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Static Response of an Isotropic and Orthotropic Thin Plate Under Different Excitation

Sabah Moussaoui and Mourad Belgasmia

Department of Civil Engineering Setif University, Algeria
mourad.belgasmia@yahoo.fr

ABSTRACT

This article presents a static analysis of the orthotropic and isotropic thin plates in pure bending under uniform and concentrated load, by finite elements method. For space modeling the isoparametric quadrilateral element with 4 nodes Q4, and 3 degrees of freedom per node is used. The formulation of the element is founded on a linear theory of the orthotropic plates, like, the kinematics assumptions of Mindlin-Reissner which hold in account the effect of transverse shearing. The linear stiffness matrix is evaluated numerically by using the technique of selective numerical integration. The load vector is evaluated in an exact way by Gauss type diagrams.

Key words: Finite Element Formulation, Orthotropic Thin Plate, Quadrilateral Element, Numerical Integration, Selective Integration

INTRODUCTION

The researchers are led to use numerical methods to deal with the increasing complexity of the structural mechanics problems. The finite element method is the most powerful and general analysis method of structures. It allows a detailed analysis of the behavior of complex structures, which are difficult to calculate by the usual procedures of the strength of materials, such as plates and shells. The plates (thin, thick, isotropic, orthotropic) are structures which are frequently used in various fields: aeronautics, civil engineering, nuclear thermal power stations, structures oriented in their plans, such as plates are more difficult to calculate numerically, they are often subjected to statistical and dynamic stresses. The objective of this work primarily concerns the static analysis of the orthotropic and isotropic plates under uniform and concentrate load using finite element.

THE PLATES THEORY

A plate is an elastic solid in which the dimension according to the thickness, is small compared to both others, and which has a symmetry plane in the middle of the thickness, often called average area [1], [9],[10].

Displacements Field Description

In Hencky theory, we give a displacements model based on three independent variables: transverse displacement and two rotations as follows [9], [5]:

$$u = z\beta_x(x, y) \qquad v = z\beta_y(x, y) \qquad w = w(x, y) \tag{1}$$

Deformation Field Description

The strain tensor is:

$$\{\varepsilon\} = \left\{\{\varepsilon_f\}^T, \{\varepsilon_c\}^T\right\}^T = \left\{z\{\chi\}^T, \{\gamma\}^T\right\}^T \qquad \{\chi\}\text{: Curvature vector} \tag{2}$$

Contribution de bending effect: Contribution of shear effect:

$$\{\varepsilon_f\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = z\{\chi\} = z\begin{Bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{Bmatrix} \qquad \{\varepsilon_c\} = \{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \beta_x + \dfrac{\partial w}{\partial x} \\ \beta_y + \dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{3}$$

EQUATIONS MOTION FORMULATION

The overall formulation of the problem is to look for the whole structure the matrix expression strain energy and the work of applied forces, according to the movements of all nodes of the structure. This requires the assembly of the elementary characteristics (stiffness matrix, vector equivalent forces) for all elements [14].

The total potential energy of the structure can be obtained by summation of the component total potential energy as shown in equation (4)

$$V = \sum_{éléments} V^e = \sum_{éléments} \left\{ \frac{1}{2} \{q\}^{eT} [K]^e \{q\}^e - \{q\}^{eT} \{F\}^e \right\} \quad (4)$$

$\{q\}^T$ Is the row displacement vector at nodes of the structure:

$$\{q\}^T = \left\{ q_1^T \ldots q_i^T \ldots q_m^T \right\} \quad (5)$$

With: $\{q_i\}$: sub- displacement vector at node i .

We can define for each element a matrix relationship in order to establish a correspondence between the element node's displacements $\{q\}^e$ and the structure nodes's one $\{q\}$, as following:

$$\{q\}^e = [\mathrm{B}]^e \{q\} \quad (6)$$
$$(n_e \times 1) = (n_e \times N).(N \times 1)$$

With : $[\mathrm{B}]^e$: Matrix location element .

n_e : Number of degree of freedom of element . N : Number of degree of freedom of structure.

Each relationship represented by the equation (6) can identify or locate degree of freedom of each element in all degree of freedom of the structure.

Using the equations (4) and (6), we can write:

$$V = \sum_{éléments} \left\{ \frac{1}{2} \{q\}^T [\mathrm{B}]^{eT} [K]^e [\mathrm{B}]^e \{q\} - \{q\}^T [\mathrm{B}]^{eT} \{F\}^e \right\} \quad (7)$$

Knowing that : $V = \frac{1}{2}\{q\}^T [K]\{q\} - \{q\}^T \{F\}$ (8)

With : $[K] = \sum_{éléments} [\mathrm{B}]^{eT} [K]^e [\mathrm{B}]^e$ (9)

$$\{F\} = \sum_{éléments} [\mathrm{B}]^{eT} \{F\}^e \quad (10)$$

$[K]$: Stifness matrix of the structure. $\{F\}$: Equivalent force vector of the structure.

In the case of concentrate load applied in node of the structures (represented as {P} vector), the expression of $\{F\}$ becomes:

$$\{F\} = \{P\} + \sum_{éléments} [\mathrm{B}]^{eT} \{F\}^e \quad (11)$$

These expressions allow us to obtain by direct application of the principle of virtual work, the equilibrium equations system of nodes. Indeed, we have:

$$\delta U = \delta W$$
$$\{\delta q\}^T [K] \{q\} = \{\delta q\}^T \{F\} \quad (12)$$

D'où : $[K]\{q\} = \{F\}$ (13)

FINITE ELEMENT PLATE WITH TRANSVERSE SHEAR

The formulation of finite elements of plate in bending and shear is based on the theory of Reissner-Mindlin. Indeed, their conformity requires only continuity C^0 de w, β_x *et* β_y [2], [8], [9].

We consider a quadrilateral type element in which we apply the isoparametric formulation [2], [6], [11], [9]. The approximations of w, β_x *et* β_y are:

$$w = N^T(\xi,\eta)W \qquad \beta_x = N^T(\xi,\eta)\hat{\beta}_x \qquad \beta_y = N^T(\xi,\eta)\hat{\beta}_y \tag{14}$$

The interpolation functions used are usually related to interpolation of the isoparametric quadrilaterals. In the case of the linear quadrilateral, we have:

$$N^T = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \text{ with: } \quad N_i(\xi,\eta) = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{15}$$

$\xi_i \, ou \, \eta_i$: taking the values (+1) or (-1) according to the considered node

$$W = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} \qquad \hat{\beta}_x = \begin{Bmatrix} \beta_{x1} \\ \beta_{x2} \\ \beta_{x3} \\ \beta_{x4} \end{Bmatrix} \qquad \hat{\beta}_y = \begin{Bmatrix} \beta_{y1} \\ \beta_{y2} \\ \beta_{y3} \\ \beta_{y4} \end{Bmatrix} \tag{16}$$

By substitution (14), in the relations of deformations (3), we obtain the interpolation deformation matrix of bending and shear:

$$\{\chi\} = \{\bar{\varepsilon}_f\} = \begin{bmatrix} 0 & \frac{\partial N^T}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N^T}{\partial y} \\ 0 & \frac{\partial N^T}{\partial y} & \frac{\partial N^T}{\partial x} \end{bmatrix} \begin{Bmatrix} W \\ \hat{\beta}_x \\ \hat{\beta}_y \end{Bmatrix}$$

$$\{\varepsilon_c\} = \{\gamma\} = \begin{bmatrix} \frac{\partial N^T}{\partial x} & N^T & 0 \\ \frac{\partial N^T}{\partial y} & 0 & N^T \end{bmatrix} \begin{Bmatrix} W \\ \hat{\beta}_x \\ \hat{\beta}_y \end{Bmatrix}$$

(17)

We can write :
$$\{\chi\} = \{\bar{\varepsilon}_f\} = \lfloor \bar{\beta}_f \rfloor \{q\} \qquad \{\gamma\} = \lfloor \beta_\gamma \rfloor \{q\} \tag{18}$$

Stiffness matrix

The expression of the deformation energy makes possible the calculation of stiffness matrix [2], [7], [9], we have:

$$U = U_F + U_C$$

$$U = \frac{1}{2}\int_{S^e} \{\chi\}^T [D_f] \{\chi\} dx\,dy + \frac{1}{2}\int_{S^e} \{\gamma\}^T [D_c] \{\gamma\} dx\,dy = \frac{1}{2}\{q\}^T [K] \{q\} \tag{19}$$

with :
$$[D_f] = \frac{h^3}{12} \begin{bmatrix} \frac{E_1}{1-\nu_{21}\nu_{12}} & \frac{E_2\nu_{12}}{1-\nu_{21}\nu_{12}} & 0 \\ \frac{E_2\nu_{12}}{1-\nu_{21}\nu_{12}} & \frac{E_2}{1-\nu_{21}\nu_{12}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \qquad [D_c] = h\,k \begin{bmatrix} G_{13} & 0 \\ 0 & G_{23} \end{bmatrix} \tag{20}$$

$\lfloor D_f \rfloor$: Bending stiffness matrix, E_1, E_2 : Young moduli in the directions (X), (y) respectively. ν_{12}, ν_{21} : Poisson's ratios, $[D_c]$: Shear stiffness matrix G_{12}, G_{13}, G_{23} : moduli of rigidity, k : Coefficient of correction of transverse shearing.

After substitution of the expressions of the deformations in the deformation energy, we obtain:

$$[K] = [K_f] + [K_c] = \int_S \lfloor \bar{\beta}_f \rfloor^T [D_f] \lfloor \bar{\beta}_f \rfloor \, dx\,dy + \int_S \lfloor \beta_\gamma \rfloor^T [D_c] \lfloor \beta_\gamma \rfloor \, dx\,dy \tag{21}$$

$$[K] = \int_{-1}^{+1}\int_{-1}^{+1} \lfloor \bar{\beta}_f \rfloor^T [D_f] \lfloor \bar{\beta}_f \rfloor \det[J] \, d\xi \, d\eta + \int_{-1}^{+1}\int_{-1}^{+1} \lfloor \beta_\gamma \rfloor^T [D_c] \lfloor \beta_\gamma \rfloor \det[J] \, d\xi \, d\eta \tag{22}$$

$[J]$: is the Jacobienne matrix of the geometrical transformation, The stifness matrix $[K]$ is evaluated numerically by selective numerical integration; $[K_f]$: is integrated with (2 X 2) points of Gauss, $[K_c]$: is obtained by reduced integration .

The integration of Gauss at two points is as following:

$$\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)\, d\xi\, d\eta = \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j f(\xi_i, \eta_j) \tag{23}$$

Equivalent Loads Vector

The potential energy of external forces [10], [4],[14] expressed using surface forces and volume, as following

$$W = \int_{V^e} \{u\}^{e^T}\{f_v\}dV + \int_{S^e}\{u\}^{e^T}\{f_s\}dS = \{q\}^{e^T}\{F\}^e \tag{24}$$

We have : $\{u\}^e = [N]^e\{q\}^e$ (25)

$$W = \langle q\rangle^e \int_{V^e}[N]^{e^T}\{f_V\}\,dV + \langle q\rangle^e\int_{S^e}[N]^{e^T}\{f_S\}\,dS = \langle q\rangle^e\{F\}^e \tag{26}$$

With : $\{F\}^e = \int_{V^e}[N]^{e^T}\{f_V\}dV + \int_{S^e}[N]^{e^T}\{f_S\}\,dS$ (27)

For the distributed uniform load f_z along z, the equivalent vector load has the usual form

$$\{F\}^e = \int_{S^e}[N]^{e^T} f_z\, dS \tag{28}$$

RESULTS AND DISCUSSION

Static Response of Simply Supported Isotropic Thin Plate

We consider an isotropic square thin plate simply supported, subjected to a uniform load, the geometry of the structure and the properties of material are represented on the Fig.1 for this problem. The Fig. 2 represents the static evolution response of normal displacement for different mesh and then compared to analytical response given by [3, 14].

$$w_c = 0.00406\frac{P_0 L^4}{D} \qquad \text{Where: } D = \frac{Eh^3}{12(1-\nu^2)} \qquad \nu = 0.3$$

D : Flexion coefficient stiffness , ν : Poisson ratio , w_C : Normal displacement

Data **:** a = 1.2 m; E = 210 × 10^9 N/m^2 ; h = 0.01 m. ν = 0.3; P_0 = 750 N/m^2

Static Response of Embedded Isotropic Thin Plate

We consider an isotropic thin square plate embedded on four sides, subjected to a uniform load, the geometry of the structure and the properties of material are represented on the Fig. 3 for this problem. The Fig. 4 represents the static evolution response of normal displacement for different mesh and then compared to analytical response given by [3, 14].

$$w_c = 0.00126\frac{P_0 L^4}{D}$$

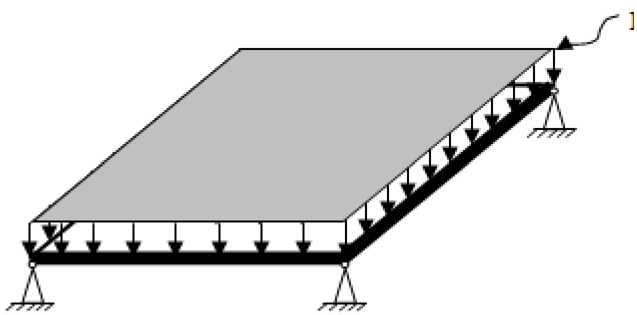

Fig.1 Simply Supported Thin Plate under a Uniform Load

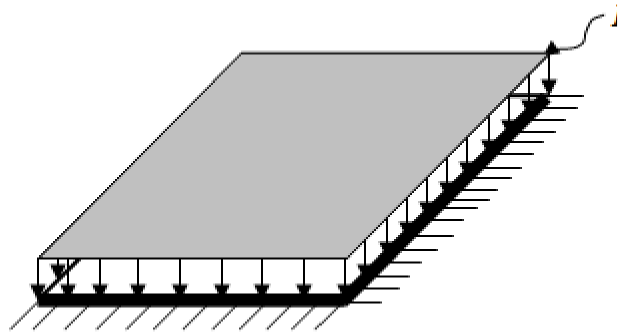

Fig.3 Embedded Thin Plate Under a Uniform Load

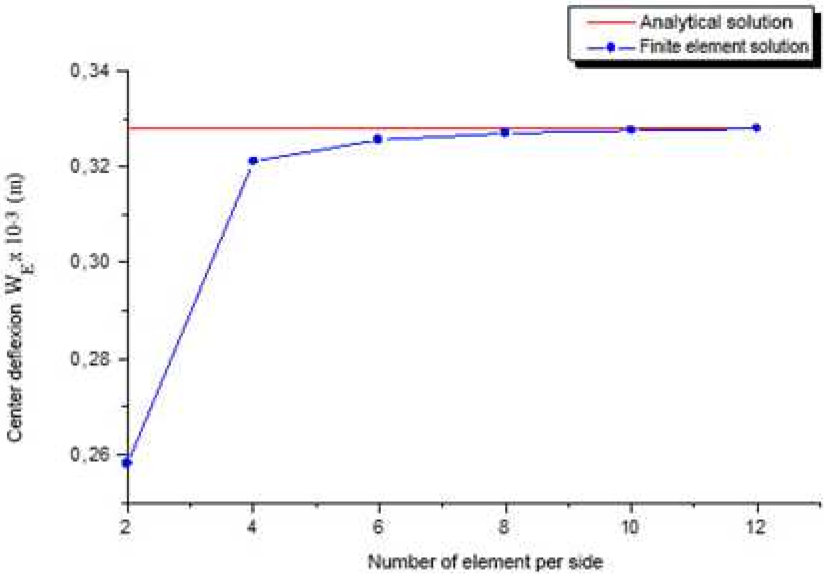

Fig. 2 Static Evolution Response of Normal Displacement for Different Mesh of Simply Supported Plate

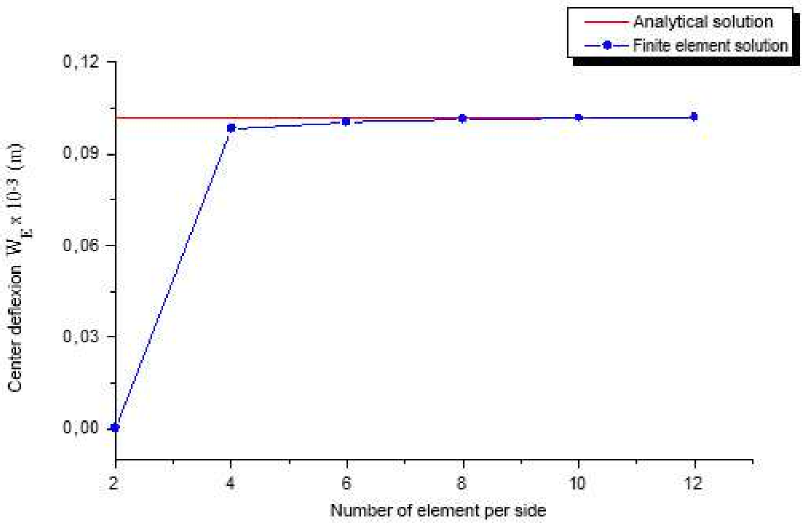

Fig. 4 Static Evolution Response of Normal Displacement for Different Mesh of Embedded Plate

Static Response of Simply Supported Orthotropic Thin Plate under Uniform Load

We consider anorthotropic square thin plate simply supported [12], subjected to a uniform load, the geometry of the structure and the properties of material are represented on the Fig. 5. The plates was divided into 100 elements, each side having 10 elements of equal length, and have compared our numerical results of the deflection along the two half-axes of symmetry of the plate with the exact solution, (Fig. 6). In order to study the convergence , we represent in Fig. 7 a normal displacement evolution in the center of the plate vs different mesh cases(2x2 , 4x4, 6x6, 8x8, and10x10), our numerical results are compared with the analytical solution given by [13].

$$w = 0.9391 \times 10^{-3}\,\frac{P_0\,a^4}{D_{22}} = K \cdot \frac{P_0\,a^4}{D_{22}} \times 10^{-3}$$

With : $D_{22} = \dfrac{E_2\ h^3}{12\ (1 - \nu_{12}\,\nu_{21})}$ $\quad \nu_{21} = \dfrac{\nu_{12}\,E_2}{E_1}$

Data: a = 2 m ; $E_1 = 2.068 \times 10^{11}$ N/m^2; $E_2 = E_1/15$; $\nu_{12} = 0.3$; $P_0(x,y) = 800$ N/m^2.

$G_{12} = G_{23} = G_{13} = 6.055 \times 10^{8}\ N/m^2$; h = 0.01 m;

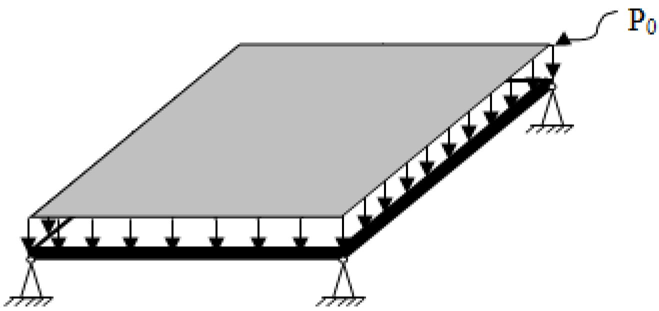

Fig. 5 Simply Supported Orthotropic Thin Plate under Uniform Load P_0

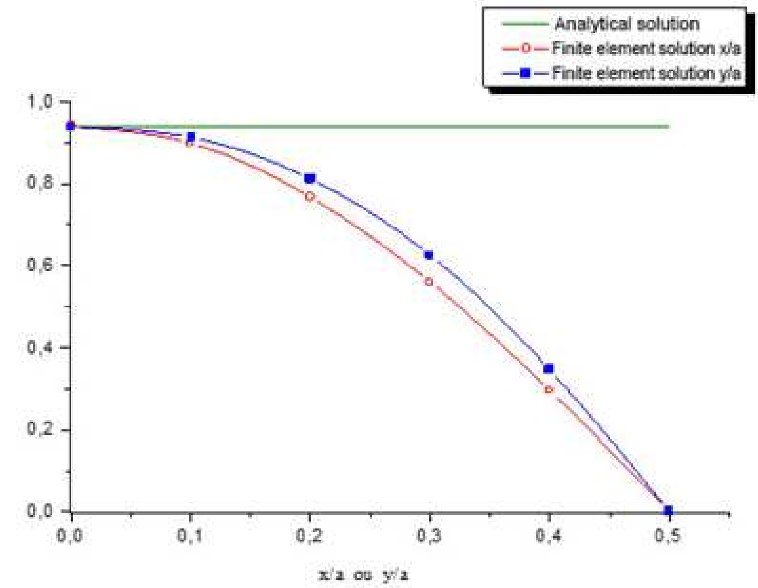

Fig. 6 Simply Supported Orthotropic Thin Plate Deflection along the Two Half-Axes of Symmetry

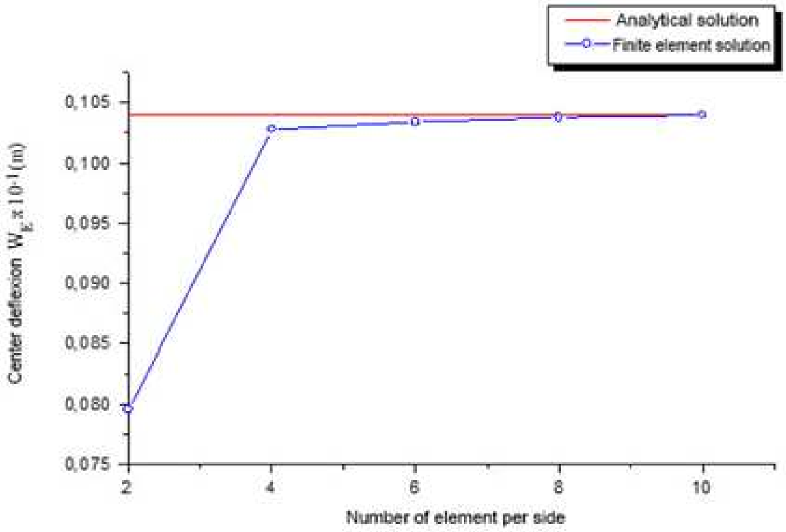

Fig. 7 Convergence of Finite Element Solution Depending on Mesh for a Simply Supported Orthotropic Plate

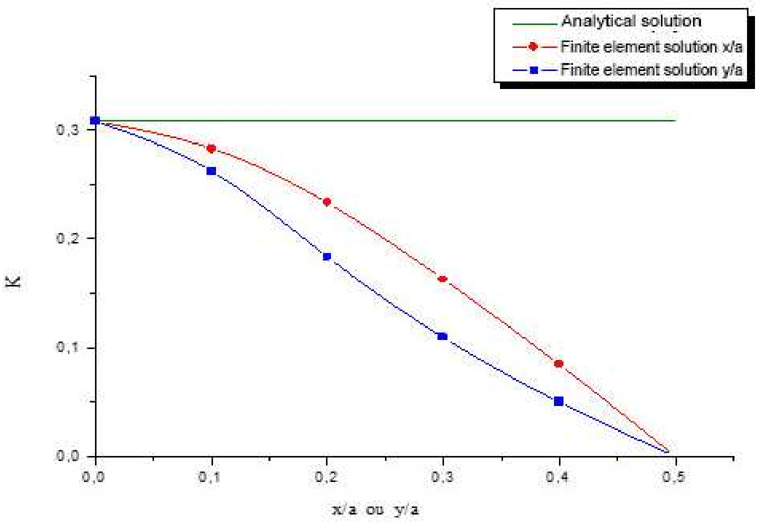

Fig. 9 Simply Supported Orthotropic Thin Plate Deflection along the Two Half-Axes of Symmetry under Concentrated Load

Using the previous example (Fig. 7), but in this case, the plate is under a concentrated load P_0 applied in the middle. Our numerical results of the deflection along the two half-axes of symmetry are compared with the analytical solution given by [13] (Fig. 8).

$$w = 0.3084 \times 10^{-2} \frac{F a^2}{D_{22}} = K \frac{F a^2}{D_{22}} \times 10^{-2}$$

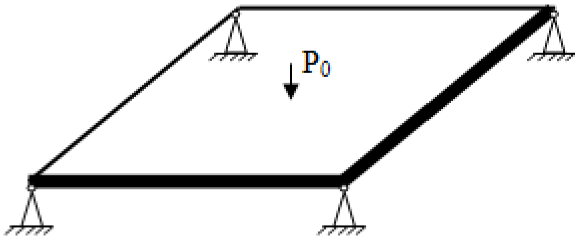

Fig. 8 Simply Supported Orthotropic Thin Plate Under Concentrated Load P_0

Static Response of Embedded Orthotropic Thin Plate under Uniform Load

Fig. 10 and Fig. 11 represent an embedded orthotropic thin plate under uniform load and normal displacement evolution in the center of the plate vs different mesh cases respectively. Our numerical results are compared with the analytical solution given by [13].

Data –
a = 2 m ;
$E_1 = 2.068 \times 10^{11}$ N/m^2;
$E_2 = E_1/15$;
$\nu_{12} = 0.3$;
$G_{12} = G_{23} = G_{13} = 6.055 \times 10^8 \; N/m^2$; h = 0.01 m;
$P_0(x,y) = 800$ N/m^2.

$$w = 0.18645 \times 10^{-3} \frac{P_0 a^4}{D_{22}}$$

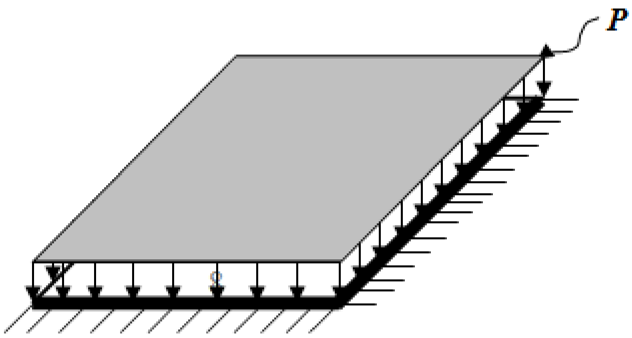

Fig. 10 Embedded Orthotropic Plate under Uniform Load

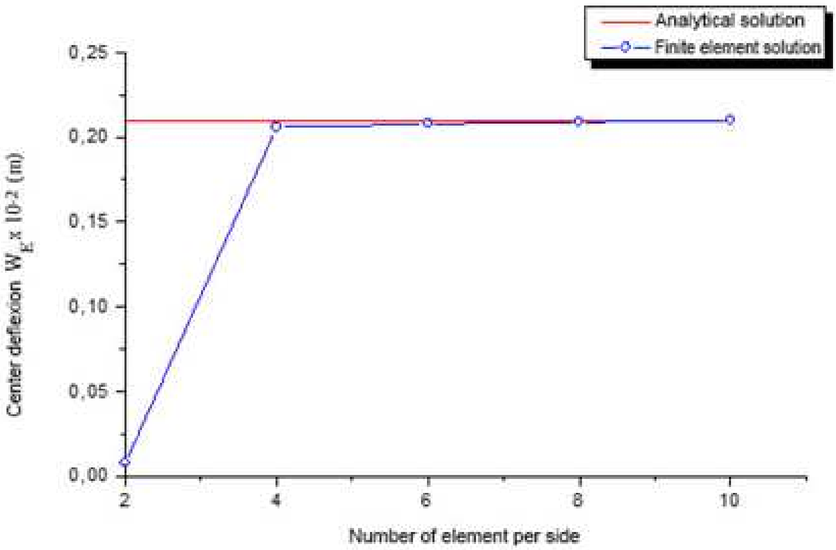

Fig. 11 Convergence of Finite Element Solution Depending on Mesh for Embedded Orthotropic Plate under Uniform Load

Static Response of Embedded Orthotropic Thin Plate under Concentrated Load

Using the previous example but in the case, the plate is under a concentrated load F applied in the middle. Our numerical results represented by the normal displacement evolution in the center of the plate vs different mesh cases are given in Fig. 13.

$$w = 1.17180 \times 10^{-3} \frac{F a^2}{D_{22}}$$

Data - a = 2 m ;
$E_1 = 2.068 \times 10^{11}$ N/m^2;
$E_2 = E_1/15$; $\nu_{12} = 0.3$;
$G_{12} = G_{23} = G_{13} = 6.055 \times 10^8 \; N/m^2$;
h = 0.01 m;
$P_0(x,y) = 800$ N/m^2.

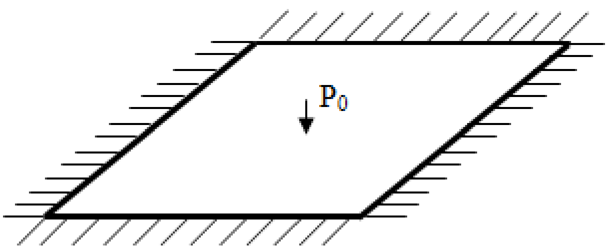

Fig. 12 Embedded Orthotropic Plate under Concentrated Load

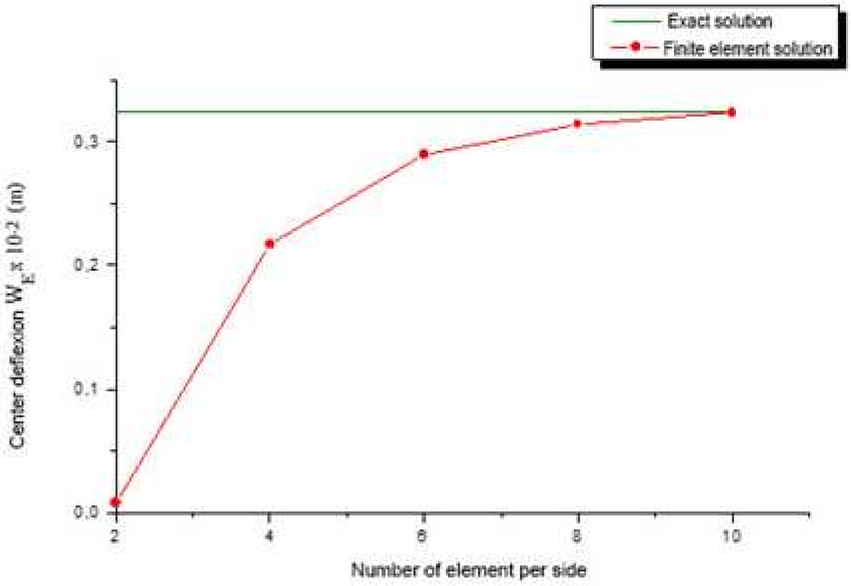

Fig. 13 Convergence of Finite Element Solution Depending on Mesh for Embedded Orthotropic Plate under Concentrated Load

Through the different results, presented in this article, we can make the following remarks:

- The modelisation results of an isotropic and orthotropic plate, whatever the type of the support, show a clear convergence with those of the analytical one.
- The convergence of a simply supported and embedded isotrpic and orthotropic plate are reached respectively for 8x8 elements mesh (see Figs 2 and 6) and 6x6 elements mesh (Fig. 4 and Fig. 11).

CONCLUSION

This work aimed the evaluation of the quadrilateral element Q4 with 12 d.d.l. The various analyses of the static behavior of isotropic and orthotrpic plate allowed us to highlight the direct influence of boundary conditions on the response of the plate and to conclude that the accuracy of solutions increases with mesh refinement.

REFERENCES

[1] KJ Bathe, *Finite Element Procedures*, Prentice-Hall, Englewood Cliffs, New Jersey, **1996**.
[2] JL Batoz and G Dhatt, *Modélisation des Structures par Éléments Finis*, Vol.2, Ed. Hermès, Paris, **1990**.
[3] N Belheine, *Analyse Linéaire Statique et Dynamique (Transitoire) des Plaques Minces par Éléments Finis*, Mémoire de Magistère, Université Mentouri de Constantine, Janvier, **2001**.
[4] RW Clough and J Penzien, *Dynamique Des Structures*, Tome I, Editions Pluralis, ***1980***.
[5] JC Craveur, Modélisation *des Structures Calcul Par Élément Finis*, Masson, Paris, **1996**.
[6] G Dhatt and G Touzot, *Une présentation de la Méthode Des Éléments Finis* , 2nd Edition, Editeur Maloine SA, France, **1984**.
[7] F Frey F, *Analyse des Structures et Milieux Continus Mécanique des Solides*, Vol. 3, 1st Edition, Lausanne, **1998**.
[8] P Geoffroy, *Développement et Évaluation d'un Élément Fini Pour l'analyse Non Linéaire Statique et Dynamique de Coques Minces*, Thèse de Doct.-Ing., Université de Technologie de Compiègne, Avril **1983**.
[9] JF Imbert, *Analyse des Structures par Éléments Finis*, 3rd Edition, Editions Cépaduès, Toulouse, Janvier **1991**.
[10] P Patrick, *Dynamique des Structures Application Aux Ouvrages de Génie Civil*, Lavoisier, Paris, **2005.**
[11] JN Reddy, An *Introduction to the Finite Element Method*, Editeurs Slaughter TM, Hazlett S, Ed. McGraw-Hill, Etats-Unis, **1984**.
[12] M Robert, *Mechanics of Composite Materials*, Edition McGraw-Hill, Etats-Unis, **1975**.
[13] G Shi and G Bezine, A General Boundary Integral Formulation for the Anisotropic Plate Bending Problems, *Journal of Composite Materials*, **1988,** 22, 694-716.
[14] S Timoshenko and S Woinowsky-Krieger, *Théorie des plaques et coques* , Librairie Polytechnique CH. Béranger, **1988**.