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On The Fekete-Szegö Problem for Generalized Class $M_{\alpha,\gamma}(\beta)$ Defined By Differential Operator

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Abstract: In this study the classical Fekete-Szegö problem was investigated. Given $f(z)=z+a_2z^2+a_3z^3+\ldots$ to be an analytic standartly normalized function in the open unit disk $U=\{z\in\mathrm{C}:|z|<1\}$. For $|a_3-\mu a_2^2|$, a sharp maximum value is provided through the classes of $\overline{S}^{*}_{\alpha,\gamma}(\beta)$ order β and type α under the condition of $\mu\geq 1$.

Diferansiyel Operatör ile Tanımlanmış Genelleştirilmiş $M_{\alpha,\gamma}(\beta)$ Sınıfı için Fekete-Szegö Problemi

Anahtar Kelimeler
Yalınkat fonksiyonlar,
Analitik,
Yıldızıl,
Konveks,
Fekete Szegö problemi

Özet: Bu çalışmada, Fekete-Szegö problemi çalışılmıştır. $f(z)=z+a_2z^2+a_3z^3+\ldots$ $U=\{z\in\mathrm{C}:|z|<1\}$, açık birim diskinde normalize edilmiş analitik fonksiyonların bir sınıfı olsun. $\mu\geq 1$ koşulu altında α tipli β mertebeli $\overline{S}^{*}_{\alpha,\gamma}(\beta)$ sınıfı ile ilgili, $|a_3-\mu a_2^2|$ için kesin maksimum değeri elde edilmiştir.

1. Introduction, Preliminaries and Definition

Let A indicate the family of analytic functions in the unit disk $U=\{z\in\mathrm{C}:|z|<1\}$ as given below,

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

In addition, it is well known that the class of functions which are univalent in $U=\{z\in\mathrm{C}:|z|<1\}$ is shown by S. Strongly starlike functions of order β and type α is defined over the class A of all analytic functions $f(z)$ in the form (1). Such functions are denoted by $\overline{S}^{*}_{\alpha,\gamma}(\beta)$,if they fulfill,

$$\left|\arg\left(\frac{\gamma I^{n-2}f(z)+(1-\gamma)I^{n-1}f(z)}{\gamma I^{n-1}f(z)+(1-\gamma)I^{n}f(z)}-\alpha\right)\right|<\frac{\pi}{2}\beta \tag{2}$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0\leq\beta<1)$ and $z\in U$. Over the class of S which is being analytic univalent functions, upper value of $|a_3-\mu a_2^2|$ is calculated by Fekete-Szegö [1] when μ is real. For the functions of various subclasses of S, the maximum value of $|a_3-\mu a_2^2|$ is examined by many several authors. Some of these references are given here ([see, e.g., 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17]).

Nalinakshi and Parvatham in [18] defined differential operator for all integer values of n as follows:

$$I^n f(z)=z+\sum_{k=2}^{\infty}k^{-n}a_kz^k. \tag{3}$$

They observed that

$$I^{-n}f(z)=z+\sum_{k=2}^{\infty}k^{n}a_kz^k=D^nf(z) \tag{4}$$

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where D is an operator defined in [19]. Also, we know that

$I^{-1}f(z)=zf'(z)=Df(z)$ and $I^m(I^n f(z))=I^{m+n}f(z)$ (5)

Definition 1.1. Given $0\le\alpha<1$, $0\le\gamma\le1$ and $\beta>0$, and also let $f\in S$. Then $f\in M_{\alpha,\gamma}(\beta)$ if and only if there exists $g\in\overline{S}^*_{\alpha,\gamma}(\beta)$ such that

$$\mathrm{Re}\left(\frac{\gamma I^{n-2}f(z)+(1-\gamma)I^{n-1}f(z)}{\gamma I^{n-1}g(z)+(1-\gamma)I^{n}g(z)}\right)>0\ ,\ (z\in U)\quad(6)$$

for the function $g(z)=z+b_2z^2+b_3z^3+\dots$.

Note that $M_{0,0}(\beta)=R_0(\beta)$ is the classes close-to-convex functions given by [9] and $M_{0,0}(1)=R_0(1)$ is defined by Kaplan [20] for the class of normalized functions.

The main goal of this study is to calculate sharp upper value of $|a_3-\mu a_2^2|$ for the class defined by using differential operator I^n, which is given Eqs. (6).

2. Key Lemma and Derivation of Main Theorem

First of all, we have to consider the following lemma to find our main results [21].

Lemma 2.1. Let h be in P, that is, h be analytic in the unit disc and represented by

$h(z)=z+c_2z^2+c_3z^3+\dots$ and $\mathrm{Re}\{h(z)\}>0$ for $z\in U$, then

$$\left|c_2-\frac{c_1^2}{2}\right|\le 2-\frac{|c_1^2|}{2}\ .\quad(7)$$

Theorem 2.2. Given $0\le\alpha<1$, $0\le\gamma\le1$, $\beta\ge1$ and $\mu\ge1$, also let the function f which is given by the series of (1) be an element of the class $M_{\alpha,\gamma}(\beta)$. Then a sharp inequality given below is obtained for modulus of $a_3-\mu a_2^2$:

$$\begin{aligned}|a_3-\mu a_2^2|&\le\frac{1}{3^{1-n}(1+2\gamma)}\\&\times\left(\frac{\beta^2\mu3^{1-n}(2-\alpha)(1+2\gamma)-\beta^2 2^{1-2n}(1+\gamma)^2(1-\alpha)(3-\alpha)}{2^{-2n}(1-\alpha)^2(2-\alpha)(1+\gamma)^2}\right)\\&+\frac{1}{3^{1-n}(1+2\gamma)}\times\left(\frac{\left(3^{1-n}\mu(2\gamma+1)-2^{1-2n}(1+\gamma)^2\right)(1-\alpha)}{2^{-2n}(1-\alpha)(1+\gamma)^2}\right.\\&\left.+\frac{2\beta\left(3^{1-n}\mu(2\gamma+1)-2^{1-2n}(1+\gamma)^2\right)}{2^{-2n}(1-\alpha)(1+\gamma)^2}\right).\end{aligned}\quad(8)$$

Proof. Let $f(z)\in M_{\alpha,\gamma}(\beta)$, it is seen from Eqs. (6) that

$$\begin{aligned}&\gamma I^{n-2}f(z)+(1-\gamma)I^{n-1}f(z)\\&\quad=\left(\gamma I^{n-1}g(z)+(1-\gamma)I^{n}g(z)\right)q(z).\end{aligned}\quad(9)$$

For $z\in U$, $q\in P$ denoted by,
$q(z)=1+q_1z+q_2z^2+q_3z^3+\dots$. Equating coefficients we obtain

$$\begin{aligned}2^{1-n}(1+\gamma)a_2&=q_1+2^{-n}(1+\gamma)b_2\\3^{1-n}(1+2\gamma)a_3&=q_2+2^{-n}(1+\gamma)b_2q_1+3^{-n}(2\gamma+1)b_3.\end{aligned}\quad(10)$$

It is also seen from (2) that

$$\begin{aligned}&\gamma I^{n-2}g(z)+(1-\gamma)I^{n-1}g(z)-\alpha\left(\gamma I^{n-1}g(z)+(1-\gamma)I^{n}g(z)\right)\\&=g(z)\left(p(z)\right)^{\beta}\end{aligned}\quad(11)$$

where $z\in A$, $p\in P$ and

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\dots.\quad(12)$$

So, Eqs. (13) is attained by equating coefficients,

$$\begin{aligned}2^{-n}(1+\gamma)(1-\alpha)b_2&=\beta p_1\\3^{-n}\left[(2-\alpha)(2\gamma+1)\right]b_3&=\beta\left(p_2+\frac{\beta(3-\alpha)+\alpha-1}{2(1-\alpha)}p_1^2\right).\end{aligned}\quad(13)$$

From (10) and (13) we have

$$\begin{aligned}&a_3-\mu a_2^2\\&=\frac{1}{3^{1-n}(1+2\gamma)}\left(q_2-\frac{q_1^2}{2}\right)+\frac{\left[2^{1-2n}(1+\gamma)^2-\mu3^{1-n}(2\gamma+1)\right]}{3^{1-n}2^{2-2n}(2\gamma+1)(1+\gamma)^2}q_1^2\\&+\frac{\beta}{3^{1-n}(2-\alpha)(2\gamma+1)}\left(p_2-\frac{p_1^2}{2}\right)\\&+\frac{\beta^2\left[2^{1-2n}(1+\gamma)^2(1-\alpha)(3-\alpha)-\mu3^{1-n}(2-\alpha)(2\gamma+1)\right]}{3^{1-n}2^{2-2n}(1+\gamma)^2(2-\alpha)(2\gamma+1)(1-\alpha)^2}p_1^2\\&+\frac{\beta\left[2^{1-2n}(1+\gamma)^2-\mu3^{1-n}(2\gamma+1)\right]}{2^{1-2n}3^{1-n}(1+\gamma)^2(1-\alpha)(2\gamma+1)}p_1q_1.\end{aligned}\quad(14)$$

$\mathrm{Re}\{a_3-\mu a_2^2\}$ can be estimated, under the assumption of positiveness of $a_3-a_2^2$. The following equations related to (15) is calculated by using Lemma 2.1, Eqs. (14) and under the condition of $0\le\phi<2\pi$, $p_1=2re^{i\theta}$, $q_1=2re^{i\phi}$, $0\le r\le1$, $0\le R\le1$ and $0\le\theta<2\pi$. Simply calculations of Eqs. (15) is given below:

$$\begin{aligned}
&3^{1-n}(1+2\gamma)\operatorname{Re}(a_3-\mu a_2^2)\\
&=\operatorname{Re}\left(q_2-\frac{q_1^2}{2}\right)+\frac{\left[2^{1-2n}(1+\gamma)^2-\mu 3^{1-n}(2\gamma+1)\right]}{2^{2-2n}(1+\gamma)^2}\operatorname{Re}q_1^2\\
&+\frac{\beta}{(2-\alpha)}\operatorname{Re}\left(p_2-\frac{p_1^2}{2}\right)\\
&+\frac{\beta^2\left[2^{1-2n}(1+\gamma)^2(1-\alpha)(3-\alpha)-\mu\beta 3^{1-n}(2-\alpha)(2\gamma+1)\right]}{2^{2-2n}(1-\alpha)^2(2-\alpha)(1+\gamma)^2}\operatorname{Re}p_1^2\\
&+\frac{\beta\left[2^{1-2n}(1+\gamma)^2-\mu 3^{1-n}(2\gamma+1)\right]}{2^{1-2n}(1-\alpha)(1+\gamma)^2}\operatorname{Re}p_1q_1\\
&+\frac{\beta\left[2^{1-2n}(1+\gamma)^2-\mu 3^{1-n}(2\gamma+1)\right]}{2^{1-2n}(1-\alpha)(1+\gamma)^2}\operatorname{Re}p_1q_1\\
&\le 2(1-R^2)\\
&+\frac{\left[2^{1-2n}(1+\gamma)^2-\mu 3^{1-n}(2\gamma+1)\right]}{2^{-2n}(1+\gamma)^2}R^2\cos 2\theta\\
&+\frac{2\beta}{(2-\alpha)}\left(1-r^2\right)\\
&+\frac{\beta^2\left[2^{1-2n}(1+\gamma)^2(1-\alpha)(3-\alpha)-\mu\beta 3^{1-n}(2-\alpha)(2\gamma+1)\right]}{2^{-2n}(1-\alpha)^2(2-\alpha)(\gamma+1)^2}r^2\cos 2\phi\\
&+\frac{2\beta\left[2^{1-2n}(1+\gamma)^2-\mu 3^{1-n}(2\gamma+1)\right]}{2^{-2n}(1-\alpha)(1+\gamma)^2}rR\cos(\theta+\phi)\\
&\le\left(\frac{3^{1-n}\mu(2\gamma+1)}{2^{-2n}(1+\gamma)^2}-4\right)R^2\\
&+\frac{2\beta\left[3^{1-n}\mu(2\gamma+1)-2^{1-2n}(1+\gamma)^2\right]}{2^{-2n}(1-\alpha)(1+\gamma)^2}rR\\
&+\frac{\beta^2\mu 3^{1-n}(2-\alpha)(1+2\gamma)-\beta^2 2^{1-2n}(1+\gamma)^2(1-\alpha)(3-\alpha)}{2^{-2n}(1-\alpha)^2(2-\alpha)(1+\gamma)^2}r^2\\
&-\frac{2^{1-2n}\beta(1-\alpha)^2(1+\gamma)^2}{2^{-2n}(1-\alpha)^2(2-\alpha)(1+\gamma)^2}r^2+\frac{2(\beta-\alpha)+4}{(2-\alpha)}\\
&=\psi(r,R)
\end{aligned}\tag{15}$$

Let α, β and μ be fixed and $\psi(r,R)$ be partially differentiable under the condition of $0\le\alpha<1$, $\beta\ge1$ and $\mu\ge1$. Then equation (16) given below is attained

$$\begin{aligned}
&\psi_{rr}\psi_{RR}-(\psi_{rR})^2\\
&=2^{2-4n}\beta(1+\gamma)^4\left[4\beta+2+\alpha\left(2\alpha\beta+2\alpha-4-7\beta\right)\right]\\
&-3^{1-n}\mu\beta(1+\gamma)^2(1+2\gamma)2^{-2n}\left[6\beta+2+\alpha\left(2\alpha\beta+2\alpha-4-8\beta\right)\right]\\
&<0.
\end{aligned}\tag{16}$$

As a result, $\psi(r,R)$ takes the maximum value on the boundaries. Thus the final inequality can be as follows:

$$\begin{aligned}
&\psi(r,R)\le\psi(1,1)\\
&=\frac{\beta^2\mu 3^{1-n}(2-\alpha)(1+2\gamma)-\beta^2 2^{1-2n}(1+\gamma)^2(1-\alpha)(3-\alpha)}{2^{-2n}(1-\alpha)^2(2-\alpha)(1+\gamma)^2}\\
&+\frac{\left(3^{1-n}\mu(2\gamma+1)-2^{1-2n}(1+\gamma)^2\right)(1-\alpha)}{2^{-2n}(1-\alpha)(1+\gamma)^2}\\
&+\frac{2\beta\left(3^{1-n}\mu(2\gamma+1)-2^{1-2n}(1+\gamma)^2\right)}{2^{-2n}(1-\alpha)(1+\gamma)^2}.
\end{aligned}\tag{17}$$

The inequality given by Eqs. (8) is gotten when we take $p_1=q_1=2i$ and $q_1=q_2=-2$.

3. Conclusions

The following remarks and corollary can be calculated for some particular values of related parameters.

Setting $\alpha=0$ in Theorem 2.2., we obtain the result of Jahangiri [22] as Corollory 3.1.

Corollary 3.1. Let f be given by the series of (1) and in the class of $K(\beta)$. Then the following inequality provides sharpness of the result for $\beta\ge1$, and $\mu\ge1$:

$$\left|a_3-\mu a_2^2\right|\le\beta^2(\mu-1)+\frac{(3\mu-2)(1+2\beta)}{3}.\tag{18}$$

Remark 3.2. When we choose $n=0$ and $\gamma=\lambda$ in Theorem 2.2., our results are reduced to that by Orhan and Kamali [23] .

Remark 3.3. When we choose $\gamma=0$ and $n=0$ in Theorem 2.2., our results are reduced to that by Frasin and Darus [24].

Remark 3.4. When we choose $\gamma=0$, $\alpha=0$ and $n=0$ in Theorem 2.2., our results are reduced to that by Jahangiri [22].

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