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Asymptotic Estimates for Real Zeros of Random Polynomials

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ABSTRACT

This paper provides asymptotic estimates strong result for real zeros of random algebraic polynomial for the expected number of real zeros of a random algebraic polynomial of the form. The strong result for the lower bound was obtained in the general case by their lower bound was $\mu \log n/\log\left\{\frac{k_n}{t_n}\log n\right\}$ *Which is obtained by taking* $\varepsilon_n = \mu/\log\left\{\frac{k_n}{t_n}\log n\right\}$ *in our present result .This result is better than that of Dunnage since our constant is (1/√2) Times his constant and our error term is smaller . the proof is based on the convergence of an integral of which an asymptotic estimation is obtained .*

Key words: Independent, identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots, domain of attraction of the normal law, slowly varying function

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INTRODUCTION

We shall suppose that $\xi_v(\omega)$'s real-valued random variables defined on the probability space (Ω , m, P) .The random events to be considered in the proof correspond to P-measurable subsets of this space. The probability that an event E occurs will be denoted by $P(E)$. Let N_n be the number of real roots of $f(x,\omega)\sum_{v=0}^{n}\xi_v(\omega)x^v$. In Mishra et al [1] we have shown that for $n > n_0, N_n$ is at least $\varepsilon_n \log n$ outside an exceptional set of measure at most $\mu/(\varepsilon_{n_0} \log n_0)$ where { ε_n } is any sequence tending to zero such that $\varepsilon_n^2 \log n$ tends to infinity as n tends to infinity. We have assumed that the ξ_v's have a common characteristics function exp $(-C|t|\alpha)$ where $\alpha \geq 1$ and C is a positive constant.

In the present work we have proved the same result in the general case. We assume that the ξ_v's are any random variables with finite variance and third absolute moment. Our previous result holds in the case of a special characteristic function which has infinite variance $(1 < \alpha < 2)$.

The strong result for the lower bound was obtained in the general case by Samal and Mishra [2]. Their lower bound was $\mu \log n/\log\left\{\frac{k_n}{t_n}\log n\right\}$ which is obtained by taking $\varepsilon_n = \mu/\log\left\{\frac{k_n}{t_n}\log n\right\}$ in our present result, where k_n, t_n have the same meaning as in our present work.

We claim that our strong result for the lower bound in the general case is the best estimation done so far. We shall use [x] to denote the greatest integer not exceeding x.

THEOREM 1

Let $f(x,\omega)=\sum_{v=0}^{n}\xi_v(\omega)x^v$ be a polynomial of degree n whose coefficients are independent random variables with expectation zero. Let σ_v^2 be the variance and τ_v^3 be the third absolute moment of $\xi_v(\omega)$.Take $\{\varepsilon_n\}$ to be a sequence tending to zero such that $\varepsilon_n^2 \log n$ tends to infinity as n tends to infinity. Let $t_n = \min_{0\le v\le n}\sigma_v$, $k_n = \max_{0\le v\le n}\sigma_v$ and $p_n = \max_{0\le v\le n}\tau_v$.Then there exists an integer n_0 and a set $A(\omega)$ of measure at most $\mu/\varepsilon_{n_0}\log n_0$ such that, for $n>n_0$ and all ω not belonging to $A(\omega)$, the equations $f(x,\omega)=0$ have at least $\varepsilon_n \log n$ real roots, provided $\lim\frac{p_n}{k_n}$ and $\lim\frac{k_n}{t_n}$ are finite.

PRELIMINARY LEMMAS

LEMMA 1

Suppose $X_1, X_2,...X_n$ are independent random variables with expectation zero and that A_v^2 is the variance and B_v^3 is the third absolute moment of X_v.

Let

$$\mu_n^2=\sum_{v=1}^{n}A_v^2,\qquad \lambda_v=\begin{cases}\dfrac{B_v^3}{A_v^2} & \text{if } A_v\neq 0\\ 0 & \text{if } A_v=\Lambda_n=\max_{1\le v\le n}(\lambda_n)\end{cases}$$

Also let $F_n(t)$ be the distribution function of $\frac{1}{\mu_n}\sum_{v=1}^{n}X_v$ and

$$\phi(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp\left(-\frac{1}{2}u^2\right)du$$

Then

$$\sup_t |F_n(t)-\phi(t)|\le 2\left(\Lambda_n/\mu_n\right)$$

This result is due to Esseen [5] and Berry [4].

LEEMA 2

Let $\eta_1,\eta_2,\eta_3,...$ be a sequence of independent random variables identically distributed with $V(\eta_i)<1$ for all i. Then, for each $\varepsilon>0$

$$p\left\{\sup_{k\ge k_0}\left|\frac{1}{k}\sum_{i=1}^{k}\{\eta_i-E(\eta_i)\}\right|\ge\varepsilon\right\}\le\frac{D}{\varepsilon^2 k_0'}$$

Where D is a positive constant. This form of the strong law of large number is a consequence of the Hajek-Renyi inequality [3].

Proof of the Theorem

Take $\beta_n=\frac{t_n}{k_n}\exp\left\{\frac{C_1}{\varepsilon_n^2\log n}\right\}$ where C_1 is a constant to be chosen later.

Let A and B be constants such that 0<B<1 and A >1. Let

$$M_n=\left[2\beta_n^2\left(\frac{k_n}{t_n}\right)^2\frac{Ae}{B}\right]+1. \tag{1}$$

So

$$\mu\left(\frac{k_n}{t_n}\right)^2\beta_n^2\le M_n\le\mu'\left(\frac{k_n}{t_n}\right)^2\beta_n^2.$$

We define

$$\phi(x)=x^{[\log x]+x}$$

Let k be the integer determined by

$$\phi(8k+7)M_n^{8k+7} \le n < \phi(8k+11)M_n^{8k+11}. \qquad (2)$$

Obviously $$\mu_1 \sqrt{\frac{\log n}{\log\left(\frac{k_n}{t_n}\beta\right)}} \le k \le \frac{\mu_2 \log n}{\log\left(\frac{k_n}{t_n}\beta_n\right)} \qquad (3)$$

which implies $$\frac{\mu_1}{\sqrt{C_1}}\varepsilon_n \log n \le k \le \frac{\mu_2}{C_1}(\varepsilon_n \log n)^2.$$

We consider $$f(x,\omega) = U_m(\omega) + R_m(\omega)$$

at the points $$x_m = \left\{1 - \frac{1}{\phi(4m+1)M_n^{4m}}\right\}^{1/2} \qquad (4)$$

for $m = [k/2]+1, [k/2]+2, \ldots k$, where

$$U_m(\omega) = \sum_1 \xi_v(\omega)x_m^v, \; R_m(\omega) = \left(\sum_2 + \sum_3\right)\xi_v(\omega)x_m^v$$

the index v ranging from $\phi(4m-1)M_n^{4m-1}+1$ to $\phi(4m+3)M_n^{4m+3}$ in $\sum_1$, from 0 to $\phi(4m-1)M_n^{4m-1}$ in $\sum_2$, and from $\phi(4m+3)M_n^{4m+3}+1$ to n in $\sum_3$.

Let $$V_m = \frac{1}{2}\left(\sum_1 \sigma_v^2 x_m^{2v}\right)^{1/2}.$$

We define the events E_m as the sets of ω for which $U_{2m}(\omega) > V_{2m}$ and $U_{2m+1}(\omega) < -V_{2m+1}$ and the events F_m as the sets of ω for which $U_{2m}(\omega) < -V_{2m}$ and $U_{2m+1}(\omega) > V_{2m+1}$. Obviously the sets of ξ_v's in $U_{2m}(\omega)$ and the sets of ξ_v's in $U_{2m+1}(\omega)$ are disjoint. Thus $U_{2m}(\omega)$ and $U_{2m+1}(\omega)$ are independent random variables. Let S_m^+, S_m^- be the sets of ω in which respectively $U_m(\omega) > V_m, U_m(\omega) < -V_m$.

Hence $E_m \cup F_m = \left(S_{2m}^+ \cap S_{2m+1}^-\right) \cup \left(S_{2m}^- \cap S_{2m+1}^+\right).$

Since the two sets within the braces on the right hand side are disjoint and since $U_{2m}(\omega)$ and $U_{2m+1}(\omega)$ are independent random variables,

$$P(E_m \cup F_m) = P(S_{2m}^+)P(S_{2m+1}^-) + P(S_{2m}^-)P(S_{2m+1}^+).$$

If σ^2 is the variance of $U_{2m}(\omega)$ then $\sigma^2 = 4V_{2m}^2$.

So $\sigma = 2V_{2m}$. Let $F_{2m}(t)$ be the distribution function of $\frac{U_{2m}(\omega)}{\sigma}$.

Hence $$P\{U_{2m}(\omega) < -V_{2m}\} = P\{U_{2m}(\omega)/\sigma < -\tfrac{1}{2}\} = F_{2m}(-\tfrac{1}{2}).$$

Here we shall apply Lemma 1.

In our case $$B_v^3 = \tau_v^3 x_{2m}^{3v}, A_v^2 = \sigma_v^2 x_{2m}^{2v}$$

So $$\lambda_v = \left(\tau_v^3 / \sigma_v^2\right)x_{2m}^v \; , \; \Lambda_n = \max_{0\le v\le n}\left(\tau_v^3 / \sigma_v^2\right)x_{2m}^v \le \frac{p_n^3}{t_n^2} \text{ and } \mu_n = \sigma = 2V_{2m} \; .$$

Therefore $$\sup_t |F_{2m}(t) - \phi(t)| \le \frac{p_n^3}{t_n^2}\frac{1}{V_{2m}}$$

Hence $$P(S_{2m}^-) = F_{2m}(-\tfrac{1}{2}) \ge \phi(-\tfrac{1}{2}) - \left|F_{2m}(-\tfrac{1}{2}) - \phi(-\tfrac{1}{2})\right| > \phi(-\tfrac{1}{2}) - \frac{p_n^3}{t_n^2}\frac{1}{V_{2m}}$$

Similarly the other probabilities can be calculated.

Therefore $P(E_m \cup F_m) \geq \left\{1-\phi\left(\tfrac{1}{2}\right)-\frac{p_n^2}{t_n^2 V_{2m}}\right\}\left\{\phi\left(-\tfrac{1}{2}\right)-\frac{p_n^3}{t_n^2 V_{2m+1}}\right\}+\left\{\phi\left(-\tfrac{1}{2}\right)-\frac{p_n^3}{t_n^2 V_{2m}}\right\}\left\{1-\phi\left(\tfrac{1}{2}\right)-\frac{p_n^3}{t_n^2 V_{2m+1}}\right\}$

It can be easily shown as in [5] that $V_m^2 > \frac{t_n^2}{4}\phi(4m+1)M_n^{4m}\left(B/A\right)e^{-1}$ when n is large (5)

So $V_m^2 > \frac{t_n^2}{4}(8m+1)M_n^{8m}(B/A)e^{-1}$

The least value of m is [k/2] +1. Hence $V_{2m} > t_n A_n$

where $A_n \to \infty$ as $n \to \infty$, since $M_n > 1$ and $8m+1 > \mu k > \mu' \varepsilon_n \log n$.

Since $\lim_{n\to\infty}\left(p_n/t_n\right)$ is finite, it follows that $\frac{p_n^3}{t_n^2 V_{2m}} < \frac{p_n^3}{t_n^3}\frac{1}{A_n}$ tends to zero as n tends to infinity.

Therefore $P(E_m \cup F_m)$ is greater than a quantity which tends to $2\phi\left(-\tfrac{1}{2}\right)\left\{1-\phi\left(\tfrac{1}{2}\right)\right\}$ as n tends to infinity.
Denote this last expression by δ .

LEMMA 3

There is a set Ω_m of measure at most $\frac{1}{m^2\beta_n^2}+\frac{16\,Ae}{B}\left(\frac{k_n}{t_n}\right)^2 \exp\left\{-(4m+1)^2 M_n^2\right\}$

such that if $\omega \notin \Omega_m$ and $n > n_0$ then $|R_m(\omega)| < V_m$ for $m = \left[k/2\right]+1, \left[k/2\right]+2, \ldots k.$

Proof of the Theorem

$$R_m(\omega) = \left(\sum_2 + \sum_3\right)\xi_v(\omega)x_m^v.$$

By Tchebycheff's inequality, we have

$$P\left\{\left|\sum_3 \xi_v(\omega)x_m^v\right| \geq \frac{1}{2}V_m\right\} \leq \frac{4k_n^2}{V_m^2}\sum_3 x_m^{2v} .$$

Proceeding as in Lemma 2 of [4], we now get that the above probability does not exceed

$$\frac{16\ Ae}{B}\left(\frac{k_n}{t_n}\right)^2 \exp\left\{-(4m+1)^2 M_n^2\right\}$$

Again, by using the same inequality

$$P\left\{\left|\sum_2 \xi_v(\omega)x_m^v\right| > m\beta_n\left(\sum_2 \sigma_2^v x_m^{2v}\right)^{1/2}\right\} < \frac{1}{m^2\beta_n^2} .$$

Thus if $\omega \notin \Omega_m$ where

$$P(\Omega_m) < \frac{1}{m^2\beta_n^2}+\frac{16Ae}{B}\left(\frac{k_n}{t_n}\right)^2 \exp\left\{-(4m+1)^2 M_n^2\right\}$$

we have $$|R_m(\omega)| < \tfrac{1}{2}V_m + m\beta_n\left(\sum_2 \sigma_v^2 x_m^{2v}\right)^{1/2}$$

Now, by using (1) and (5) and following the procedure of Lemma 2 of [4], we have

$$m\beta_n\left(\sum_2 \sigma_v^2 x_m^{2v}\right)^{1/2} < \tfrac{1}{2}V_m$$

We have shown earlier that $P(E_m \cup F_m) = \delta_m > \delta > 0$.

Let η_m be a random variable such that it takes value 1 on $E_m \cup F_m$ and zero elsewhere. In other words

$$\eta_m = \begin{cases} 1 & \text{with probability } \delta_m \\ 0 & \text{with probability } 1-\delta_m \end{cases}$$

The η_m 's are thus independent random variables with $E(\eta_m)=\delta_m$ and $V(\eta_m)=\delta_m-\delta_m^2<1$.

Let ρ_m be defined as follows:

$$\rho_m = \begin{cases} 0 & \text{if } |R_{2m}(\omega)|<V_{2m} \text{ and } |R_{2m+1}(\omega)|<V_{2m+1} \\ 1 & \text{otherwise.} \end{cases}$$

Let $\theta_m = \eta_m - \eta_m \rho_m$.

The conclusion of [4] gives that the number of roots in (x_{2m_0}, x_{2k+1}) must exceed $\sum_{m=m_0}^{k} \theta_m$

Where $m_0 = \left[\frac{1}{2}k\right]+1$

Now we appeal to Lemma 2.

We have $$\left|\sum_{m=m_0}^{k}\{\theta_m - E(\eta_m)\}\right| \le \left|\sum_{m=m_0}^{k}\{\eta_m - E(\eta_m)\}\right| + \sum_{m=m_0}^{k}\rho_m .$$

Let $A(\omega)$ be the set of ω for which

$$\sup_{k-m_0+1\ge k_0} \frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\theta_m - E(\eta_m)\}\right| > \varepsilon,$$

$B(\omega)$ be the set of ω for which

$$\sup_{k-m_0+1\ge k_0} \frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\eta_m - E(\eta_m)\}\right| > \tfrac{1}{2}\varepsilon$$

and $C(\omega)$ be the set of ω for which

$$\sup_{k-m_0+1\ge k_0} \frac{1}{k-m_0+1}\sum_{m=m_0}^{k}\rho_m > \tfrac{1}{2}\varepsilon.$$

$$E(\rho_m) = P\{(|R_{2m}|\ge V_{2m}) \cup (|R_{2m+1}| \ge V_{2m+1})\} \le P(|R_{2m}|\ge V_{2m}) + P(|R_{2m+1}|\ge V_{2m+1}).$$

By Lemma 3,

$$P(|R_{2m}|\ge V_{2m}) < \frac{1}{4m^2\beta_n^2} + \frac{16Ae}{B}\left(\frac{k_n}{t_n}\right)^2 \exp\{-(8m+1)^2 M_n^2\} < \frac{1}{m^2\beta_n^2} + \frac{16Ae}{B}\left(\frac{k_n}{t_n}\right)^2 \exp(-m^2 M_n^2).$$

Similarly

$$P(|R_{2m+1}|\ge V_{2m+1}) < \frac{1}{m^2\beta_n^2} + \frac{16Ae}{B}\left(\frac{k_n}{t_n}\right)^2 \exp(-m^2 M_n^2).$$

Hence by using (2.1), we have

$$E(\rho_m) < \frac{\mu}{m^2\beta_n^2} + \mu'\left(\frac{k_n}{t_n}\right)\exp(-m^2 M_n^2) < \mu''/(m^2\beta_n^2) < \mu''/m^2 .$$

Therefore

$$\frac{1}{k-m_0+1}\sum_{m+m_0}^{k} E(\rho_m) < \mu''/m_0^2$$

and so

$$P\{C(\omega)\} < \sum_{k-m_0+1\ge k_0} P\left\{\frac{1}{k-m_0+1}\sum_{m=m_0}^{k}\rho_m > \tfrac{1}{2}\varepsilon\right\} < \frac{2\mu''}{\varepsilon}\sum_{k-m_0+1\ge k_0}\frac{1}{m_0^2} .$$

Again by Lemma 2, we have

$$P\{B(\omega)\} < \frac{4D}{\varepsilon^2 k_0} .$$

Since

$$\sup_{k-m_0+1\geq k_0} \frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\theta_m - E(\eta_m)\}\right| \leq \sup_{k-m_0+1\geq k_0} \frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\eta_m - E(\eta_m)\}\right| + \sup_{k-m_0+1\geq k_0} \frac{1}{k-m_0+1}\sum_{m=m_0}^{k}\rho_m$$

it follows that

$$A(\omega) \subseteq B(\omega) \cup C(\omega).$$

Hence calculation as in [5] gives

$$P\{A(\omega)\} < \frac{\mu'}{k_0} < \frac{\mu''}{\varepsilon_{n_0} \log n_0} .$$

Thus if

$$\omega \notin A(\omega)$$
$$\frac{1}{k-m_0+1}\sum_{m=m_0}^{k}\theta_m > \frac{1}{k-m_0+1}\sum_{m=m_0}^{k}E(\eta_m) - \varepsilon$$

for all k such that $k - m_0 + 1 \geq k_0$.

So $$N_n > \tfrac{1}{2}(\delta-\varepsilon)k > \tfrac{1}{2}(\delta-\varepsilon)\frac{\mu_1}{\sqrt{c_1}}\varepsilon_n \log n$$

for all k such that $k - m_0 + 1 \geq k_0$ or in other words, for all $n > n_0$

Now the theorem follows by taking $C_1 = \frac{1}{4}\mu_1^2(\delta-\varepsilon)^2$

CONCLUSION

We conclude that random variables with finite variance and third absolute moment with characteristic function has infinite variance $(1<\alpha<2)$.By taking the lower bound was $\mu \log n / \log\left\{\frac{k_n}{t_n}\log n\right\}$ which is obtained by taking $\varepsilon_n = \mu/\log\left\{\frac{k_n}{t_n}\log n\right\}$ in our present result, where k_n, t_n. In the above polynomial of degree n whose coefficients are independent random variables with expectation zero with σ_v^2 be the variance and τ_v^3 be the third absolute moment of $\xi_v(\omega)$ Taking $\{\varepsilon_n\}$to be a sequence of the polynomial tending to zero, such that $\varepsilon_n^2 \log n$ tends to infinity as n tends to infinity. Hence the Asymptotic Estimates for Real Zeros of Random Polynomials $\varepsilon_n \log n$.

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