Modeling Predator Population assuming that the Prey follows Richards Growth Model

Purnachandra Rao Koya[1*], Ayele Taye Goshu[2] and Mohammed Yiha Dawed[3]

[1, 2, 3] School of Mathematical and Statistical Sciences, Hawassa University, Hawassa, Ethiopia

Email: *drkpraocecc@yahoo.co.in*[1], *Ayele_taye@yahoo.com*[2], *mhmmdyiha@yahoo.com*[3]

*author for correspondence

Abstract: In this paper, the predator-prey problem is studied with the assumptions that interaction of a predation leads to a little or no effect on growth of the prey population by relaxing the classical assumptions and the prey growth rate parameter is a positive valued function of time. The prey population is assumed to follow Richard's model. Then the respective predator growth model is constructed and studied by simulation. Further analysis shows that for this model, the predator population size either converges to a finite positive limit or to 0 or diverges to $+\infty$. It is shown algebraically and illustrated pictorially that there is a condition at which both the predator and prey population models converge to the same finite limit. Moreover, equilibrium points are identified and stability analysis is made and the details are included.

Keywords: Predator, Prey, Koya-Goshu, Population Growth, Richards.

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1. Introduction

Mathematics and Biology have a strong relationship. Biology produces interesting problems, Mathematics provides models to understand those problems and Biology returns to test the Mathematical models. Several interesting problems are available in biology that can be modeled mathematically. Interesting features of the biological processes such as population growths and interaction can be revealed through mathematical modeling. This paper attempts to study some theoretical and mathematical aspects of the well-known predator-prey problem. In this paper, we apply growth functions to predator –prey population growths. We consider the prey population size follows known growth model Viz., Richards and construct the corresponding growth model for the predator [1-6].A generalized mathematical model for biological growth is studied in [7-9].

The model generalizes the commonly known models such as Generalized Logistic, Particular Case of Logistic, Richards, Von Bertalanffy, Brody, Logistic, Gompertz, Generalized Weibull, Weibull, Monomolecular, Mitscherlich and many more new models and includes many more new models.

The growth models are so flexible to be useful in modeling problems.

In this paper, we apply some of these growth models to the predator-prey population dynamics. We consider that the growth of prey population size or density follows biological growth models and construct the corresponding growth models for the predator. The same authors [10] applied Logistic and Von Bertalanffy models to the prey model and derived the model equation to describe the population growth of the corresponding predator.

In the next sections, the Lotka–Volterra predator-prey model and the newly proposed approach are presented in Section 2. The case of Richards prey model is considered in Section 4. The paper ends in Section 5 with concluding remarks.

2. Prey-predator Models: Classical and Modified

The mathematical models describing two – species interactions are, in general, first order nonlinear ordinary

Corresponding Author: Purnachandra Rao Koya, Hawassa University, Hawassa, Ethiopia. Email: *drkpraocecc@yahoo.co.in*, Cell: + 251 932549318.

differential equations. The well-known model named as Lotka – Volterra is given as

$$\frac{dV}{dt} = a\,V - b\,VP \qquad (1a)$$

$$\frac{dP}{dt} = -cP + d\,VP \qquad (1b)$$

Here $a,\ b,\ c$ and $\ d$ are assumed to be positive constants.

The model (1) assumes the following: (i) In absence of predator, prey population would grow at a natural rate, (ii) In absence of prey, predator population would decline or grow at a natural rate, (iii) When both the predator and prey are present, there occur, in combination with these natural rates of growth and decline, a decline in the prey population and a growth in the predator population each at a rate proportional to the frequency of encounters between individuals of the two populations. We often assume further that the frequency of such encounters is proportional to the product of the populations.

Other assumptions available in the literature include [11-13] and are as follows: (i) The prey population finds ample food at all times, (ii) The food supply of the predator population depends entirely on the size of the prey population, (iii) The rate of change of population is proportional to its size, (iv) During the process, the environment does not change in favor of one species and the genetic adaptation is sufficiently slow and (v) Predators have limitless appetite.

It is suggested in literature [10, 11, and 14] that the assumption that the parameters a, b, c, d are constants will lead to oversimplification of the system and that be not realistic. Thus it is recommended in [14] that these parameters can be functions of time.

Some studies indicate that predation may have no effect on prey coexistence [15-18]. According to the authors, predators have very little effect on the competitive interactions.

This means the ecosystem is so that the prey and predator interactions can lead to various outcomes. The present paper considers the case when the interaction leads to a little or no effect predator population on growth of the prey population.

In this paper we assume that presence of predator population has a little or no effect on growth of the prey population, that is $b \approx 0$. Further it is assumed that the parameter a is positive valued function of time. Thus the

classical assumptions are relaxed and the proposed predator-prey model is defined as follows:

$$\frac{dx}{dt} = r(t)\,x \qquad (2a)$$

$$\frac{dy}{dt} = -v\,x + s\,xy \qquad (2b)$$

Here x denotes population size or density of prey; y denotes population size or density of predator in the two-species system. Here $r(t)$ is a relative growth rate function and is a positive valued function of its time t. The other parameters $s,\ u,\ v$ are all positive constants.

The prey equation (2a) is a first order differential equation whose solutions are studied to be biological growth models in [8]. This helps us to select the prey growth model from the large family of growth functions and solve the corresponding predator equation. This procedure provides more options for researchers and practitioners those are working in field of population dynamics.

The steps for the general approach include: (i) assume that there is prior information about the prey population that $x(t)$ is a known growth function, (ii) Assume that the impact of predator on prey population growth is negligible, (iii) Predator population declines in absence of prey (iv) The predator population grows with a rate proportional to a function of both x and y, i.e., $\dot{y} = -v\,y + s\,xy$ (v) Solve for predator's population size y.

This proposed model is also considered by the same authors [10] and they assumed that the prey population $x(t)$ follows Logistic and Von Bertalanffy growth functions and constructed corresponding growth functions to describe the predator population $y(t)$.

The idea is to consider the prey population to follow a known growth model among the family of Koya-Goshu models, and then construct the corresponding growth model for the predator population. This is can be helpful, for example, for managing the ecosystem. Here we consider Gompertz and Richards growth models for prey population and solve for the respective predator population sizes.

3. Richards prey and the corresponding predator

In this section, we assume that the growth of prey follow Richards biological growth function and construct the corresponding predator growth model.

Thus, the prey population growth is assumed to be described by Richards [19] (Richards 1959) as

$$x(t) = A\,(1 - B\,e^{-kt})^m \qquad (3)$$

Here $B = 1 - (A_0/A)^{1/m}$, $A_0 = x(0)$ is the initial prey population, A is asymptotic growth value of the prey population, and k is absolute growth rate parameter. A single point of inflection occurs when the growth reaches the weight $f(a) = [(m-1)/m]^m\,A$ at time $a = (1/k)\log\{m[1-(A_0/A)^{1/m}]\}$. Its relative growth rate function is given by $r_t = mk[\ (A/x(t))^{1/m} - 1]$. Further analyses are available in [7, 8, and 9].

Now, by substituting (3) in (2b), we determine the corresponding predator population growth function $y(t)$ as

$$\log y = -vt + sA \int (1 - B\,e^{-kt})^m \ dt \qquad (4)$$

The integral (4) can't be evaluated analytically easily. However, here the solution is suggested to consider in three different ways Viz., (1) Taylor series method, and (2) Hypergeometric function solution. These solutions are mentioned and discussed below:

3.1. Taylor series solution to the predator population

Here we consider for simplicity and as a special case that the Richards m as a positive integer. Other cases of m (that is other than positive integer case) can be considered to evaluate the integral (4). Further, for m any positive integer the integral in (4) can be evaluated using Taylor series expansion of the integrand and the predator function $y(t)$ can be expressed as

$$y(t) = Y_0\,e^{(As-v)t}\,.\,e^{-\,(As/k)\,\sum_{i=0}^{m-1} P(m)} \qquad (5)$$

Or equivalently

$$y(t) = Y_0 \left[\frac{1-(A_0/A)^{1/m}}{1-(x/A)^{1/m}}\right]^{(As-v)t} .\,e^{-\,(As/k)\sum_{i=0}^{m-1} P(m)} \qquad (6)$$

Here in the equations (5), (6) and (7) and in the text what follows we denote $P(m)$ to represent the following expression:

$$P(m) = (1/(m-i))\left[(x/A)^{(m-i)/m} - (A_0/A)^{(m-i)/m}\right]$$

Here $Y_0 = y(0)$ is the initial predator population size. It is interesting to observe that both population growth functions of prey and predator are related as (7):

$$y = y_0 \left[\frac{1-(x/A)^{1/m}}{1-(A_0/A)^{1/m}}\right]^{(v-As)/k} e^{-\,(As/k)\sum_{i=0}^{m-1} P(m)} \qquad (7)$$

Further, it is to be mentioned here that all the three equations (5), (6) and (7) represent the same, i.e., expression for the predator population growth function $y(t)$, except that they are expressed differently as these versions of expression are useful to use directly in different occasions to draw some useful conclusions. Detailed derivations of the equations (5), (6) and (7) are given in Appendix.

It is found that the predator model either declines and converges to 0 or converges to a finite positive limit or diverges to $+\infty$ depending on the values set to the birth parameter s and death parameter v of the predator. The three cases are presented below:

Case I ($As = v$): In this case, the predator growth function given by (6) takes the form

$$y(t) = Y_0.\,e^{-\,(As/k)\sum_{i=0}^{m-1} P(m)} \qquad (8)$$

The limit of the predator growth function given in (8), as time goes to infinity, is computed to be

$$\lim_{t\to\infty} y(t) = Y_0 \,.\, e^{-\left(\frac{As}{k}\right)\sum_{i=0}^{m-1}\left(\frac{1}{m-i}\right)\left\{1-\left(\frac{A_0}{A}\right)^{\frac{m-i}{m}}\right\}} \qquad (9)$$

It can be interpreted, considering the equations (3) and (9), that the predator population decays with an exponential rate of (As/k) and converges to a lower asymptote of $y_0\,e^{-\frac{As}{k}\sum_{i=0}^{m-1}\left(\frac{1}{m-i}\right)\left\{1-\left(\frac{A_0}{A}\right)^{\frac{m-i}{m}}\right\}}$ while the prey population grows following Richards model and reaches the upper asymptote A. The situation at which the prey population upper asymptote and the predator population lower asymptote coincide at A and continue to maintain the same population sizes is found to be:

$$s = (k/A)\frac{\log y_0 - \log A}{\sum_{i=0}^{m-1}(1/(m-i))\left[1-(A_0/A)^{(m-i)/m}\right]} \qquad (10)$$

The equation (10) is arrived at by equating the population sizes of both the prey (3) and Predator (8). Also Figure 1(a) illustrates the occasion where both the populations converge to a same asymptote. When initial size of the predator is smaller than A, then there is an increment of its growth to converge to the same size of prey. See Figure 4(c) − 4(d).

Case II ($As < v$): In this case, the predator population decays and eventually dies down to 0, while the prey population follows Richards, as assumed, and reaches an upper asymptote A (see Figure 1b).

Case III ($As > v$): In this case, the predator population grows higher and higher and eventually diverges to $+\infty$, while the prey population grows according to the Richards curve with an upper asymptote of A (see Figure 1c)).The predator population growth has minimum value at time point t_{min} that is a function of the parameters. It is given by

$$t_{min} = \left(\tfrac{1}{k}\right)\left\{log\left(\frac{1-\left(\frac{A_0}{A}\right)^{1/m}}{1-\left(\frac{v}{As}\right)^{1/3}}\right)\right\} \qquad (11)$$

Then the values of prey and predator populations are respectively

$$x(t_{min}) = v/s \qquad (12)$$

and

$$y(t_{min}) =$$
$$y_0 \left[\frac{1-\left(\frac{A_0}{A}\right)^{\frac{1}{m}}}{1-\left(\frac{v}{As}\right)^{\frac{1}{m}}}\right]^{\frac{A\,s-v}{k}} e^{-\frac{A\,s}{k}\sum_{i=0}^{m-1}\left(\frac{1}{m-i}\right)\left[\frac{v^{\frac{m-i}{m}}-(A_0\,s)^{\frac{m-i}{m}}}{(A\,s)^{\frac{m-i}{m}}}\right]} \qquad (13)$$

Note that Brody and Von Bertalanffy functions are special case of Richards with $m = 1$ & 3 [20-21].

3.2. Hypergeometric function solution to the predator population

For any real number m , the predator population function $y(t)$ in equation (4) can be expressed in terms of hypergeometric function as follows:

$$y(t) = e^{-vt+C} e^{-As\left[\frac{\left(1-B\,e^{-kt}\right)^m \;_2F_1\left(-m,-m;1-m;\frac{e^{kt}}{B}\right)}{k\,m\left(1-(e^{kt}/B)\right)^m}\right]} \qquad (14)$$

Here 2F1 is a Hypergeometric function and is defined as follows [22]: Here it is appropriate to introduce briefly the definition and description of the hypergeometric function. A hypergeometric function is the sum of a hypergeometric series, which is defined as follows:

Definition-1 A series $\sum c_n$ is called hypergeometric if the ratio $\frac{c_{n+1}}{c_n}$ is a rational function of n . By factorization this means that, for all $n = 0, 1, 2, \ldots..$

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)\,(n+a_2)\ldots(n+a_p)\,z}{(n+b_1)\,(n+b_2)\ldots(n+b_p)\,(n+1)} \qquad (15)$$

Definition-2 The hypergeometric function $pFq\left(a_1, a_2, \ldots, a_p;\; b_1,\; b_2, \ldots, b_q\;;z\right)$ is defined by means of a hypergeometric series as

$$pFq\begin{pmatrix}a_1, a_2, \ldots, a_p \\ b_1,\; b_2, \ldots, b_q\end{pmatrix};z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n\,(a_2)_n\ldots(a_p)_n}{(b_1)_n\,(b_2)_n\ldots(b_q)_n}\cdot\frac{z^n}{n!}$$
(16)

Sometimes the most general hypergeometric function pFq is called a generalized hypergeometric function. Then the words "hypergeometric function" refer to the special case

$$2F1(a,b;c;z) = 2F1\begin{pmatrix}a,\; b \\ c\end{pmatrix};z\right) = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!}$$
(17)

Note that if $a = -N$ with $N \in \{0, 1, 2, \ldots\}$, then we have

$$(a)_n = (-N)_n = (-N)\left(-N+1\right)\left(-N+2\right)\ldots\left(-N+n-1\right) = 0 \qquad (18)$$

Here $n = N+1,\ N+2,\ N+3, \ldots$. Hence, for all $N \in \{0, 1, 2, \ldots\}$ we will have

$$2F1\begin{pmatrix}-N, b \\ c\end{pmatrix};z\right) = \sum_{n=0}^{\infty}\frac{(-N)_n\,(b)_n}{(c)_n}\cdot\frac{z^n}{n!} \qquad (19)$$

Otherwise, the series (19) converges for $|z| < 1$ and also converges for $|z| > 1$ whenever $Re\,(c - a - b) > 0$.

4. Simulation Study

The simulation study is designed in by varying the model parameters: A_0 , A, k for prey and y_0, s, v for predator population; and the Richards model with three cases.

Model: Richards

Prey's parameters: $A = 100$, $A_0 = 20$, $k = 0.1 \ or \ 0.01$

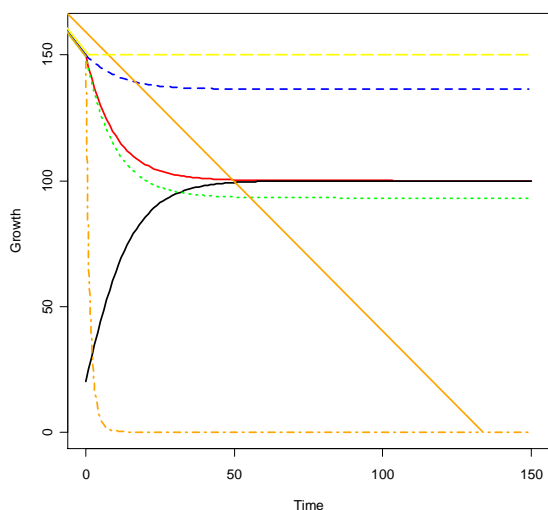**Predator's parameters:** $y_0 = 1.5A \ or \ 0.5A \ or \ 0.5A_0$

Cases Case I: $(A\,s = v)$ Case II: $(A\,s < v)$ Case III: $(A\,s > v)$

Case I: $s = 0.00005$ & $v = 0.005$; $s = 0.0001$ & $v = 0.01$; $s = 0.01$ & $v = 1.0$; $s = 10E - 10$ & $v = 10E - 8$
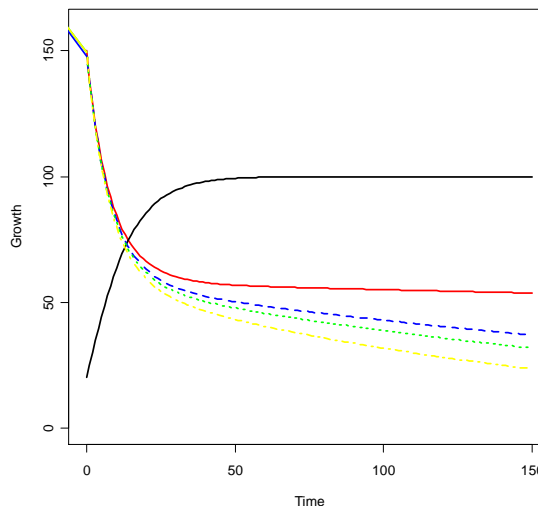
Case II: $s = 0.001$ & $v = 0.105$; $s = 0.001$ & $v = 0.130$; $s = 0.001$ & $v = 0.140$; $s = 0.001$ & $v = 0.160$

**Case III:** $s = 0.001$ & $v = 0.08$; $s = 0.001$ & $v = 0.07$; $s = 0.001$ & $v = 0.06$; $s = 0.001$ & $v = 0.04$

The specifications are as follows: **k_seq** $= 0.01, 0.04, 0.07$, and 0.1.

Figure (1b) Plots of predator population dynamics with prey population growth following Richards's model representing the Case II.

Figure (1a) Plots of predator population dynamics with prey population growth following Richards's model representing the Case I.
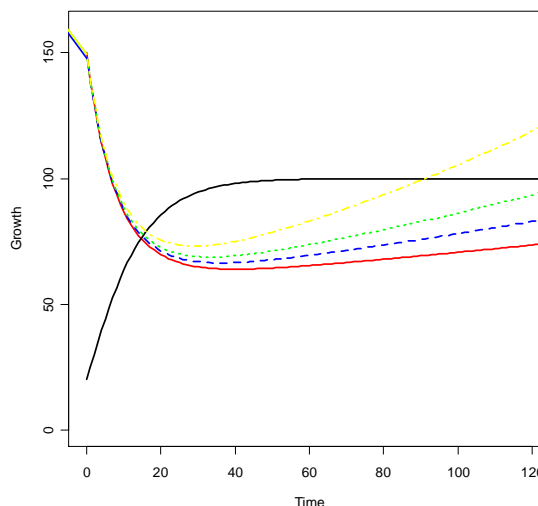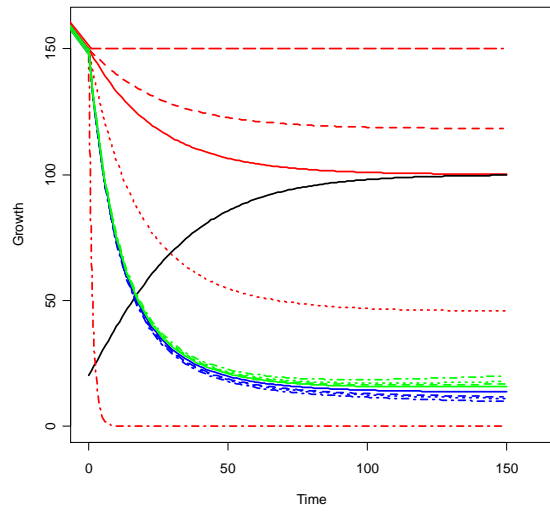
Figure (1c) Plots of predator population dynamics with prey population growth following Richards's model representing the Case III.

In Case III, see Figure 1c), the death/birth of predator is less than A. As shown by simulations plotted in Figure 1 &2, the predator population declines for some time and then increases to infinity. The minimal point at which the curve turns or gets minimum value is found to be: $t_{min} = \frac{1}{k} log \left(\frac{\frac{A}{A_0} - 1}{\frac{As}{v} - 1} \right)$. Then the values of prey and predator populations are $x(t_{min}) = v/s$ and $y(t_{min}) = y_0 \left(\frac{A - A_0}{As - v} \right)^{\frac{As - v}{k}} \left(\frac{A_0}{v} \right)^{\frac{v}{k}} (s)^{\frac{As}{k}}$, respectively.
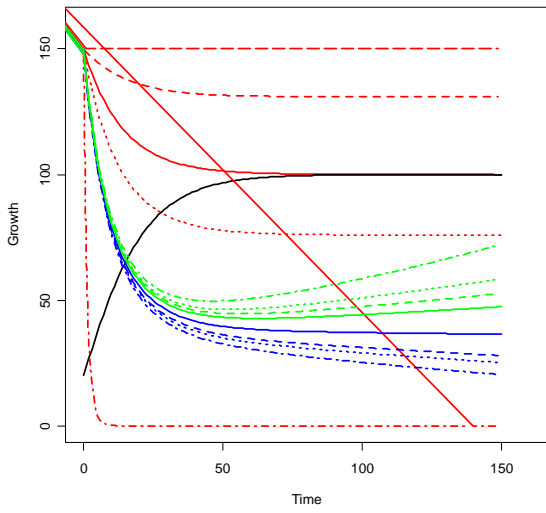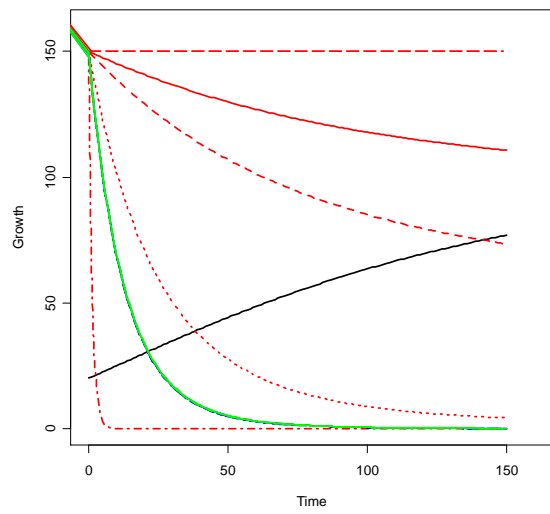
Figure (2a) Plots of all three cases for the predator population dynamics with prey population growth following Richards model where $k = 0.1$, $Y_0 = 1.5A$

Figure (2c) Plots of all three cases for the predator population dynamics with prey population growth following Richards model with $k = 0.01$, $Y_0 = 1.5A$
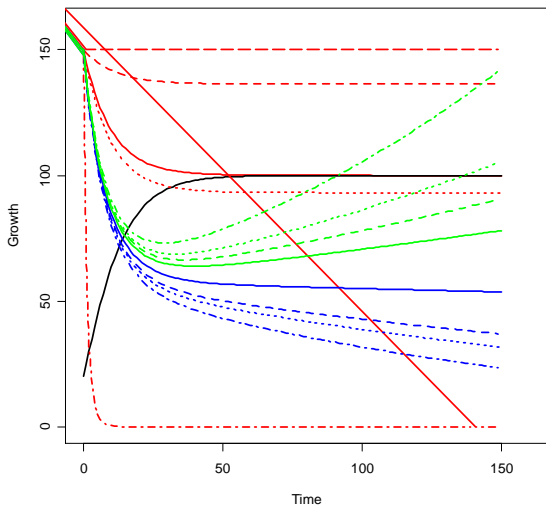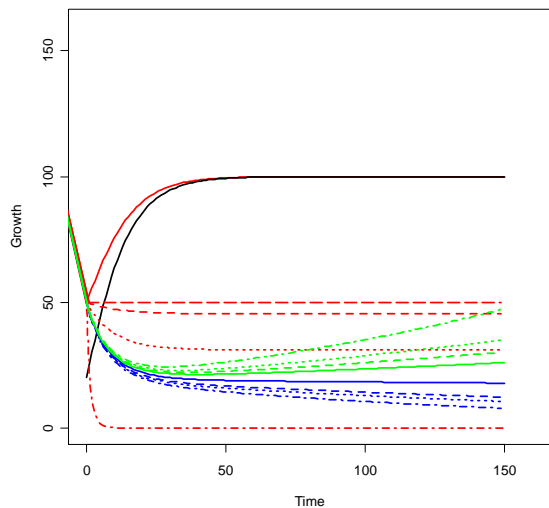

Figure (2b) Plots of all three cases for the predator population dynamics with prey population growth following Richards model with $k = 0.01$, $Y_0 = 1.5A$

Figure (2d) Plots of all three cases for the predator population dynamics with prey population growth following Richards model with $k = 0.1$, $Y_0 = 0.5A$
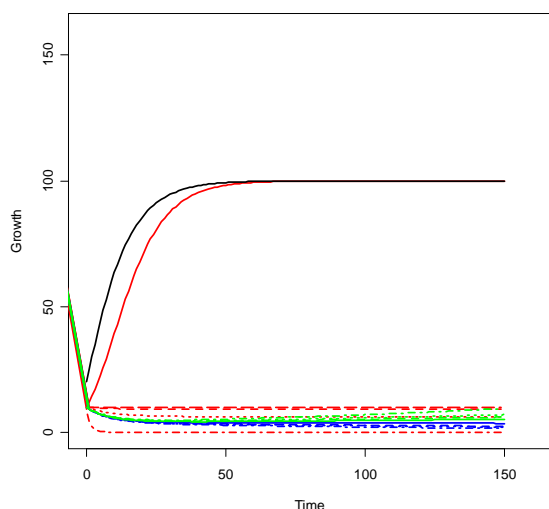
Figure (2e) Plots of all three cases for the predator population dynamics with prey population growth following Richards model with $k = 0.1$, $Y_0 = 0.5 A_0$

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Figure (2f) Plots of all three cases for the predator population dynamics with prey population growth following Richards model with $k = 0.1$, $Y_0 = 0.5 A_0$

In summary, it is observed that the predator population either converges to a finite limit or converges to 0 or diverges to $+\infty$ depending on the selection of the parameters. We show by simulation study that for a particular value of the birth parameter s, the population sizes of both the prey and predator will converge to the same asymptote.

5. Analysis of phase diagram and Equilibrium points

The newly proposed predator – prey model (2) in its full form can be expressed, in case of Richards prey, as the system of equations $(dx/dt) = mk\left[(A/x)^{1/m} - 1\right]x$ and $(dy/dt) = -vy + sxy$. The two equilibrium points of this system are found to be $(x_1^*, y_1^*) = (0,0)$ and $(x_2^*, y_2^*) = (A, 0)$ since at both these points the necessary and sufficient conditions $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ are satisfied. Also the Jacobian matrix of the system of equations is $J(x, y) = \begin{bmatrix} a_{11} & 0 \\ sy & -v + sx \end{bmatrix}$ where $a_{11} = mk\left[(1 - (1/m))(A/x)^{1/m}\right]$. We now analyze the nature of the equilibrium points below and the summery is tabulated in Table 1:

Nature of the equilibrium point $(x_1^*, y_1^*) = (0,0)$: The Jacobean matrix at this point takes the form $J(x_1^*, y_1^*) = J(0,0) = \begin{bmatrix} b_{11} & 0 \\ 0 & -v \end{bmatrix}$ where $b_{11} = mk\left[(1 - (1/m))(A/A_0)^{1/m} - 1\right]$ and the corresponding eigenvalues are $\lambda_{11}^* = b_{11}$ and $\lambda_{12}^* = -v$. Recall that the parameters k, A, A_0 and v are all positive quantities and thus here arises two cases.

Case I. $m < 0$ and $\left(1 - (1/m)\right)^m < (A/A_0)$: In this case, both the eigenvalues λ_{11}^* and λ_{12}^* are real and negative and hence the equilibrium point is stable.

Case II. $m > 0$ and $\left(1 - (1/m)\right)^m < (A/A_0)$: In this case, both the eigenvalue λ_{11}^* and λ_{12}^* are real but opposite in sign and hence the equilibrium point is unstable.

Nature of the equilibrium point $(x_2^*, y_2^*) = (A, 0)$: The Jacobean matrix at this point takes the form $J(x_2^*, y_2^*) = J(A, 0) = \begin{pmatrix} -k & 0 \\ 0 & -v + sA \end{pmatrix}$ and the corresponding eigenvalues are $\lambda_{21}^* = mk - m - k$ and $\lambda_{22}^* = -v + sA$. Recall that the parameters k, v, s and A are all positive quantities and thus here arises nine cases.

Case I. $k = (m/(m - 1))$ and $(As = v)$: In this case both the eigenvalues λ_{21}^* and λ_{22}^* are zero and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is stable.

Case II. $k = (m/(m - 1))$ and $(As < v)$: In this case the eigenvalue λ_{21}^* is zero while λ_{22}^* is negative and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is stable. Here we note that, $X(t) = c_1 V_1(t) e^{0t} + c_2 V_2(t) e^{\lambda_{22}^* t}$ where $V_1(t)$ and $V_2(t)$ are eigenvectors corresponding to the eigenvalues $\lambda_{21}^* = 0$ and $\lambda_{22}^* = Negative\ value$. Since λ_{22}^* is a negative $X(t) \to c_1 V_1(t) e^{0t}$ as $t \to \infty$.

Case III. $k = (m/(m - 1))$ and $(As > v)$: In this case the eigenvalue λ_{21}^* is zero while λ_{22}^* is positive and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is unstable.

Case IV. $k < (m/(m - 1))$ and $(As = v)$: In this case the eigenvalue λ_{21}^* is negative while λ_{22}^* is zero and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is stable.

Case V. $k < (m/(m - 1))$ and $(As < v)$: In this case both the eigenvalues λ_{21}^* and λ_{22}^* are negative and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is stable.

Case VI. $k < (m/(m - 1))$ and $(As > v)$: In this case the eigenvalue λ_{21}^* is negative while λ_{22}^* is

positive and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is unstable.

Case VII. $k > (m/(m-1))$ **and** $(As = v)$: In this case the eigenvalue λ_{21}^* is positive while λ_{22}^* is zero and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is unstable.

Case VIII. $k > (m/(m-1))$ **and** $(As < v)$: In this the eigenvalue λ_{21}^* is positive while λ_{22}^* is negative and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is unstable.

Case IX. $k > (m/(m-1))$ **and** $(As > v)$: In this case both the eigenvalues λ_{21}^* and λ_{22}^* are positive and hence the equilibrium point $(x_2^*, y_2^*) = (A, 0)$ is unstable.

Table 1. Summary of Stability of the equilibrium points – the case of Richards prey

Equilibrium Point	Eigenvalues	Cases	Sign of the eigenvalues	Nature of the point
(x_1^*, y_1^*) $= (0,0)$	$\lambda_{11}^* = k[(m-1)(A/A_0)^{1/m} - m]$ $\lambda_{12}^* = -v$	**Case I.** $m < 0$ and $(1 - (1/m))^m < (A/A_0)$	Both λ_{11}^* and λ_{12}^* are real and negative in sign	Stable
		Case II. $m > 0$ and $(1 - (1/m))^m < (A/A_0)$	λ_{11}^* and λ_{12}^* are real but opposite in sign	Unstable
(x_2^*, y_2^*) $= (A, 0)$	$\lambda_{21}^* = mk - m - k$ $\lambda_{22}^* = -v + sA$	**Case I.** $k = (m/(m-1))$ and $(As = v)$	Both λ_{21}^* and λ_{22}^* are zero	Stable
		Case II. $k = (m/(m-1))$ and $(As < v)$	λ_{21}^* is zero while λ_{22}^* is negative	Stable
		Case III. $k = (m/(m-1))$ and $(As > v)$	λ_{21}^* is zero while λ_{22}^* is positive	Unstable
		Case IV. $k < (m/(m-1))$ and $(As = v)$	λ_{21}^* is negative while λ_{22}^* is zero	Stable
		Case V. $k < (m/(m-1))$ and $(As < v)$	Both λ_{21}^* and λ_{22}^* are negative	Stable
		Case VI. $k < (m/(m-1))$ and $(As > v)$	λ_{21}^* is negative while λ_{22}^* is positive	Unstable
		Case VII. $k > (m/(m-1))$ and $(As = v)$	λ_{21}^* is positive and λ_{22}^* is zero	Unstable
		Case VIII. $k > (m/(m-1))$ and $(As < v)$	λ_{21}^* is positive and λ_{22}^* is negative	Unstable
		Case IX. $k > (m/(m-1))$ and $(As > v)$	Both λ_{21}^* and λ_{22}^* are positive	Unstable

6. Conclusions

In this paper, we studied the some theoretical mathematical aspects of the known predator-prey problem by relaxing the classical assumptions as: the interaction of the predator-prey populations leads to a little or no effect on growth of the prey population and that the prey growth rate parameter is a positive valued function of time. We have constructed predator growth models considering that the prey follows biological growth model viz., Richards's

model. It is observed that the predator population either converges to a finite limit or 0 or diverges to $+\infty$ irrespective of the fact that the prey population continuously g

rows and eventually converges to upper asymptote. There is a situation at which both prey and predator populations converge to the same amount, irrespective of their initial population sizes. There is also a situation where the predator population declines for some time and then start

to increase and diverge to infinity. In general, the analytic and simulation studies have revealed some insights to the problem addressed in this paper so that it can be applied to the real-world situations. The authors are working on applying other Biological models to be followed by prey population and trying to construct the corresponding growth models of predator populations and the results will be published soon.

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Appendix
Derivation of predator model given that prey follows Richards growth model

Consider the predator equation

$\frac{dy}{dt} = -vy + s\,xy$

$\Rightarrow \frac{dy}{y} = [-v + s\,x]dt$

$\Rightarrow \log y = -vt + s \int x\,dt$ \qquad (i)

We now substitute the Richards function for the prey growth. That is, $x(t) = A\,(1 - B\,e^{-kt})^m$ and $B = 1 - (A_0/A)^{1/m}$. Thus, (i) take the form

$\log y = -vt + As \int (1 - B\,e^{-kt})^m dt$ \qquad (ii)

Now, put $w = 1 - B\,e^{-kt}$ $\Rightarrow$ $dw = Bk\,e^{-kt} = k(1 - w)dt \Rightarrow -\frac{dw}{k(w-1)} = dt$ in (ii), to get

$\log y = -vt - \frac{As}{k} \int \frac{w^m}{w-1} dt$ \qquad (iii)

For m positive integer and applying long division, the expansion for $[w^m/(w - 1)]$ reduces to $[w^m/(w - 1)] = \left[\sum_{i=0}^{m-1} w^i + \frac{1}{w-1} \right]$.

Hence, (iii) reduces to

$$log\ y = -vt - \frac{As}{k}\int\left[\sum_{i=0}^{m-1} w^i + \frac{1}{w-1}\right]dt$$

$$= -vt - \frac{As}{k}\left[\sum_{i=1}^{m}\frac{w^i}{i} + \log(w-1)\right] + C \qquad (iv)$$

Now, put in (iv) the initial condition $y = Y_0$, $w = 1 - B = (A_0/A)^{1/m}$ at $t = 0$ to fix the integral constant C. Thus, $\quad log\ y_0 = -(As/k)\left[\sum_{i=1}^{m}(1/i)(A_0/A)^{i/m} + \log\big((A_0/A)^{1/m} - 1\big)\right] + C$ Now, (vi) takes the form,

$$log(y/y_0) = -vt - I_1 - I_2 \qquad (v)$$

Where $\quad I_1 = (As/k)\sum_{i=1}^{m}\left[\big(w^i - (A_0/A)^{i/m}\big)/i\right]$ and $I_2 = (As/k)\log\left[(w-1)/\big((A_0/A)^{1/m} - 1\big)\right]$. But recall the simplifications that $\log\left[(w-1)/\big((A_0/A)^{1/m} - 1\big)\right] = -kt$ and also $w = (x/A)^{1/m}$. Thus, (v) takes the form

$$log(y/y_0) = -vt - I_1 + sAt$$

$$y = y_0\ e^{(sA-v)t}\ e^{-I_1} \qquad (vi)$$

But, $x = A(1 - B\ e^{-kt})^m$

$$\Rightarrow e^{-kt} = \left[\big(1 - (x/A)^{1/m}\big)/B\right]$$

$$= \left[\big(1 - (x/A)^{1/m}\big)/\big(1 - (A_0/A)^{1/m}\big)\right]$$

$$\Rightarrow e^{-(v-sA)t} =$$

$$\left[\big(1 - (x/A)^{1/m}\big)/\big(1 - (A_0/A)^{1/m}\big)\right]^{(v-As)/k}$$

Thus, (vi) takes the required form as

$$y = y_0\left[\frac{1 - \left(\frac{x}{A}\right)^{\frac{1}{m}}}{1 - \left(\frac{A_0}{A}\right)^{\frac{1}{m}}}\right]^{\frac{v-As}{k}}\ .$$

$$e^{-\frac{As}{k}\left[\left(\frac{\left(\frac{x}{A}\right)-\left(\frac{A_0}{A}\right)}{m}\right)+\left(\frac{\left(\frac{x}{A}\right)^{\frac{m-1}{m}}-\left(\frac{A_0}{A}\right)^{\frac{m-1}{m}}}{m-1}\right)+\ ...+\left(\left(\frac{x}{A}\right)^{\frac{1}{m}}-\left(\frac{A_0}{A}\right)^{\frac{1}{m}}\right)\right]}$$

(vii)

Author Profile

Dr. Purnachandra Rao Koya received M. Sc. And Ph. D. degrees in Mathematics from Bhavnagar University, Gujrat, India in 1990 and 1998 respectively. He served (i) Saurashtra University 91993-'95) (ii) Nagabhairava Educational Academy, Gundlapally (1997-'98) (iii) Bapatla Engineering College (1998-2001) (iv) Chirala Engineering College (2001-'11) and (v) Mother Teresa Institute of Science and Technology, Sathupally (2011-'12). He has been at Hawassa University since October 2012.