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New Possible Formulas for Irregular Arc Length Determination

Kamel, Ayman [1] and Sedky, A [2]
[1] Microbiology Department, Faculty of Agriculture, Al-Azhar University,
Nasr City, Cairo, Egypt
Aymankamel91@yahoo.com

[2] Physics department, Faculty of Science, Assiut University,
Assiut 71515, Egypt
sedky1960@yahoo.com

Abstract: We report here a new possible two formulas for obtaining the length of irregular arc **ℓ** in terms of their base **b** and height **h**. The first formula is obtained by applying the law of cosines and intersecting chord theory and **ℓ** is given by $\frac{2.18\times10^{-3}(b^2+4h^2)}{h}\cos^{-1}(1-\frac{32h^2b^2}{b^4+8b^2h^2+16h^4})$. While the other is obtained by applying Pythagorean theory and **ℓ** is given by $\sqrt{4h^2+b^2}+Kh$ with an error of $\frac{K(2h-b)}{2}$, where K is a constant and equal 0.313165528 and can be used only in case of 2h > b. Finally, the earth circumference is calculated by using the two formulas and their values are 39910.0252 Km and 39999.5504 Km, which is consistent with the reported elsewhere (39992.1984 km).

Keywords: New formulas, Arc length, Irregular curve, Constant and Error.

1. Introduction

The arc length ℓ has been considered for a portion of the circumference of a regular circle and it is given by (ℓ = θr in radian, ℓ = θ πr/180 in degree), where θ is the central angle and r is the radius. But a rectification is normally taken as the length of an irregular arc curve. Archimedes had pioneered a way for finding the area beneath a curve as an irregular arc, but few researches believed that it is impossible for the curves to have definite length, as well as straight lines. So, the researchers began to inscribe polygons within the curves and they are able to compute the length of the sides for a somewhat accurate measurement of the length. By using more segments, and by decreasing the length of each segment, they could obtain more accurate approximation [1,2]. In particular, by inscribing a polygon of many sides in a circle, they find approximate values of π.

Although, a lot of methods have been used for obtaining the length of some specific irregular curves, the advent of infinitesimal calculus led to general integral formulas that provide a closed form solutions in most of cases [1]. However, a rectification has been obtained by geometrical methods of several transcendental curves such as logarithmic spiral, cycloid, and catenaries [1-5]. After that, Williams credited Neale's discovery for the first rectification of a nontrivial algebraic curve which is called semicubical parabola [2]. During this period, Van Heuraet obtained the length of irregular arc (a semicubical parabola) by using an integral form in terms of the area under a curve [3]. Similar results are obtained Fermat by applying a general theory on the curved lines [4]. Unfortunately a specific integration is necessary for the arc length as follows [5-8]; $\ell=\int_{t_1}^{t_2}\sqrt{1+(\frac{dy}{dx})^2}$ for Cartesian coordinates,

Corresponding Author: Ayman kamel, Al-Azher University, aymankamel91@yahoo.com, +20107249183;
A.Sedky, Assiut University, sedky1960@yahoo.com, +201157803325

$$\ell = \int_{t_1}^{t_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$ for polar coordinates,

$$\ell = \int_{t_1}^{t_2} \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$ for cylindrical coordinates and

$$\ell = \int_{t_1}^{t_2} \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2 \sin^2 \varphi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2}$$

for spherical coordinates.

As discussed above, we could not find a simple formula for obtaining the length of irregular arc. Therefore, we report here new possible formulas for obtaining the length of irregular arc in terms of their base and height. The first method is obtained by applying the law of cosines and intersecting chord theory, while the other is obtained by applying Pythagorean theory.

2. Mathematical Results

2.1. First Method

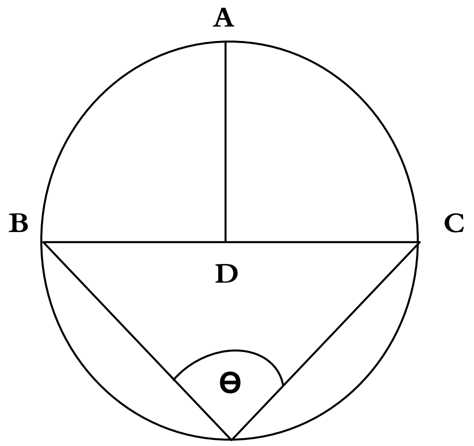

Figure 1: Irregular curve of base b and height h

Let us consider irregular arc curve of length ℓ = BAC, base b = BC and height h = AD as shown in Figure 1. This arc should be started with minimum value and then increased up to optimum value, and it descends again to the minimum. Assuming BC = 2AD = 2R = b, AD = h, and applying the law of cosines and intersecting chord theorem [9-11], we found that;

$$b^2 = r^2 + r^2 - 2\cos\theta \times r \times r$$

$$b^2 = 2r^2 - 2\cos\theta \times r^2 \qquad (1)$$

$$b^2 = 2r^2(1 - \cos\theta)$$

$$\cos\theta = \left(1 - \frac{b^2}{2r^2}\right) \qquad (2)$$

$$\theta = \cos^{-1}\left(1 - \frac{b^2}{2r^2}\right)$$

Then $r = \frac{(b^2 + 4h^2)}{8h}$ but $\ell = \frac{(\pi\theta r)}{180}$ therefore

$$\ell = \frac{(\pi\theta r)}{180} = \frac{\pi\theta(b^2 + 4h^2)}{8h \times 180} \qquad (3)$$

Substituting about r and Θ from equation (2) in equation (3), we find that:

$$\ell = \left[\frac{\pi(b^2 + 4h^2)}{180 \times 8h}\right]\cos^{-1}\left(1 - \frac{b^2}{2\left(\frac{b^2 + 4h^2}{8h}\right)^2}\right)$$

$$\ell = \frac{2.18 \times 10^{-3}(b^2 + 4h^2)}{h}\cos^{-1}\left(1 - \frac{32h^2b^2}{b^4 + 8b^2h^2 + 16h^4}\right) \qquad (4)$$

By using equation (4), one can calculate the length of irregular arc curve in terms of their base and height.

2.2. Second Method

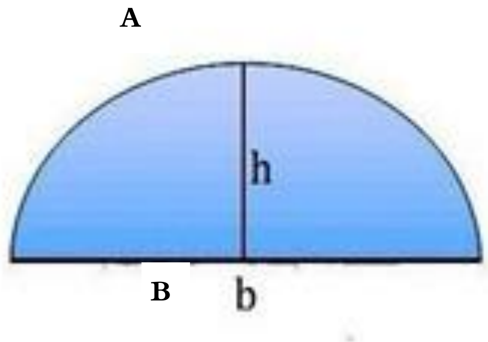

Figure 2: The regular arc length with base and height

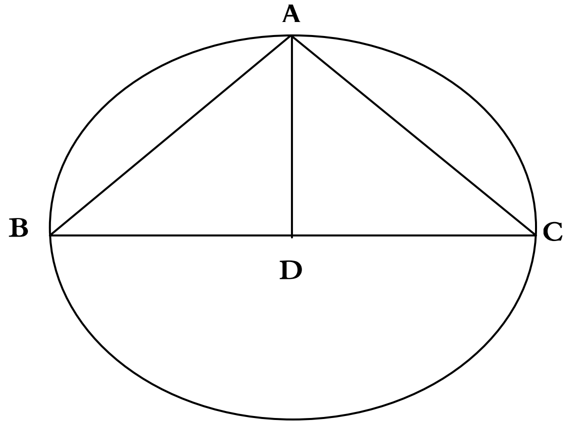

Figure 3: The irregular arc length with base and height

As shown in Figure 3, and according to Pythagorean theory[12,13], $AB = \sqrt{(AD)^2 + (BD)^2}$,

$$AC = \sqrt{(AD)^2 + (CD)^2} \qquad (5)$$

By assuming that $BD = DC = \frac{BC}{2}$, therefore,

$$AB + AC = \sqrt{4(AD)^2 + (BC)^2} \qquad (6)$$

By applying equation (6), the length of irregular arc ABC is given in Table 1. It is clear from Table 1 that the difference between (ABC Arc) - (AB + AC) is given by Kr, where K is constant and equal 0.313165528 or (π - 2√2).

Table 1: Arc length, lengths of triangle ABC and the difference between them

r = AD = BC/2	**1 cm**	**2 cm**	**3 cm**	**4 cm**
Arc length ABC	3.14 cm	6.28 cm	9.42 cm	12.57 cm
(AB + AC)	2.83 cm	5.66 cm	8.49 cm	11.31 cm
(ABC Arc) - (AB + AC)	0.31 cm	0.62 cm	0.93 cm	1.26 cm

Based on the above calculations, the irregular arc length can be written as;

$$ABC = \sqrt{4(AD)^2 + (BC)^2} + K(AD)$$

$$\ell = \sqrt{4h^2 + b^2} + Kh \qquad (7)$$

It is found from equation (7) that the error Kh is gradually decreases to zero as the diameter b increases up to 2h. This is represented by a straight line, see Figure 4.

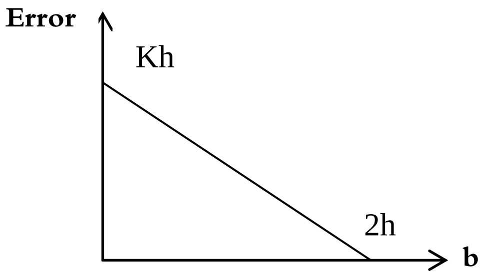

Figure 4: The error versus the base b of the arc

Based on the above Figure, the slope can be given by the difference between any two points which helps us for obtaining the error as follows;

Consider the three points are (x, y), (x_1,y1) and (x_2,y_2) as (x,y), (0,Kh) and (2h,0). Therefore, the slope must be constant, and it is given by the following;

$$\frac{(hK - 0)}{(0 - 2h)} = \frac{(y - hK)}{(x - 0)}$$

$$\frac{(hK)}{(-2h)} = \frac{(y - hK)}{x} \qquad (8)$$

Equation (8) can be rewritten as;

$2y = 2hK - Kx$, where x = b and y is the error.

The error is given by,

$$y = error = \frac{K(2h - b)}{2} \qquad (9)$$

Then, equation (7) can be rewritten as;

$$\ell = \sqrt{4h^2 + b^2} + Kh - \frac{K(2h - b)}{2} \qquad (10)$$

It is clear that the error is tends to zero when b = 2h as shown in Figure 4. Therefore, the error can be used only in case of 2h > b.

To clarify the above results, we are used the formulas for calculating the earth circumference which is 24850 miles = 39992.1984 km as reported (1mile = 1.609344 Km) [14]. However, we takes the base b = D_1= 12756.26 Km, where D_1 is the diameter measured along the equator, and h = D_2/2 = 6357 Km, where D_2 is the diameter measured across the poles. The values of the earth circumference are 39910.0252 Km by using the first formula and 39999.5504 Km by using the other. For more sure to our formulas, we are calculated the earth circumference in terms of the following Ramanujan approximation for the circumference of ellipse;

$$\ell = (a + b)(1 + \frac{3(\frac{a-b}{a+b})^2}{10 + \sqrt{4 - 3(\frac{a-b}{a+b})^2}}$$ with a = D_1/2 and b = D_2/2. It is found that the earth circumference is 40024 Km.

3. Conclusion

A possible two formulas for obtaining the length of irregular arc **ℓ** in terms of their base **b** and height **h** are investigated. The length of irregular arc is given by $\frac{2.18 \times 10^{-3}(b^2 + 4h^2)}{h} \cos^{-1}(1 - \frac{32h^2b^2}{b^4 + 16h^4 + 8h^2b^2})$ for the first formula, and by $\sqrt{4h^2 + b^2} + Kh$ with an error of $\frac{K(2h - b)}{2}$ for the other; where **K** is constant and equal 0.313165528, and can be used only in case of **2h > b**. By using the present formulas, the earth circumference values are 39910.0252 Km and 39999.5504 Km, which is consists with the reported elsewhere (39992.1984 km).

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