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Determination of the Search Direction in Quadratic Constrained Optimization

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ABSTRACT

This paper provides a method for computing search direction for constrained nonlinear optimization problems. We discuss some important differences in formulation and solution that arise in quadratic programming based methods for nonlinearly constrained optimization with particular emphasis on the treatment of inequality constraints. Some issues including incompatibility or ill-conditioning of the constraints determination of active set and estimation of Lagrange multipliers are discussed.

Key words: Nonlinearly constrained optimization, Active set, Lagrange multiplier estimates, Quadratic programming sub problem

INTRODUCTION

We have considered method in this paper to solve the inequality constrained nonlinear program i.e.

P.I. Minimize $f(x) \; : x \in R^n$

Subject to $c_i(x) \geq 0, \; i = 1,2,..., p$

where $f(x)$ is the objective function and $c_i(x)$ are the constraint functions which are two times continuously differentiable.. A typical iteration of a method to solve PI includes the following procedure if *x* is the current iterate.

- Compute a search direction p by solving a sub-problem.
- Determination of a step α, such that specified properties hold at $x + \alpha p$. Following these steps $x + \alpha p$ becomes the new iterate.

We shall examine and compare two extremes of QP. At one extreme, an equality constrained QP (EQP) is solved and at the other extreme the sub-problem is an inequality constrained QP (IQP). The two approaches are different in several ways. Between the two extremes, there are many variations in formulation of QP, some of which are shortly noted. We shall use the following notations throughout this paper i.e.

$g(x) = \nabla f(x), G(x) = \nabla^2 f(x), a_i(x) = \nabla c_i(x)$ and

$G(x) = \nabla^2 c_i(x), \partial(x)$ represents the set of active constraints at *x*. Let

$\partial_i(x) = \nabla \hat{C}_i(x)$ and $\hat{G}_i(x) = \nabla^2 \hat{C}_i(x)$. Matrix $A(x)$ will denote the matrix whose *i*th column is $a_i(x)$. The solution of PI is x^*. It will be supposed that the first and second order Kuhn-Tucker conditions hold at x^* i.e. there exist Lagrange multipliers $\{\lambda_i^*\}$ corresponding to the active constraints, such that

$$g(x^*) = \hat{A}(x^*)\lambda^* \qquad (1a)$$

$$\lambda_i^* \geq 0, \; i = 1,2,...,t \qquad (1b)$$

Let $Z(x)$ denote a matrix whose columns form a basis for the set of vectors orthogonal to $\hat{A}(x)$. Then the matrix

$$z(x^*)^T = c(x^*) - \sum_{i=1}^{t} \lambda^* G\left(x^*\right) z\left(x^*\right) \qquad (2)$$

is positive definite.

We close this section by describing research related to the proposed research work. The continuously rising success of interior point techniques applied to linear programming has stimulated research in various related fields. Alizadeh, Haeberly, Jarre and Overton [2] consider a problem similar to this study. Their models allow only equality constraints and no inequalities. Algorithmically these authors use mostly interior point based techniques to solve the problem. Alizadeh [1] propose a potential reduction method and shows a polynomial running time to find an ϵ-optimal solution. Jarre [17] uses a barrier approach and works directly on the dual.

There are a number of alternative active-set methods available for solving a quadratic programming problem with constraints of the methods specifically designed for convex quadratic programming, [3],[8 11],[15],[16],[18-21] and [22]. Only the methods of Boland [5] and Wong [24] are dual active set methods. The primal active-set method proposed in this study is motivated by the methods of Fletcher [12], Gould and Gill [13-14] and Wong [24], which may be viewed as methods that extend the properties of the simplex method to generate quadratic programming. Alternative approaches that use a parametric active-set method have been proposed by Best [4], Ritter [21]. The use of shifts for the bounds have been suggested by Cartis and Gould[6] in the context of interior methods that are shown to be convergent for strictly convex quadratic program have been considered by Curtis, Han and Robinson[7].

METHODS FOR QUADRATIC PROGRAMS

In order to discuss QP sub-problems we want to present a brief over view of some aspects of solving QP.

Equality Constrained QP

We consider the problem

Minimize $$\frac{1}{2} p^T H p + p^T d,\ p \in R^n \tag{3a}$$

Subject to $$\hat{A}^T p = \hat{b} \tag{3b}$$

where $\hat{A}$ matrix contains t columns, Let *r* be the rank of $\hat{A}$ and let the *r* columns of a matrix *Y* form a basis for the range space of $\hat{A}$. Similarly the (*n-r*) columns of a matrix *Z* are supposed to form a basis for the set of vectors orthogonal to the columns of $\hat{A}$.

i.e. $$\hat{A}^T Z = y^T Z = 0 \tag{4}$$

The solution of (3), p^* is given by

$$p^* = Y p_Y^* + Z p_Z^* \tag{5}$$

Using (5), (3b) gives.

$$\hat{A}^T p^* = \hat{A}^T Y p_Y^* = \hat{b} \tag{6}$$

The vector p_z* is determined by minimizing quadratic form (3a) with respect to the remaining (n-r) degrees of freedom. Using (5) into (3a), differentiating with respect to know p_Z and equating the derivative to zero, we get the linear system.

$$z^T H Z p_Z^* = -z^T d - z^T H Y p_Y^* \tag{7}$$

If $Z^T HZ$ is non singular p_Z^* is unique if $Z^T HZ$ is positive definite (5) is the desired solutions of (3). If $Z^T HZ$ is positive semi define p_Z^* is not unique. If $Z^T HZ$ is indefinite, p_Z^* defined by (5-7) is not a local minimum of (3) and the quadratic function (3a) is not bounded below.

There are many other ways to find solution of (3). The advantage of this method is that the determination of whether the solution to (3) is well defined can be made during the computation. This procedure also provides an extremely reliable estimate of the rank.

Inequality Constrained QP

Here we consider the problems as

Minimize $$\frac{1}{2} p^T H p + p^T d, \quad p \in R^n \tag{8a}$$

Subject to $$A^T p \ge b \tag{8b}$$

Where, A has m columns. In general the solution of (8) must be found by iteration. Each iteration contains the two procedures i.e. finding of a search direction and of step length. To find search direction, some subset of the constraints 8(b) at any current point say is $\tilde{p}$ considered as active set. Let $\overline{A}$ matrix contain the columns of A corresponding to the active constraints and Let $\overline{b}$ be the vector of corresponding elements of b so that

$$\overline{A}^T \overline{p} = \overline{b} \tag{9}$$

Vector $\hat{p}$ will denote the solution of EQP (8a) Let $\overline{d}$ denote the gradient of the function (8a) at $\overline{p}$ i.e. $\overline{d} = H\overline{p} + d$ If $\overline{p} \neq \hat{p}$ The search direction δ^* is the solution of EQP for next iterate, we need

$$\overline{A}^T (\overline{p} + \delta^*) = \overline{b} \tag{10}$$

Then δ^* solves,

$$\text{Minimize} \quad \frac{1}{2}\delta^T H \delta + \delta^T \overline{d} \tag{11a}$$

$$\text{Subject to} \quad \overline{A}^T \delta = 0 \tag{11b}$$

and can be used using (5), (6) and (7)

If $Z^T \overline{d} = 0, \overline{p} = \hat{p}$,the Lagrange multipliers of the EQP are the solution of the compatible system $\overline{A}\hat{\lambda} = d + H\hat{p}$ If $\hat{\lambda} > 0$ for all i, $\hat{p}$ is optimal for (8).

Nonlinearly Constrained Optimization

In this section we consider in some detail various aspects of the resulting procedures to compute search direction. In the nonlinear case, the constraints may be transformed. A linear approximation of a smooth nonlinear constraint c_i at the point *x* can be derived by Taylor's series.

$$c_i(x+p) = c_i(x) + a_i(x)^T p + \frac{1}{2} p^T G_i(x) p + O(\|p\|^3 \tag{12}$$

Using only linear terms, we have

$$c_i(x+p) \approx c_i(x) + q_i(x)^T p \tag{13}$$

Various options have been proposed for the RHS of QP constraints. The quadratic function of the sub problem is usually based on the Lagrangian function because of its essential role in the second order optimality conditions for nonlinear constraints. If a QP has only equality constraints with matrix $\hat{A}$, then its solution is unaltered if the linear term of the objective function includes a term of the form $\hat{A}$ δ. Hence, since the gradient of the Lagrangian function is $g(x) - \hat{A}(x)\lambda_i, g(x)$ alone is usually taken as the linear term of the objective function. In the case of inequality constrained QP, the solution will vary depending on whether *g*(*x*) or $g(x) - \hat{A}(x)\lambda_i$ is used as the linear term of the objective function.

Incompatible Constraints

The first difficulty that can occur in the formulation of the linear constraints of the QP is incompatibility i.e. the feasible region of the sub-problem is empty even through that of the original problem is not in practice, in-computability appears to be more likely with an IQP sub-problem, for two reasons First, by definition IQP sub-problem contains more constraints second and probably more important is the linearization of an inactive constraint represents a restriction involving the boundary of the feasible region that is made at a point for removed from the boundary.

with an EQP approach, the constraints are of the form,

$$\hat{A}^T p = \hat{d} \tag{14}$$

if (9) is incompatible the columns of $\hat{A}$ must be linearly dependent some algorithms include a flexible strategy to specify $\overline{d}$, which can be invoked to eliminate or reduce the likelihood selecting active constraints can attempt to exclude constraints whose gradients are linearly dependent. As a first step in this direction some pre-assigned strategies do not allow more than n constraints to be considered active incompatibility leads to a more complicated situation with an inequality constrained QP subproblem in the equality case, incompatibility can be determined during the process of solving,

Minimize $\left\| A^T p - \hat{d} \right\|_2^2$ and an alternative definition of p typically makes use of the quantities already computed with inequalities however, incompatibility is determined only at the end of an iterative procedure to find a feasible point. Obviously many other strategies are possible and will undoubtedly be proposed. It seems clear that there is a danger of great inefficiency with an inequality constrained QP sub problem unless the computational effort expended to discover incompatibility can be exploited in the same way as in the equality constrained case.

Conditioning of the Constraint

In the case of a pre-assigned active set strategy, the columns of $\hat{A}$ can be nearly linearly dependent. If the original constraints were linear the sub problem then would represent the intersection of the constraints or possibly x^* itself is ill defined. The effect of ill-conditioning in $\hat{A}$ on the QP is thus to make the constraints of questionable value. Usually $\|p^*{}_y\|$ becomes extremely large if if is $\hat{A}$ ill conditioned, and hence tends to dominate the search direction. Even if by chance $\|p^*{}_y\|$ is an acceptable size the reliability of $p^*{}_y$ is dubious because, by definition, small perturbation in the $\bar{A}^T p = \bar{d}$ can induce large relative changes in its solution since this eqn. provides an approximation to the desired behaviour of the nonlinear constraints, it is important to take precautions so that the entire sub problem is not invalidated. Moreover there is the danger that an algorithm will be unable to make any progress away from the neighbourhood in which the ill-conditioning is present.

Determining Active Set

In the nonlinear case, the active constraints at the solution are usually satisfied exactly only in the limit and hence other criteria must be employed. Any method based on the Lagrangian function includes some decisions about the active set in defining which constraints correspond to non-zero multipliers with a QP-assigned active set strategy the Lagrange multipliers from the IQP sub problem at the previous iteration determine the selection of the active set, in the sense that the set of active linear constraints at the solution of the QP is equivalent to the set of active nonlinear constraints at the solution of the original problem. Almost any sensible set of criteria will predict the active set correctly in a small neighbourhood of the solution under the assumption required to guarantee a correct prediction for the IQP sub problem.

The justification for any active set strategy arises from its reliability when the current iterate is not in a small neighbourhood of the solution. Since the prediction of the active set influences the logic of either an EQP or IQP method. It is advisable to include consistency the prediction to be reliable.

LAGRANGE MULTIPLIER ESTIMATE

Lagrange multiplier estimates are used within the EQP approaches in two ways i.e. and approximation to the Lagrangian function must be constructed and many pre-assigned active set strategies consider multiplier estimates in selecting the active set since $g(x)$ and $\hat{A}(x)$ are evaluated before defining the approximation to the Lagrangian function at x, a first order Lagrange multiplier estimate can be computed as the solution of the least square problem.

$$\text{Minimize} \quad \left\|\hat{A}(x)\lambda - g(x)_2^2\right\| \tag{15}$$

An alternate estimate which allows for the fact that $\|\bar{c}(x)\|$ is not zero is given by

$$\lambda = \lambda - \left[\left(\hat{A}\right)^T \hat{A}\right]^{-1} \hat{c} \tag{16}$$

where λ_1 is the solution of (10) and $\hat{c}$ and $\hat{A}$ are evaluated at x. When using the IQP approach, the QP multipliers from the previous iteration are used to define the new quadratic approximation to the Lagrangian function. In this way, the QP multipliers can be interpreted as providing a prediction of the active set in the sense that they define which constraints are included in and which are omitted from the Lagrangian function with either EQP or IQP sub problem the quality of the estimate critically depends not only on the correctness of the active set but also on how well H approximates the Hessian of the Lagrangian function. The right hand side of $\bar{A}\lambda = Hp^* + g$ can be interpreted as a prediction of the Lagrangian function at $x + p^*$ and the value of the estimate from the above mentioned eqn. hence depends on the fact that $x + p^*$ is a better point than *x*. consequently the value of the estimate is questionable when a unit step is not taken along p^*, which is often the case except very near the solution.

CONCLUSION

As a consequence computing the search direction by solving a QP is not assured for all algorithms in constrained optimization. Certainly a QP based formulation overcomes some of the disadvantage of alternative methods and QP based method typically work extremely well in the neighborhood of the solution. It may be through that the greater effort involved in solving an IQP would cause less major iteration to be required to solve the original problem. The main objective of this paper has been to point considerations of importance in evaluating any proposed method that included a QP.

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