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Original Research Article

Fixed Points of Quasi Contractive Mappings Using Picard Iteration

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ABSTRACT

A map has fixed point at p if fixed point theorems have useful applications in Analusis some of the iterative methods which have been studied are related to S. Banach, W.R. Mann, J. Riemer Mann, W.G. Daston and a host of other mathematicians.
Studies by Prof. Ishikawa and by Prof. B.E. Rhoads, throw new light on the iteration process of W.R. Mann, Prof. Ishikawa studies by the following iteration process.
For a subset E of an Ailbert space H, if and only if the sequence generated by where (c_n) and (d_n) are real sequence in [0, 1].

Key words: Fixed Point, Metric Space, Picanel Iteration.

2. 1. INTRODUCTION

If, quasi contractive mappings with the 'constant of contractivity' K < 1. We note that in the proof of theorem 1 (section 1.2) it k = 1. We tail to derive the expression (1) and moreover in (2) the requirement that (1-dj-k^2) is positive no longer holds. However if we take the Picand iteration and quasi contractive mappings with constant of contractivity k = 1. We can prove convergence to the fixed point. This is what we intend to prove in the present chapter.

With regard to the fixed points of quasi contractive mappings (QC_1) Shih and Yeh proved the following theorem.

Theorem - A

Let T be a continuous mapping of a non empty compact metric space (x, d) into itself such that for some positive integer m. the iterated map T^m satisfies.

d ($T^m x$, $T^m y$) < max {d (x, y), d($T^m x$, x), d(y, $T^m y$), d(x, $T^m y$), d (y, $T^m x$)}... .(1)

For all x, y $\in$ x with x $\neq$ y then

(a) T has a unique fixed point $\bar{x} \in x$.

(b) The sequence of iterates ($T^k x$) converges to for any x $\in$X.

(c) Given $\lambda \in$ (0, 1) there exists a metric d λ, topological equivalent to d such that d_λ (T_x, T_y) $\leq$ λd_λ (x, y) for all x, y X.

We now extend the results (a) (b) to a pair of mappings in the following theorem.

2.2. MAIN RESULT

Theorem

Let S and T be a pair of commuting mappings of a compact metric space (x, d) into itself and satisfy for some positive integer m, the inequality

d ($S^m x$, $T^m y$) < Max {d (x, y), d(x, $S^m x$), d(y, $T^m y$), d(x, $T^m y$), d(y, $S^m x$)}..... (2)

For all x, y $\in$ X with x $\neq$ y. Suppose further that ST is continuous. Then S and T have a unique common fixed point $\bar{x}$. Further $\bar{x}$ is the unique fixed point of S and T and the sequence $((ST)^n, x)$ converges to for any x$\in$X.

2.3 REMARKS

Remark (1)

It is easily seen that the case S = T of the above theorem reduces to theorem A ((a), (b)). It is observed that in theorem A, T is required to be continuous. It is further shown by [61] thus an Example that the existence of fixed point cannot be guaranteed if T is a not continuous.

But in case of the present theorem if S = T then the condition that T^2 be continuous is required. So we conclude from our theorem that even if T is not continuous and T^2 is continuous then the existence of the fixed point is guaranteed. The following example illustrates this point.

Example - 1

Define T: [0, 1] [0, 1] by Tx = 0 $0 \le x < 1$ and $T_1 = \frac{1}{2}$ with the usual metric. Here T is discontinuous at x = 1 but T^2 is discontinuous being equal to zero for all $x \in [0, 1]$. T satisfies (2) (with S = T and m = 1) and possesses the fixed point zero.

It may be pertinent to mention that in example given by S Kamal not only T. but also T^2 is discontinuous.

Remark (ii)

The following example shows that commutatively is necessary in the above theorem.

Ex - 2

Let (x, d) be a metric space, where

x = {1, 2, 3, 4} and d is defined by

d (x, x) = 0 for all $x \in X$, d(1, 2) = 1=d (3, 4)

d(1, 3) = 2=d(2, 4), d (1, 4) = $\sqrt{s}$ = d(2, 3)

Let S and T be self mappings on x defined by

$S_1 = 3, S_2 = 3, S_3 = 2, S_4 = 2$

$T_1 = 4, T_2 = 1, T_3 = 4$ and $T_4 = 1$

Here S and T satisfy the inequality (2) with m = 1 but posses no fixed points. It is easily verified that

$$S_{T_1} = 2,\ S_{T_2} = 3,\ \ S_{T_3} = 2,\ S_{T_4} = 3$$

$$T_{S_1} = 4,\ T_{S_2} = 4,\ \ T_{S_3} = 1,\ T_{S_4} = 1$$

And So on X

ST ≠ TS

2.4 PROOF OF THE THEOREM

Case - 1

We shall prove the case for m = 1 ST being a self map on x we have

$(ST)^{k+1}x \le (ST)^k X$ k = 0, 1, 2.............. (3)

Let as write

F = ST

and $H = H = \prod_{k=1}^{\infty} F^k x = \prod_{k=1}^{\infty} (ST)^k x.$

Since X is compact and I is continuous $(F^k x)$ is a sequence of compact subsets on x having finite intersection property. So H is a nonempty compact subset of X.

Now we proceed to prove that F(H) = H. First of all if X ∈ H, then $X \in F^k X$ for all k = 1, 2....... And this proves that F (H) ⊂H.

Again if X ∈ H, then for any k, $X \in F^{k+1} X$ = F $(F^k X)$

So there exists an X_k (Say) in $F^k X$ such that F_{x_k} = X, k = 1, 2,.......by compactness of X we can choose a subsequence of (X_k) which we shall again denote by (X_k) such that $X_k \to X_0 \in X$ for some X_0 since $\{X_k, X_{k+1}......\}$ is contained in $F^k X$ and $F^k X$ is compact. $X_0 \in F^k X$ for k = 1, 2........ so, Again F being continuous.

$$F_{X_0} = F(\lim_k X_k) = \lim_k F_{X_k} = X$$

Hence $X = F_{X_0} \in F(H)$ Thus H F (H) implying the desired result F (H) = H

It remains to show that H is a singleton set. If not then there exist distinct points Z_1, Z_2 in H such that

S (H) = diameter of H

= sup {d (x, y); X, Y H}

= d (Z_1, Z_2)

F (H) = H implies that there exist distinct points X_1 and X_2 in H such that

$$F_{X_1} = F_{X_2} \ \ and \ \ F_{X_2} = ST_{X_2} = Z_2$$

Now for each k = 1, 2........ We have

T $(ST)^k X$ = (TS) T $(ST)^{k-1} X$.

= $(TS)^2$ T $(ST)^{k-2}$ X

= ....

= $(TS)^{k}.TX$

$\subseteq (TS)^kX = (ST)^kX$

by commutatively, Hence $X_1 \in (ST)^kx$ implies that

$T_{X_1} \in T(ST)^kX \subseteq (ST)^kX$

For each k, Hence H. Similarly $S_{X_2} \in H$ but by inequality (2)

$d(Z_1, Z_2) = d(STX_1, STX_2)$

$= d(STX_1, TSX_2)$

$<$ Max {d (TX,).d(TX_1, STX_1), d(SX_2, TSX_2)} if $TX_1 = T_{X_2}$

d(TX_1, TSX_2), d(SX_2, STX_2)}

and this contradicts. [(4)] Hence H is a singleton set. Let H = {X}

Now F^nx being compact, there exist $X_n, Y_n \in F^nX$

Such that $(F^nX) = d(X_n, Y_n)$. The sequence (X_n) Thus obtained lies in the compact space X and here we choose a subsequence (X_{n_k}) of X_n, converging to X_0 for some $X_0 \in X$ and rename of as (X_n). Then $X_n \in F_nX$ and $X_n \to X_0$ as $n \to \infty$..

Now $\{X_n, X_{n+1}, ...\} \le F^nX$ and F^nX is compact. Hence

X_0 F^nX for n = 1, 2, ...

Thus $X_8 \in \overset{\infty}{\underset{n=1}{\Gamma}} F^n x = H$ in similar fashion we have an $Y_0 \in X$.

Such that $Y_n \to Y_0$ as $n \to \infty$ and $Y_0 \in H$. But H = {$\bar{x}$}.

a singleton set. Hence $X_0 = \bar{x} = Y_0$ Thus

$$\lim_{n\to\infty} 8(F^nX) = \lim_{n\to\infty} d(X_n, Y_n) = d(X_0, Y_0) = 0$$

Now for any $x \in X$ and any +ve integer n

$$d(F^nX, \bar{X}) \le 8(F^nX) \to 0, n \to \infty$$

Which implies that

$F^nX = (ST)^nX \to \bar{x}$ as $n \to \infty$ and F(H) = H gives that is the unique fixed point of F=ST.

Hence $F(\bar{T}_x) = TS(T\bar{x}) = T(ST\bar{x}) = T\bar{x}$

Since F has a unique fixed point $T\bar{x} = \bar{x}$. Further $S\bar{x} = ST\bar{x} = \bar{x}$. Hence $\bar{x}$ is a common fixed point of S and T.

If Y is any common fixed point of S and T. Then $F_y = T_{Sy} = T_y = y$ would imply $y = \bar{x}$. Again if z is any fixed point of S then an application of (2) gives

$$d(Z, \bar{x}) = d(Sz, T\bar{x}) < d(Z, \bar{x})$$

Hence $Z = \bar{x}$. Similarly $\bar{x}$ is the unique fixed point of T.

This completes the proof of the theorem in the case m = 1.

Case – II

Suppose m > 1.

By commutatively of S and T, we see that $S^mT^m = (ST)^m = T^mS^m$ and more over S^mT^m is continuous. Hence it follows from the case m = 1 that S^m and T^m have a unique common fixed point $\bar{x}$, say i.e. $S^m\bar{x} = \bar{x} = T^m\bar{x}$.

Further $\bar{x}$ is the unique fixed point of S^m and T^m and $((ST)^mX)$ converges for m > 1 to $\bar{x}$ for each $x \in X$ as $n \to \infty$. Thus $S\bar{x} = S(S^m) = S(S^m\bar{x})$ and this gives that S is a fixed point of S^m but uniqueness of $\bar{x}$ implies $S\bar{x} = \bar{x}$. Similarly $T_{\bar{x}} = \bar{x}$ more over $(ST)^m$ x $\to k$ as implies that $(ST)^m x \to \bar{x}$ as $n \to \infty$ implies that $(ST)^n X \to \bar{x}$ as $n \to \infty$ let n be any +ve integer.

We can write n = mk + r, where r = 1, 2...........m and k = 0, 1, 2.......Now

$(ST)^nX = (ST)^{mk+r}X$

$= (ST)^r (ST)^{mk}.x$

$\to (ST)^r\bar{x}$ as $(n \to \infty)$

$= \bar{x}$ (5)

$\bar{x}$ being a fixed point of ST.

This completes the proof of the theorem.

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