int_0^1 G_2(t,\tau)g(\tau)\mathrm{d}\tau, \tag{2.4}$$

$$u^{'''}(t) = -\int_0^1 \frac{\partial G_2(t,\tau)}{\partial t} g(\tau)\mathrm{d}\tau. \tag{2.5}$$

Lemma 2.2. *([8]) Assume(H_1) (H_2) hold. Then one has:*

(i) $G_i(t,s) \ge 0, \forall t,s \in [0,1]$;
(ii) $G_i(t,s) \le C_i G_i(s,s), \forall t,s \in [0,1]$;
(iii) $G_i(t,s) \ge \delta_i G_i(t,t)G_i(s,s), \forall t,s \in [0,1]$.

where $C_1 = 1, \delta_1 = 1; C_2 = \dfrac{1}{\sin\sqrt{A}}, \delta_2 = \sqrt{A}\sin\sqrt{A}$.

Lemma 2.3. *If $g(t) \in C[0,1], g(t) \ge 0$, then the unique solution $u(t)$ of the BVP (2.1) satisfies:*

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge d_1\|u\|_0, \min_{\frac{1}{4}\le t\le\frac{3}{4}} (-u^{''}(t)) \ge d_2\|u^{''}\|_0.$$

where $d_1 = \dfrac{\sqrt{A}\sin^2\sqrt{A}C_0D_1}{M_1}, d_2 = \sqrt{A}\sin^2\sqrt{A}D_2, C_0 = \displaystyle\int_0^1 G_1(s,s)G_2(s,s)\mathrm{d}s,$
$M_1 = \int_0^1 G_1(s,s)\mathrm{d}s, D_i = \min_{\frac{1}{4}\le t\le\frac{3}{4}} G_i(t,t), (i = 1,2).$

Proof. By(2.4) and (ii) of Lemma2.2, we have

$$\begin{aligned} u(t) &= \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)g(\tau)\mathrm{d}\tau\mathrm{d}s \\ &\le C_1C_2 \int_0^1 \int_0^1 G_1(s,s)G_2(\tau,\tau)g(\tau)\mathrm{d}\tau\mathrm{d}s \\ &\le C_1C_2M_1 \int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau \end{aligned}$$

So,

$$\|u(t)\|_0 \le C_1C_2M_1 \int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau.$$

Using (iii) of Lemma2.2, we have:

$$\begin{aligned} u(t) &\ge \delta_1\delta_2 \int_0^1 \int_0^1 G_1(t,t)G_1(s,s)G_2(s,s)G_2(\tau,\tau)g(\tau)\mathrm{d}\tau \\ &= \delta_1\delta_2C_0G_1(t,t) \int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau \\ &\ge \frac{\delta_1\delta_2C_0}{C_1C_2M_1} G_1(t,t)\|u(t)\|_0 \end{aligned}$$

So,

$$\begin{aligned} \min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) &\ge \frac{\delta_1\delta_2C_0D_1}{C_1C_2M_1}\|u(t)\|_0 \\ &= \frac{\sqrt{A}\sin^2\sqrt{A}C_0D_1}{M_1}\|u(t)\|_0 \\ &= d_1\|u(t)\|_0. \end{aligned}$$

By (2.6) and (ii) of Lemma2.2, we have:

$$\begin{aligned} -u''(t) &= \int_0^1 G_2(t,\tau)g(\tau)\mathrm{d}\tau \\ &\le C_2 \int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau \end{aligned}$$

So, we have:

$$\|u''(t)\|_0 = C_2 \int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau.$$

Using (iii) of Lemma2.2, We have:

$$\begin{aligned} -u''(t) &= \int_0^1 G_2(t,\tau)g(\tau)\mathrm{d}\tau \\ &\ge \delta_2 \int_0^1 G_2(t,t)G_2(\tau,\tau)g(\tau)\mathrm{d}\tau \\ &= \delta_2G_2(t,t) \int_0^1 G_2(\tau,\tau)g(\tau)\mathrm{d}\tau \\ &\ge \frac{\delta_2G_2(t,t)}{C_2}\|u''(t)\|_0. \end{aligned}$$

So,

$$\begin{aligned}\min_{\frac{1}{4}\le t\le\frac{3}{4}}(-u''(t)) &\ge \frac{\delta_2 D_2}{C_2}\|u''(t)\|_0\\ &= \sqrt{A}\sin^2\sqrt{A}D_2\|u''(t)\|_0\\ &= d_2\|u''(t)\|_0\end{aligned}$$

□

Let X be a Banach space and $K\subset X$ in a cone. Suppose $\alpha,\beta:X\to R^+$ are two continuous convex functionals satisfying $\alpha(\lambda u)=|\lambda|\alpha(u),\beta(\lambda u)=|\lambda|\beta(u)$, for $u\in X,\lambda\in R$, and $\|u\|\le M\max\{\alpha(u),\beta(u)\}$, for $u\in X$ and $\alpha(u)\le\alpha(v)$ for $u,v\in K,u\le v$,where $M>0$ is a constant.

Theorem 2.1. *([10]) Let $r_2>r_1>0,L>0$ be constants and*

$$\Omega_i=\{x\in X:\alpha(x)<r_i,\beta(x)<L\},i=1,2,$$

two bounded open sets in X. Set

$$D_i=\{x\in X:\alpha(x)=r_i\},i=1,2.$$

Assume $T:K\to K$ is a completely continuous operator satisfying

(A_1) $\alpha(Tx)<r_1,x\in D_1\bigcap K;\alpha(Tx)>r_2,x\in D_2\bigcap K;$
(A_2) $\beta(Tx)<L,x\in K;$
(A_3) *there is a $p\in(\Omega_2\bigcap K)\setminus\{0\}$ such that $\alpha(p)\neq 0$ and $\alpha(x+\lambda p)\ge\alpha(x)$, for all $x\in K$ and $\lambda\ge 0$.*

Then T has at least one fixed point in $(\Omega_2\backslash\overline{\Omega}_1)\bigcap K$.

3 The main results

Let $X=C^4[0,1]$ be the Banach space equipped with the norm $\|u\|=\max\limits_{t\in[0,1]}\mid u(t)\mid+\max\limits_{t\in[0,1]}\mid u'(t)\mid+\max\limits_{t\in[0,1]}\mid u''(t)\mid+\max\limits_{t\in[0,1]}\mid u'''(t)\mid$, and $K=\{u\in X:u(t)\ge 0,u''(t)\le 0,\min\limits_{\frac{1}{4}\le t\le\frac{3}{4}}u(t)\ge d_1\|u\|_0,\max\limits_{\frac{1}{4}\le t\le\frac{3}{4}}(-u''(t))\ge d_2\|u''\|_0\}$ is a cone in X.

Define two continuous convex functionals $\alpha(u)=\max\limits_{t\in[0,1]}|u(t)|+\max\limits_{t\in[0,1]}|u''(t)|$ and $\beta(u)=\max\limits_{t\in[0,1]}|u'(t)|+$ $+\max\limits_{t\in[0,1]}|u'''(t)$, for each $u\in X$, then$\|u\|\le 2\max\{\alpha(u),\beta(u)\}$ and $\alpha(\lambda u)=|\lambda|\alpha(x),\beta(\lambda u)=|\lambda|\beta(u)$, for $u\in X,\lambda\in R$; $\alpha(u)\le\alpha(v)$ for $u,v\in K,u\le v$.

In the following, we denote

$$\begin{aligned}B&=\int_0^1 G_2(\tau,\tau))\mathrm{d}\tau,\\ F&=\int_0^1\frac{\sin\sqrt{A}\tau}{\sin\sqrt{A}}\mathrm{d}\tau\\ \eta_0&=\frac{1}{C_2B(C_1M_1+1)},\eta_1=\frac{1}{\int_{\frac{1}{4}}^{\frac{3}{4}}G_2(\frac{1}{2},\tau)\mathrm{d}\tau},\eta_2=\frac{2}{3C_2B+4F},\theta=\{\frac{d_1}{2},\frac{d_2}{2}\}.\end{aligned}$$

We will suppose that there are $L>b>\theta b>c>0$ such that $f(t,u,v,u_0,v_0)$ satisfies the following growth conditions:

(H_3) $f(t,u,v,u_0,v_0)<\dfrac{c\eta_0}{\lambda}$, for $(t,u,v,u_0,v_0)\in[0,1]\times[0,c]\times[-L,L]\times[-c,0]\times[-L,L]$,

(H_4) $f(t,u,v,u_0,v_0) \geq \dfrac{b\eta_1}{\lambda}$, for $(t,u,v,u_0,v_0) \in [\frac{1}{4},\frac{3}{4}] \times [\theta b, b] \times [-L,L] \times [-b,0] \times [-L,L] \bigcup [\frac{1}{4},\frac{3}{4}] \times [0,b] \times [-L,L] \times [-b,-\theta b] \times [-L,L]$,

(H_5) $f(t,u,v,u_0,v_0) < \dfrac{L\eta_2}{\lambda}$, for $(t,u,v,u_0,v_0) \in [0,1] \times [0,b] \times [-L,L] \times [-b,0] \times [-L,L]$.

Let $f_1(t,u,v,u_0,v_0) = f_1(t,u^*,v^*,u_0^*,v_0^*)$, where

$$u^* = \max\{\max(u,0),b\}, \qquad v^* = \max\{\max(v,-L),L\},$$
$$u_0^* = \max\{\max(u_0,-b),0\}, \qquad v_0^* = \max\{\max(v,-L),L\}.$$

We denote

$$(Tu)(t) = \lambda \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s, \tag{3.1}$$

$$(Tu)^{'}(t) = \lambda[\int_t^1 \int_0^1 G_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s$$
$$- \int_0^1 \int_0^1 sG_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s], \tag{3.2}$$

$$(Tu)^{''}(t) = -\lambda \int_0^1 G_2(t,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau, \tag{3.3}$$

$$(Tu)^{'''}(t) = -\lambda \int_0^1 \frac{\partial G_2(t,\tau)}{\partial t} f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau. \tag{3.4}$$

Lemma 3.1. *Suppose (H_1) hold. Then $T: K \to K$ is completely continuous. Suppose (H_1) (H_2) hold. Then $T: K \to K$ is completely continuous.*

Proof. For $u \in K$, by (3.1) and (3.3) with Lemma 2.2, there is $Tu > 0, (Tu)^{''} \leq 0$. so

$$\begin{aligned}\|Tu\|_0 &= \max_{t\in[0,1]} \mid \lambda \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)f_1(t,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \mid \\ &\leq \lambda \int_0^1 \int_0^1 C_1C_2G_1(s,s)G_2(\tau,\tau)f_1(t,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \\ &= \lambda C_1C_2M_1 \int_0^1 G_2(\tau,\tau)f_1(t,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}\|(Tu)^{''}\|_0 &= \max_{t\in[0,1]} \mid -\lambda \int_0^1 G_2(t,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau \mid \\ &\leq \lambda C_2 \int_0^1 G_2(\tau,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\end{aligned}$$

By Lemma 2.2, (ii) and (3.1) (3.3),we have:

$$\begin{aligned}\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) &= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \lambda \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \\ &\geq \lambda\delta_1\delta_2 \int_0^1 \int_0^1 G_1(t,t)G_1(s,s)G_2(s,s)G_2(\tau,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau \\ &\geq \lambda\delta_1\delta_2C_0G_1(t,t) \int_0^1 G_2(\tau,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau \\ &\geq \lambda\delta_1\delta_2C_0D_1 \int_0^1 G_2(\tau,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau \\ &\geq \frac{\lambda\delta_1\delta_2C_0D_1}{\lambda C_1C_2M_1}\|Tu\|_0 \\ &= d_1\|Tu\|_0,\end{aligned}$$

$$\begin{aligned}\min_{\frac{1}{4}\le t\le\frac{3}{4}}(-(Tu)^{''}(t)) &= \min_{\frac{1}{4}\le t\le\frac{3}{4}}\lambda\int_0^1 G_2(t,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\\ &\ge \lambda\delta_2\int_0^1 G_2(t,t)G_2(\tau,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\\ &\ge \lambda\delta_2 G_2(t,t)\int_0^1 G_2(\tau,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\\ &\ge \frac{\lambda\delta_2 G_2(t,t)}{C_2}\|(Tu)^{''}\|_0\\ &\ge \frac{\lambda\delta_2 D_2}{\lambda C_2}\|(Tu)^{''}\|_0\\ &= d_2\|(Tu)^{''}\|_0,\end{aligned}$$

So we can get $T(K)\subset K$. Let $B\subset K$ is bounded, it is clear that $T(B)$ is bounded. Using $f_1, G_1(t,s), G_2(t,s)$ is continuous, We show that $T(B)$ is equicontinuous. By the Arzela-Ascoli theorem, a standard proof yields $T: K\to K$ is completely continuous. □

Theorem 3.1. *Suppose* (H_1)-(H_5) *hold. Then BVP* (1.1) *has at least one positive solution* $u(t)$ *satisfying* $c<\alpha(u)<b, \beta(u)<L$.

Proof. Take $\Omega_1=\{u\in X: |\alpha(u)|<c, |\beta(u)<L|\}, \Omega_2=\{u\in X: |\alpha(u)|<b, |\beta(u)<L|\}$, two bounded open sets in X, and $D_1=\{u\in X:\alpha(u)=c\}, D_2=\{u\in X:\alpha(u)=b\}$.

By Lemma 3.1, $T: K\to K$ is completely continuous,and there is a $p\in(\Omega_2\bigcap K)\backslash\{0\}$ such that $\alpha(p)\neq 0$ for all $u\in K$ and $\lambda\ge 0$.

$$\begin{aligned}\|Tu\|_0 &= |\lambda\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f_1(t,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s|\\ &\le \lambda C_1C_2M_1\int_0^1 G_2(\tau,\tau)f_1(t,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\\ &\le \lambda C_1C_2M_1\int_0^1 G_2(\tau,\tau)\mathrm{d}\tau\times\frac{c\eta_0}{\lambda}\\ &= C_1C_2M_1Bc\eta_0,\\ \|(Tu)^{''}\|_0 &= |-\lambda\int_0^1 G_2(t,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau|\\ &\le \lambda C_2\int_0^1 G_2(\tau,\tau)\mathrm{d}\tau\times\frac{c\eta_0}{\lambda}\\ &= C_2Bc\eta_0,\end{aligned}$$

Hence, for $u\in D_1\bigcap K, \alpha(u)=c$, we get

$$\alpha(Tu)=\|Tu\|_0+\|(Tu)^{''}\|_0<C_1C_2M_1Bc\eta_0+C_2Bc\eta_0=(C_1C_2M_1B+C_2B)c\eta_0.$$

Whereas for $u\in D_2\bigcap K, \alpha(u)=b$, there is $\|u\|_0\ge\frac{b}{2}$ or $\|u^{''}\|_0\ge\frac{b}{2}$, By Lemma 2.4, we get

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}}u(t)\ge d_1\|u\|_0\ge\frac{d_1b}{2}\ \text{or}\ \min_{\frac{1}{4}\le t\le\frac{3}{4}}(-u^{''}(t))\ge\frac{d_2\xi}{c_2}\|u^{''}\|_0\ge\frac{d_2b}{2}.$$

Therefore, from (H_4) and (3.3), we have

$$\begin{aligned}
\mid (Tu)^{''}(\frac{1}{2}) \mid &=\mid \lambda \int_0^1 G_2(\frac{1}{2},\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau \mid \\
&\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G_2(\frac{1}{2},\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau \\
&\geq \lambda \times \frac{b\eta_1}{\lambda}\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(\frac{1}{2},\tau)\mathrm{d}\tau \\
&= b.
\end{aligned}$$

So,

$$\alpha(Tu) \geq \mid (Tu)^{''}(\frac{1}{2}) \mid = b.$$

By (3.2) (3.4) and (H_5), we have

$$\begin{aligned}
\|(Tu)^{'}\|_0 &= \max_{t\in[0,1]} \mid \lambda \int_t^1\int_0^1 G_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \\
&\quad - \int_0^1\int_0^1 sG_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \mid \\
&< \max_{t\in[0,1]} \mid \lambda \int_t^1\int_0^1 G_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \mid \\
&\quad + \max_{t\in[0,1]} \mid \int_0^1\int_0^1 sG_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \mid \\
&\leq \lambda \mid \int_0^1\int_0^1 (1+s)G_2(s,\tau)f_1(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \mid \\
&\leq \lambda \times \frac{\eta_2 L}{\lambda} \mid \int_0^1\int_0^1 (1+s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s \mid \\
&\leq \frac{3C_2}{2}\eta_2 L \times \mid \int_0^1 G_2(\tau,\tau)\mathrm{d}\tau \mid \\
&= \frac{3C_2 B}{2}\eta_2 L, \\
\|(Tu)^{'''}\|_0 &= \max_{t\in[0,1]} \mid -\lambda \int_0^1\int_0^1 \frac{\partial G_2(t,\tau)}{\partial t} f_1(t,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))\mathrm{d}\tau\mathrm{d}s \mid \\
&\leq \lambda \int_0^1 2\frac{\sin\sqrt{A}\tau}{\sin\sqrt{A}} |f_1(t,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau))|\mathrm{d}\tau \\
&\leq \lambda \times \frac{\eta_2 L}{\lambda} 2\int_0^1 \frac{\sin\sqrt{A}\tau}{\sin\sqrt{A}}\mathrm{d}\tau \\
&= 2F\eta_2 L.
\end{aligned}$$

Hence, for

$$\beta(Tu) = \|(Tu)^{'}\|_0 + \|(Tu)^{'''}\|_0 < \frac{3C_2B}{2}\eta_2 L + 2F\eta_2 L < (\frac{3C_2B}{2} + 2F)\eta_2 L = L.$$

Theorem 2.1 implies there is $u \in (\Omega_2 \backslash \overline{\Omega}_1)\bigcap K$ such that $u = Tu$. So, $u(t)$ is a positive solution for BVP (1.1) satisfying

$$c < \alpha(u) < b, \beta(u) < L.$$

Thus, Theorem 3.1 is completed. □

4 Conclusion

In this paper, the existence of at least one positive solutions for the fourth-order two point boundary value problem with all order derivatives is considered. By using a new cone fixed point theorem, the sufficient conditions for the existence of positive solutions of the boundary value problem are verified.

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