Understanding of Selected Geometric Concepts by Pupils of Pre-Primary and Primary Level Education

Ján Gunčaga [b, *], Štefan Tkačik [a], Katarína Žilková [b]

[a] Department of Mathematics, Faculty of Education, Catholic University in Ružomberok, Slovakia
[b] Department of Science Education, Faculty of Education, Comenius University in Bratislava, Slovakia

Abstract

Misconceptions in geometry are an essential problem in the understanding of geometric terms by primary and pre-primary aged children. Present research shows some misconceptions in geometry demonstrated in the understanding of circles, squares, triangles and oblongs for children in the last year of kindergarten and pupils in the last year of primary school. The research methodology is based on the Van Hiele's theory of understanding geometric thinking. Qualitative analysis of videotapes with recorded children's activities was the method used for pre-primary children. Whereas for the examination of 4th graders, a quantitative analysis was conducted via student-completed tests. Pupils' common misconceptions are shown and identified in our research. Based on these findings, recommendations for best pedagogical practice are suggested for teachers in primary and pre-primary levels and also for the teacher training of pre-service teachers.

Keywords: geometry, misconceptions, Van Hiele levels, pupils in primary and pre-primary level.

1. Introduction

The theory of education includes a development of examined, planned and realized activities in pre-school children and elementary school children with an aim to develop the mathematical imagination. Cognitive processes in mathematics are divided into several exact levels. These levels are structured by different patterns and they mostly reflect general pedagogical and/or psychological theories within the developmental psychology field (e.g. J. Piaget, L. S. Vygotskij, J. Bruner, etc.). A concrete description of each stage of the cognitive process in mathematics was mentioned and published by Hejný et al. (2006). In terms of the development of geometric

---

* Corresponding author
E-mail addresses: jguncaga@gmail.com (J. Gunčaga)

thinking, a well-known theory published by Dutch mathematician, Van Hiele (1986); Van Hiele (1999), revealed a definition of key levels typical of each stage of interpretation of geometry terms and characteristics.

The aim of the research is to present a pilot experience of research verification of Van Hiele's theory under the socio-cultural conditions of Slovakia. As a part of the national program VEGA 1/0440/15, named "Geometric Conceptions and Misconceptions of Preschool and School Age Children", we examined the thinking of pre-primary and primary aged children about geometrical shapes. The main purpose of the project was to recognize the conceptions of the pre-primary and younger aged children about geometrical shapes, recognize their characteristics for the shapes and subsequently identify the potential misconceptions. Moreover, we aimed to identify the levels of geometric knowledge in children based on the definition published by Van Hiele and essentially specify relevant and present levels (especially the levels of visualization and analysis) based on the information obtained from research analysis.

We present the partial results of the understanding of elementary geometrical shapes by pre-school children (5-6 year olds) and students attending the fourth year of elementary school (10 year olds).

2. The theoretical basis

The main theoretical platform for our research was not only pedagogical-psychological knowledge of children's interpretations of the world, but also the knowledge of geometric thinking development based on the Van Hiele's theory.

Theory by the Dutch mathematics teachers Pierre Van Hiele and his wife Diana Van Hiele-Geldof has many defenders in the world but criticism towards the theory has also emerged. The experts in the field are trying to identify different levels of geometric knowledge in children from the youngest age until the correct abstract levels and axiomatic geometric conceptions are reached. Different scientific studies show a big interest in this area by experts. Mostly, the research is focused on knowledge about the levels of the pupils of primary and pre-service primary teachers (Contay, Duatepe Paksu, 2012; Fujita, Jones, 2006; Marchis, 2012). Levels of geometric knowledge of primary students in Slovakia were examined by the submitter of the present study (Žilková, 2013). The results of the research correspond with results from similar research in Romania, Turkey and Scotland.

The perception or geometrical shape thinking of pre-school and younger children in Slovakia has not been scientifically explored. This is the main reason we have decided to carry out the research where the main theoretical source is the Van Hiele's theory. We were focused on the identification, distinction and selection of geometrical shape characteristics. We assumed that children's thinking would correspond with the first two Van Hiele levels.

The characteristics of all five Van Hiele levels will be mentioned shortly. Each level of cognitive process is characterized by a few important attributes that will help us to determine the cognitive level of the examined subject and to identify its perception of elementary geometric concepts.

A lot of authors (e.g. Usiskin, 1982; Musser et al., 2001; De Villiers, 2010, etc.) published different names for the levels, however, the main characteristics are the same. Therefore, based on the published materials we compiled the following names and characteristics:

Level 1 (Visualization)

In the first of the Van Hiele levels, children/pupils should be able to think about the geometric shapes based on the overall look of the shape. At this point children identify shapes because of the holistic view rather than thinking of any distinct characteristics. This level can be analysed through verification of the identification abilities of geometrical shapes according to a picture.

Level 2 (Analysis)

The second Van Hiele level, also known as descriptive-analytical, is characterized by the ability of a pupil to think only about the characteristics of the geometrical shape, mostly only about the important elements of the shape. The visual perception and comparison with its prototype is too simplistic for pupils at this stage. They describe the shape by pointing out the important attributes. However, they are not able to choose the most important attributes that will be

characteristic for the next stage of geometric thinking. Therefore, a very common answer used by children to the question of why the concrete shape is a square is that it has four angles and four sides. For younger children, their answer is that it has four corners (vertices) and four sides.

Level 3 (Abstraction)

The following Van Hiele level is often called the abstraction or relationship level. At this stage, the pupils think about the characteristics of the shape in a more general and complex way. They are able to formulate simple and abstract definitions. These pupils are able to see a difference between necessary and sufficient requirements for a concept and they have the ability to understand and create logical arguments. Pupils have the ability to classify shapes hierarchically based on the analysis of their characteristics and they are able to formulate informal arguments for the reasoning of their classification. For example, pupils are able to understand a square as a rhombus with some "special characteristics" and therefore, they understand inclusive relationships between types of shapes.

Children understand that deductive reasoning can be useful when explaining relationships that are not obvious, but the need to use deduction for verification is still missing. For example, pupils are able to conclude that in every quadrilateral the sum of the inner angles is 360° with the reasoning that, therefore, every quadrilateral can be divided into two triangles with the sum of its inner angles equal to 180°.

Level 4 (Deduction)

The fourth Van Hiele level is known as a formal deduction. Pupils are able to discover and formulate simple formal evidence of Euclidean geometry and therefore, they are able to prove the validity of claims within the axiomatic system. Children understand the differences between undefined term, definition, axiom and sentence. They are able to create a proof by using a method of creating a sequence of claims that logically explains the result as a consequence of given assumptions about the properties. However, the axiomatic system is understood as consistent and unchanging and cannot be imagined by children as non-Euclidean representations of geometric shapes.

Level 5 (Rigor)

The last Van Hiele level is known as axiomatization. Pupils at this level think formally about mathematical systems, they study geometry without concrete reference models and they are able to compare different systems such as Euclidian and non-Euclidian geometry. At this level, the geometry is understood as an axiomatic system in the broadest sense and the student's knowledge is not bound to this level. We place emphasis on the adaptation of the methods of school geometry to make children accept other geometrical representations and think abstractly about systems later on, not to make them see the Euclidian geometry as the only existing geometry.

We assumed that pre-school children and 4th grade students at elementary school would mostly show signs of the first two Van Hiele levels (visualization and analysis). We tried to map specific signs of these levels in our conditions. For the pre-school children, these signs were mainly the language specifics in their description of the shape and its characteristics as was at a level with the development of their first language. Of 4th grade students, more could be expected as the introduction of (national) professional terminology could exert an influence on the students' ability to think at the next Van Hiele level.

3. Objectives and methods of the research

The aim of the research was to recognize the ideations and geometric misconceptions of pre-school children and 10 year old pupils about the shapes and their characteristics. The mission was to seek out the evidence of misconceptions in children and present an argument for the needs of deeper analysis in the context of national curricular documents and educational approach in the field of geometric shapes and their characteristics.

The research sample consisted of two groups of children. One group contained children of pre-school age (4½ to 6 years) and the other group consisted of children attending their 4th year in elementary school (approximately 10 years old). The methodology of the research was adjusted in both groups according to their age.

The research method of pre-school children was a semi-structured interview recorded to a video file. The aim of the interview was to find out whether children knew simple shape terminology (triangle, square, oblong, and circle) and whether they were able to identify the shapes

based on their visual properties. Moreover, the questions were: How do children distinguish between different types of shapes that are in different positions or sizes? How do children assign the shape to a concrete category? The research tool was a collection of real models (manipulation tools) of the planar geometric shapes and printed templates with drawn models of the planar geometric shapes. More than 53 videos were analysed and recorded from November 2015 until January 2016. The results are described in the fourth part of the research.

The research method used on the 4th grade pupils was an unstandardized test based on the prescribed curricular standards (national educational program) in geometry that was created to fit the Van Hiele levels of geometric thinking. The sample consisted of 80 children from one chosen elementary school and the data collection took place between February and March 2017. For the purposes of the research, only the results of the items that concerned the same shapes as examined in pre-school children were published (see the 5th part of the research). Therefore, the analysis of the tasks which were mainly focused on the identification of the elementary shapes (e.g. triangle, square, oblong and circle) was ultimately published. The aim was to find out what types of misconceptions were present in the pupils and how that compared with the level of imagination in pre-school children. Mutual comparison will allow identification, not only of the most common misconceptions, but also an approximation of their intensity.

4. Analysis of the results of pre-school children

In this part of the study, the results of the pre-school children are presented. The sample consisted of 53 children from four and a half to six years old. All of the children were about to enter primary school and the indirect observed via a semi-structured interview. The analysis was applied directly to the video file coding and a coding control was implemented based on the written transcript. Both qualitative and quantitative phenomena and their categories were observed. From the qualitative point of view, language tools that were most commonly used by children when describing a geometric figure and other verbal comments were observed. Quantitative analysis was focused on the rate of success in identification of the shape.

Children's perception of four shapes (triangle, square, oblong and circle) was observed and evaluated. These shapes were chosen because they are part of the national standards defined for graduates of pre-primary education. We were focused on two aspects: the ability to define a shape that provides a clear assignment of the shape name to its model and vice versa.

Therefore, two identification skills were observed: model of the shape => name of the shape, and name of the shape => model of the shape. When attempting to elicit the name of the shape from the child, the most common question asked was: "What type of shape are you holding in your hand?" or "What can you see in the picture?" and so on. When hoping to elicit the physical, instead of verbal, identification of the shape, questions such as "Could you show us all squares that are in the picture?" or "Choose a triangle" and so on were asked. In the research activity we used two types of research tools: paper or wooden models of the geometric shapes (see Fig. 1) and printed models and non-models of the shapes in different sizes and positions (see Fig. 2).

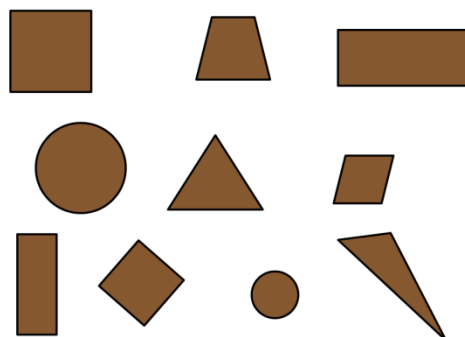**Fig. 1.** Models of the geometric shapes

Fig. 2. Models and non-models of the shapes in different positions

It was assumed that manipulation with models of the shapes would help children with the process of identification because it gives children an opportunity to rotate shapes until it evokes a

mental prototype for the shape. The set of the models contained different colours and sizes of shapes. Models of triangles and oblongs were placed in the set in different forms.

All of the examined children were familiar with the name "oblong" and they were able to select all the models of the oblongs from the set. However, the set didn't contain models of oblongs with too big of differences between adjacent sides. The name "circle" was recognized by children as well and only two children named circle as a sphere or spherule. Children of this age tend to label shapes as spatial shapes and it comes from previous experiences of manipulation with spatial shapes. This problem appeared to be much bigger when children were asked to name model of squares. However, the result can be considered very successful because 75 % of children correctly named models of squares and only 7 children identified models of squares as cubes. Models of triangles were named correctly by 80 % of the sample, where 6 children could not pick all of the triangles from the set of geometric shapes. The reason these children struggled may be due to variations in the shape of triangles especially in those triangles that didn't quite resemble the triangle prototype of an equilateral triangle.

The most difficult task for a child in shape identification is to identify/volunteer the name of a shape that is a picture on paper because then the child is not allowed to manipulate it. In the pictures used for the research, there were not just models of the 4 basic shapes but trapezoids and rhombi as well. The aim of the task was not to name these shapes because that is not included as part of the Slovak national curriculum. The other shapes were integrated into the research tool to help find out whether a child can unambiguously select 4 basic shapes.

Since the circle is a shape that can be distinguished from any angle regardless of its orientation on paper, the results for this shape were not a surprise. 46 of the children used the correct term and correctly selected two models of a circle in the picture. Two children didn't know the correct term for this shape, therefore, they named it a hole or a wheel.

The name "oblong", on the other hand, while recognized by all of the children in the real model task, in printed pictures, the rate of success dropped to 83 %. (The oblong being equally displayed in both vertical and horizontal positions.) It is clear that the manipulation of the shapes made the cognitive process much easier for pre-school children and therefore, in primary teaching, the manipulation of objects is to be recommended for its efficacy.

Squares in a standard position (with sides running horizontally and vertically) were named correctly by 38 children. In a rotated position ◇ the ability to name this shape decreased: only 29 children were able to recognize it as a square. Mostly, the children tended to name the square as a cube. For the square in a rotated position, 2 children used the term "diamond" and one child used the term "triangle". This exchange (square vs. cube) is quite common in Slovakia; not only in children, but in adults as well. The squared paper is very often called cubed paper and a checked shirt is often called a cubed shirt. These terms are incorrect but people use these terms in such a commonplace manner that they do not realize the inherent incorrectness. This is part of the reason younger children have problems with identifying squares.

In the identification of other geometric shapes, 10 children named the model of a trapezoid as a square and one child called it a cube. Similarly, the rhombus was named as a square by 18 children and in 6 cases as a cube.

The triangle was illustrated in the picture in two forms. One of them was an equilateral triangle in the standard position (prototype) and 88 % of children named the triangle in this form correctly. Second type, the obtuse triangle, differed not only in its type but in position as well. This type of a triangle was successfully named by 24 children.

From the qualitative point of view we introduce the most frequent words used by children when they didn't know the correct name for the shape:
- Triangle (obtuse): kite, peak;
- Square (in basic position): cube;
- Square (rotated): cube, diamond, triangle;
- Oblongs: square, triangle, ribbon;
- Circle: wheel, small wheel, hole.

Even the fact, that the aim of the study was not to find out the ability to name the trapezoid or rhombus, we also include the words that most often children used for the identification of the

shapes. The trapezoid was mostly called: skirt, reversed flowerpot, chimney, bucket and also square.

All of the things listed above correspond to the typical characteristics of the first Van Hiele level. It is specific to our socio-cultural conditions that instead of the word "square", the word "cube" is frequently used. It is difficult to determine whether it is simply language-specific (even though incorrectly so) or if it is a stable misconception that persists from childhood to adulthood and trickles down from the older generations to the younger one.

To verify the ability of pre-school children to define shapes with their true terminology, other printed picture templates were used (Fig. 3 – Fig. 6). For every shape one template was used with not only the models but also non-models of the shape. This variety in templates allowed the effect of the location and shape diversity to be observed as well, to see whether the sample of children would perceive the shape only holistically or if they would show signs of the next Van Hiele level.

The questions we were asking children: *Show all of the squares in the picture (triangles, oblongs, circles). How many squares can you see in the picture? Is this shape a square as well? Why is this a square? Why is this not a square?* Every interview was unique and it was modified to suit to each child's cognitive functions and communication skills.

The results will be reported according to the shapes and in the sequence that these shapes were presented to the children on printed templates (Fig. 3 – Fig. 6).
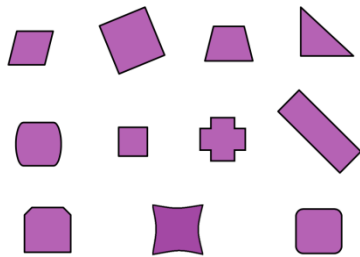
Fig. 3. Template of models and non-models for a square identification
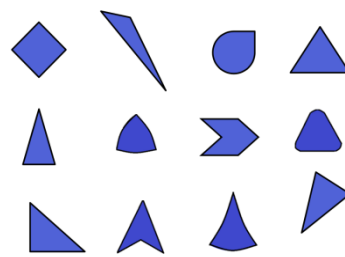
Fig. 4. Template of models and non-models for a triangle identification
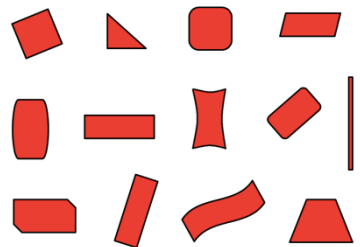
Fig. 5. Template of models and non-models for an oblong identification
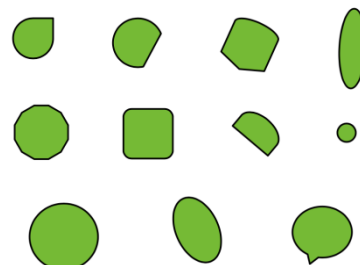
Fig. 6. Template of models and non-models for a circle identification

Models and non-models of squares

The template (Fig. 3) contains only two models of a square, one in the standard position and one moderately rotated. Other shapes were non-models of the square, although some of them had a form of a square. Quantified results of the answers of the children are listed in the table number 1.

The model of a square in its standard position was named correctly by 85 % of the children, but the rotated version was identified by 93 % of children. The results are remarkable, as it is inconsistent with previous results and theory. This disproportion can be explained by the fact that a square (Fig. 3) in a standard position was small, and therefore, the children didn't notice it and didn't react to it. Another reason might be that square is rotated only slightly and it doesn't have diagonals in a horizontal and vertical position. This magnitude of a rotation is possibly acceptable for a child because the shape evokes a square enough.

Non-models of squares can be divided into two groups. Non-models of squares that holistically evoke a square, but don't meet at least one of the defined characteristics are in the first

group. Mostly it is a deformation of the sides of the shape or deformation in the vertices. The second group of the non-models consists of triangles, oblongs, trapezoid and rhombus. We assumed that identification of the first group as a group of non-models would be much easier. This assumption was supported by our research, with the exception of rhombus identification.

Our findings showed that in first examined non-model group the biggest problem for pre-school children was to identify a shape with a square form with rounded vertices. 83 % of children identified this shape as a square, which is quite natural for children of this age. The reason is that children perceive the shape holistically and significant elements (such as sides and vertices) are neglected, or even ignored.

Table 1. Identification of squares

squares	success rate	nonmodels (1. group)	success rate	nonmodels (2. group)	success rate
	92,59%		96,30%		88,00%
	85,19%		75,00%		85,19%
			54,17%		80,00%
			45,83%		30,43%
			17,39%		

The most common arguments children used for reasoning for their identification were following characteristics of the square: "it must have 4 identical sides", or "can't be rounded" and they tended to compare it to the cube.

Models and non-models of triangles

Five models of triangles were illustrated on a template (Fig. 4): equilateral triangle, right triangle, 2 isosceles triangles (one of them in standard position, with baseline in horizontal position) and one obtuse triangle. The other shapes were not triangles. The results of the identification of the shapes are stated in table 2.

Table 2. Identification of triangles

triangles	success rate	nonmodels	success rate
	96,15%		100,00%
	96,00%		87,50%
	96,00%		84,00%
	88,46%		70,83%
	61,54%		62,50%
			54,17%
			36,36%

The observable reasoning in children for assigning the shape to the category of triangles were: "it is a triangle because it looks like a roof", "it reminds me of a triangle" or "it is peaked". Therefore, we conclude that similarity of a shape to its prototype is the most important thing for a child when it comes to reasoning. Children ignore details of a shape and they perceive shapes holistically. Most common reasons for rejection to assign the shape to triangle category where: "it is not a triangle because it is too much peaked" or "a triangle must have identical this (sides)". Some of these children accurately identified non-models of triangles and commented that "a triangle can't have a curve". This argument is typical of a higher level of visualization because significant aspects of a triangle were noticed.

Models and non-models of oblongs

Three models of oblongs were on a template (Fig. 5). The other shapes were non-models of oblongs. It is obvious from the results (Table 3) that the arrangement corresponds with perception of the oblongs and therefore it is typical of a high visualization level. Half of the children didn't correctly identify the third model of oblong as an oblong with a reason that "it can't be an oblong because it is a bar" or "a stick", "it is too thin" or "narrow", "it is a path", "line".

Table 3. Identification of oblongs

oblongs	success rate	nonmodels	success rate
	100,00%		100,00%
	96,15%		100,00%
	50,00%		95,65%
			92,31%
			62,50%
			60,87%
			52,17%
			39,13%
			37,50%
			21,74%

The biggest problem in this group of non-models of the oblongs (Table 3) was to identify a rhomboid that holistically looked like an oblong. Approximately 78 % of children identified a rhomboid as an oblong. Triangles and squares were easily and correctly identified as non-models of oblongs and they didn't have any problem with their identification.

Models and non-models of circles

Only two models of circles were situated on a template (Fig. 6) and they differed in size. A model of a circle with a larger radius was identifiable for all of the children. The children's ability to identify a smaller circle was reduced. Only 81 % of children identified a smaller circle. Therefore, *the size of a circle may have an influence on the ability to identify a shape.*

Table 4 shows the rate of success in identification of non-models of circles. The biggest problems were to identify ellipses and a regular dodecagon because these shapes were often considered to be circles. These are the shapes that resembled circles and therefore, they identified these as circles. As an interesting fact we reveal a reasoning of one child why an ellipse is not a circle: "this is not a circle because it is rectangular circle". The comparison of a shape with a prototype of a circle is evident here, moreover, the details of a shape are noticed which is characteristic for another level of cognition.

Table 4. Identification of circles

circle	success rate	nonmodels	success rate
	100,00%		100,00%
	80,77%		95,83%
			95,83%
			87,50%
			76,00%
			75,00%
			60,87%
			57,69%
			57,69%

We assume that identification of circles is the easiest for pre-school children. A circle is a shape that they come into a contact with at this age and it is not angular. That is why children justified an identification of a circle like this: "because it is rounded", "because it is bulbed" "can't have any peak", "it is like a wheel (marble, hole)". The final summary of the results with other notes from a video file analysis in a more general level is shown in Section 6.

5. The analysis of results of 4th graders

In this part we are focusing on the analysis of responses of 4th graders – pupils the age of 10. We interpreted the responses to the questions of the knowledge test from several points of view. These are the percentages correct answers in the sub-questions, the coefficient of correlation and implication graph obtained using the statistical program C.H.I.C. This graph enables us to find implicative logical relationships between the sub-answers, where the intensity of the relationship is expressed by the colour of the arrow in the graph. The values of correlation coefficient were relatively small, so it didn't indicate stronger relations between the items of the test in terms of regression analysis.

The results – task 1

The first task in the test was focused on the identification of the shape in the picture and if the pupils know correct names of the shapes.

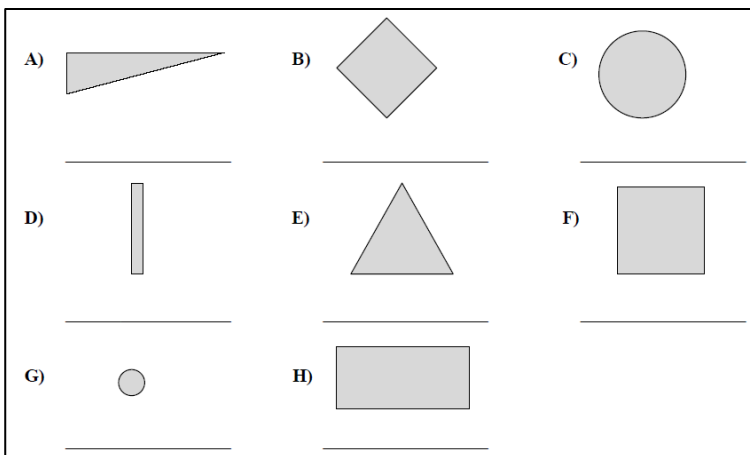*Task 1:* For each planar shape write its correct name (see Fig. 7).

Fig. 7. Plane geometrical figures for the task 1

The percentages of correct answers of 4th graders in this task are illustrated in the following picture (Fig. 8).
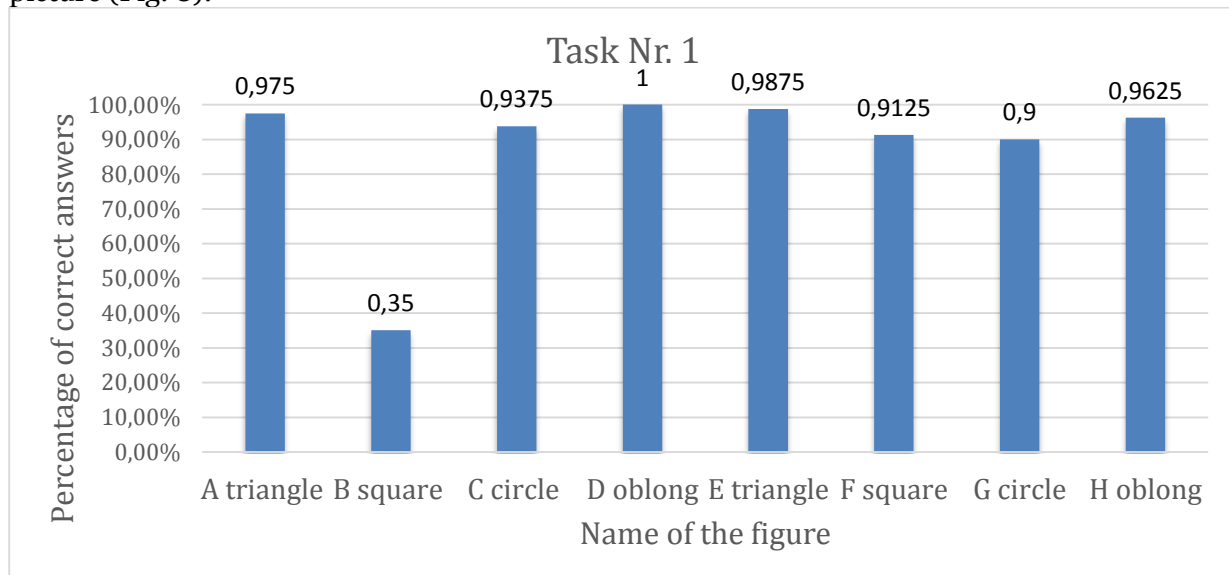

Fig. 8. Percentage of correct answers for the task 1

The least correct answers are for square ◇ (B). The reason is that the sides of a given square were not in the position as in the square ☐ (F). The square is not in the standard position, the angle of rotation is 45°and therefore pupils had problems with its identification. This phenomenon was expected but not to this extent. The success rate of 4th graders at the identification of rotated square is only at the level of 35 %, which is considered to be extremely low. When naming this shape pupils shown only the signs of the level of visualization. Interestingly, the most of correct answers in the first task is for the item (D) – more than for item (H), although both are rectangles. It is assumed that the reason is the ratio of the sides of an oblong as it is more significant for the item (D) and thus this item corresponds more to the pupils' prototype of an oblong. For the 4th graders the identification of triangles and oblongs was easier than the identification of squares, where the rotated square significantly decreased the success rate as the pupils often called it other names.

The implicative graph for the items in this task (Fig. 9) shows mutual relationships between the pupils' answers to the presented items and their strength.
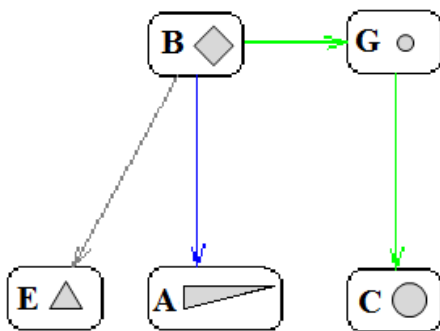

Fig. 9. The implicate graph for the items in the task 1 – the strength of implication: blue arrows 70 %, green arrows 60 %, grey arrows 50 %

From the implicative graph follows that if a pupil correctly names the item B, he also correctly named item A. A similar relationship exists between items B, G and C. This means that if the pupil correctly identified the square in non-standard position, he also correctly identified other shapes. From the result it is clear that the most difficult shape for pupils to identify, respectively to name, was the rotated square. The strongest implicative relationship is between square and triangle in non-standard positions. This result confirms the theory that the prototype shapes are easier identified and thus the shapes in non-standard positions cause problems for the pupils.

The results – task 2
The aim of the second task was to find out the ability of 4th graders to distinguish between models and non-models of a circle with the help of printed pictures.
Task 2: Mark whether the following shapes are circles (see Fig. 10). Circle the correct answer.
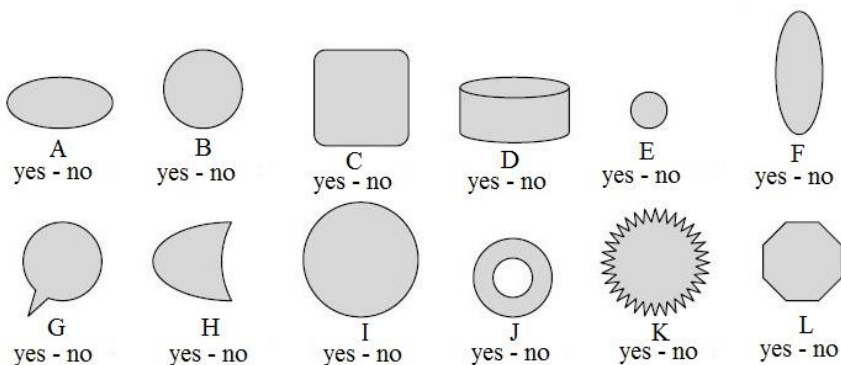

Fig. 10. Plane geometrical figures for the task 2

The results of answers to task 2, and specifically the percentage of the correct answers, are displayed in Fig. 11.
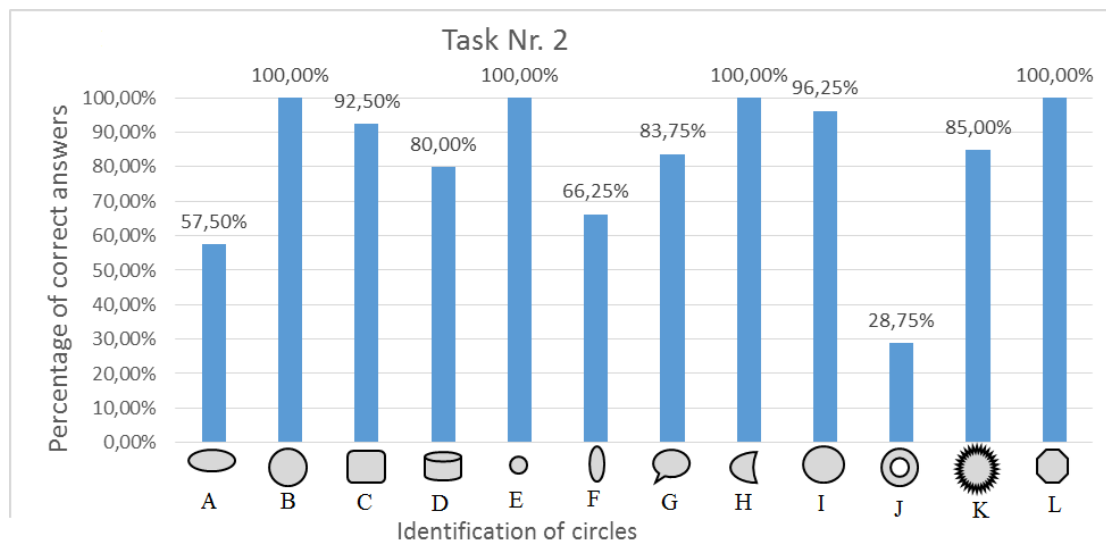

Fig. 11. Percentage of correct answers for task 2

The students were the least successful at identification of shapes A, F, J. The lowest success rate was determined in the item J, which is a very specific shape. With respect to the indisputableness of the displayed shape and low success rate we recommend to omit the shape J from the measuring tool in the future. Another reason is that it doesn't provide sufficient informative value and distinguishing ability. Very low success rate in the shapes of elliptic form (A and F) means that the pupils do not sufficiently distinguish between circle and ellipse. We assume that this phenomenon is caused by the absence of the shapes of similar type in the textbooks and workbooks of mathematics in the primary education. On the other hand, 4th graders managed to safely identify models of circles (B and E), with a slightly lower success rate than the biggest model of circle I. The result can be influenced by either the lack of concentration of the pupils or the size of the model, and we are not able to determine whether it has some effect on the result. In terms of the easiest identifiable non-models of circles we noticed H and L as the shapes with the highest success rate.

The following implicative graph illustrates the relationships between shapes created according to the pupils' answers (Fig. 12).

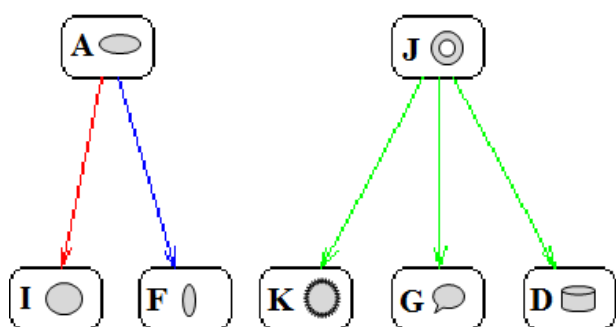**Fig. 12.** The implicative graph for the items in task 2 – the strength of implication: red arrows 90 %, blue arrows 80 %, green arrow 70 %

The results displayed with the help of implicative graph showed that there is relatively strong implicative relationship between items A => I, A => F. From this relationship and from the position of the items in the graph it is clear that the most difficult shape for identification for pupils

was elliptically formed non-model of a circle A. Its success rate in identification also determines the success rate in identification of shapes I and F. A similar but not so strong implicative relationship exists between items J, K, G and D where the most difficult from this group is the shape J described above. All of the shapes in this group are non-models of circles. Therefore, we assume that the absence of tasks designated to distinguish between models and non-models of circles (examples and counterexamples) in the textbooks of mathematics for primary education can determine the ability of pupils to improve their imagination about circles. This fact should be important for the teacher of mathematics to perform activities in math education in which the pupil can obtain personal experiences with models and non-models of the shapes.

The results – task 3

The aim of the third task was to find out the ability of 4th graders to distinguish between models and non-models of squares with the help of printed pictures.

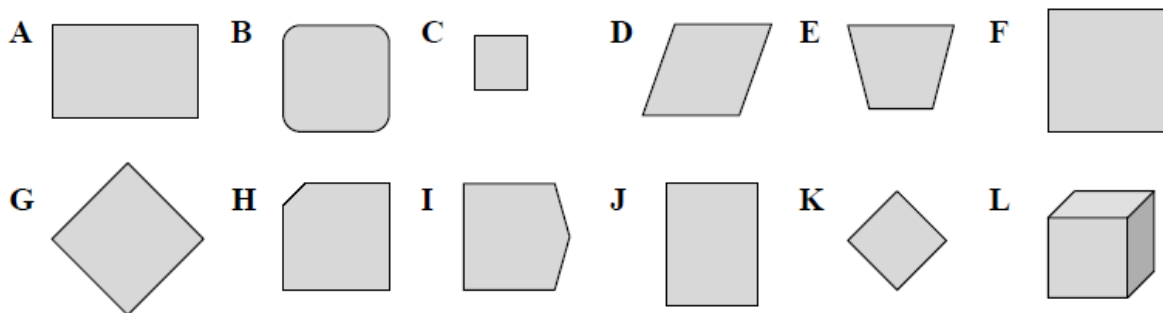*Task 3:* Which shapes are not squares? Circle the letters. (see Fig. 13)

Fig. 13. Plane geometrical figures for the task 3

The results of the answers of 4th grader, respectively the success rate of correct answers in this task can be found in the Fig. 14.
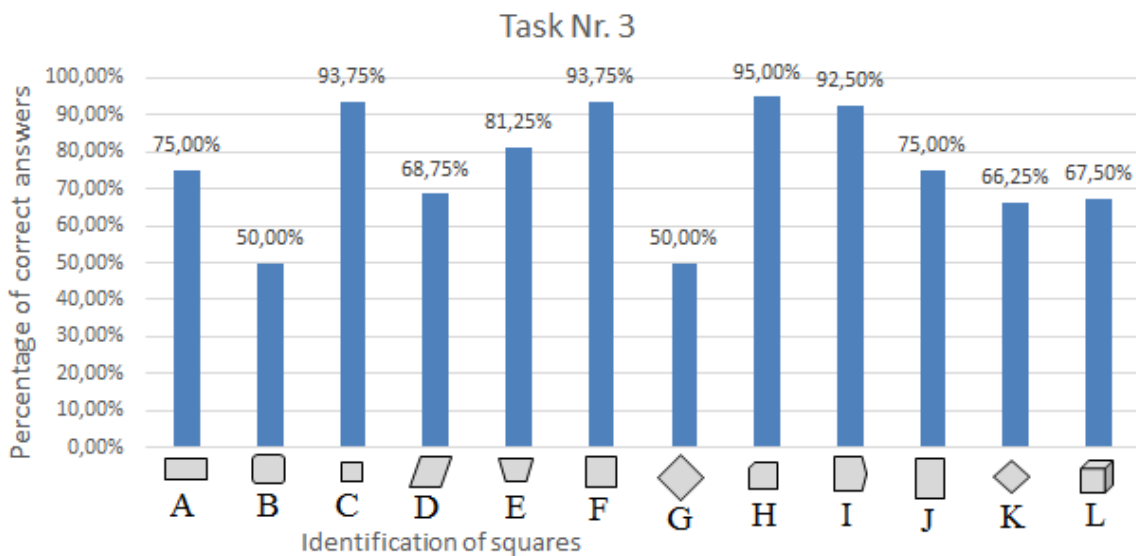
Fig. 14. Percentage of correct answers for the task 3

The most problematic representations of models and non-models of squares were shapes B, G and K. The ability to identify shape B is not to perceive the shape holistically but to notice its important elements. From this it's clear that half of the sample does not have correct visualization about squares or its important elements, which are vertices and sides. The shape B is perceived as a shape with squared form but not as a square. Half of the 4th graders still perceive the square holistically thus only at the level of visualization.

Similarly to task 1, the shapes G and K are models of a square rotated by 45°. The change in position of models of squares caused problems with their identification. Similarly, there was problem with the identification of shape L that represents spatial shape – a cube. A relatively big part of pupils (32 %) denoted a cube as a square.

Pupils didn't have problems with the identification of squares with sides in horizontally vertical position and they correctly identified small and big square (C and F).

The experience from earlier stages of education showed that pupils often denote the rhombus (shape D) as a model of a square. Looking at 4th graders' success rate of this shape (68.75 %), we find out that these misconceptions are still present. One of the reasons could be the fact that pupils do not encounter the concept of angle, especially right angle, within the primary math education in Slovakia. There are some representations of squares in a square grid, however the pupils of 4th grade are probably missing the experience of using it to compare shapes or detect perpendicularity of the shape's sides.

The implicative graph of results of tasks about models and non-models of square is in figure 15. A special group is formed from the shapes G and H, which were mentioned earlier. The question is whether the size of a shape really plays a role in correct identification of square because from the picture it is clear that the bigger rotated square was more difficult for identification than the smaller one.
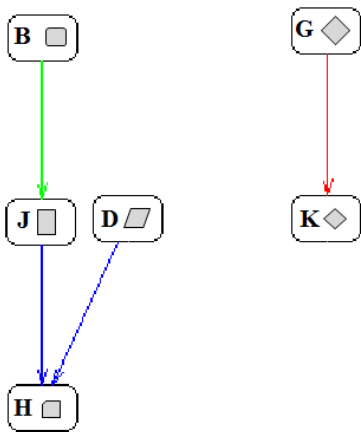

Fig. 15. The implicative graph for the items in task 3 – the strength of implication: red arrows 80 %, blue arrows 70 %, green arrows 60 %

Other relationships found in the implicative graph were in the group of shapes B, J, D and H. Again, all these shapes are non-models of a square. Shape D does not have internal angles right, shape J does not have all sides equilateral and shape H is in fact a pentagon. This means that each of them has a different property, which excludes it as a model of square. The easiest shape from these for identification was shape H and the most difficult was shape B.

It was shown that 4th graders do not know the important elements and basic properties of a square and their decisions on identification of models and non-models of squares are based only on their holistic perception.

The results – task 4

The aim of the fourth task was to find out the ability of 4th graders to distinguish between models and non-models of triangles with the help of printed pictures.

Task 4: Choose the shapes in the picture that are triangles (see Fig. 16). Circle the correct answer.
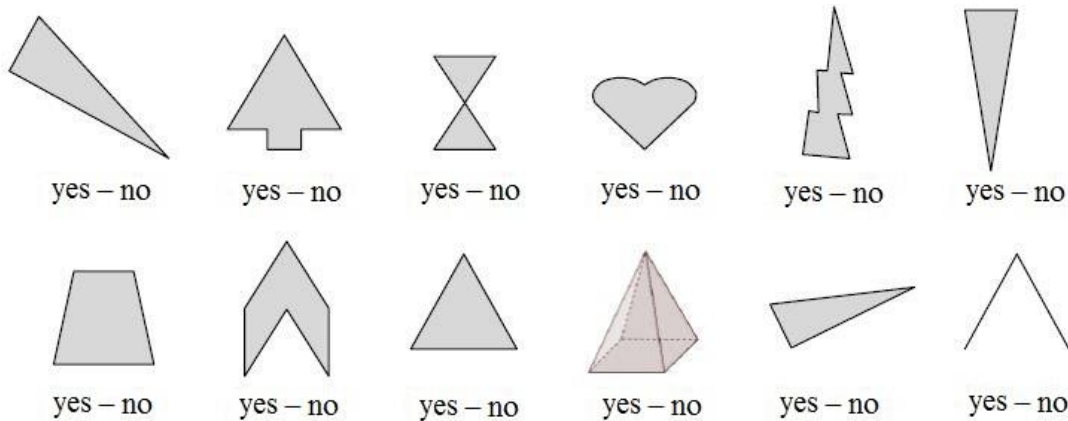
Fig. 16. Plane geometrical figures for task 4

The success rate of the 4th graders in task 4 about identification of models and non-models of triangles is illustrated in the Fig. 17.
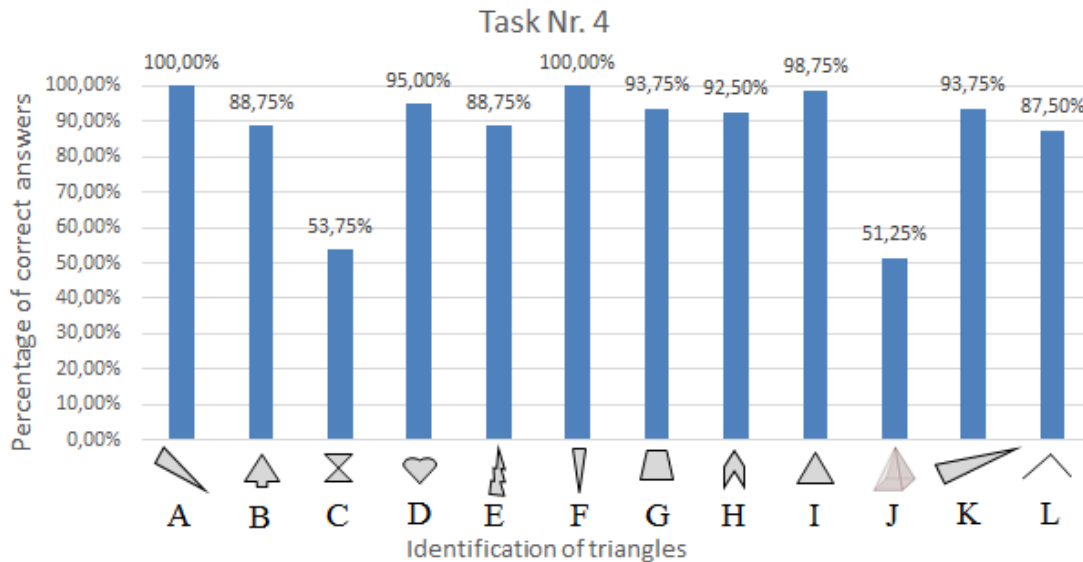

Fig. 17. Percentage of correct answers for task 4

The lowest success rate was for the shapes C and J. It is not possible to provide a relevant interpretation for the shape C as the results can be biased – pupils could perceive this shape as two triangles and therefore denoted them as models of triangles. Hence, this shape will be eliminated from the research tool in the next research. Relatively surprising information was the frequent identification of planar shape J as a model of a triangle. The fact that the faces of the pyramid are triangles could confuse the pupils.

Almost every 4th grader correctly identified models of triangles A, F and I. These shapes clearly illustrated a triangle as they presented a model of polygon with three sides and three vertices.

The implicative graph for the items of this task is illustrated in the Fig. 18.
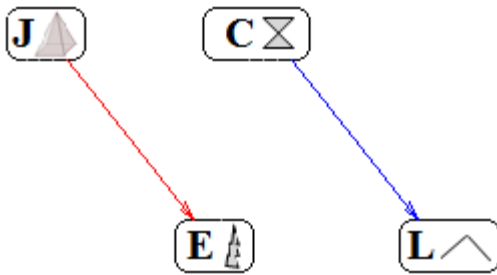
Fig. 18. The implicative graph for the items in task 4 – the strength of implication: red arrows 90 %, blue arrows 80 %

From the implicative graph follows that if a pupil knew the answer of item J, he or she also knew the correct answer for item E where the item J was more difficult for the identification. If the pupil answered correctly the item C his or her answer for the item L was correct as well. Even in this task was shown that 4th graders perceive and distinguish between models and non-models of triangles holistically. They notice the important elements of a triangle (sides and vertices) only a little, on the other, the position of a triangle is very important to them and they are not aware of the differences between the pyramid and a triangle. Besides reasons mentioned above (small proportion of non-models in the study materials in Slovakia) the results could be influenced by extremely small proportion of tasks developing the knowledge about spatial solids. In the first four years of education pupils do not get any information concerning solids; therefor they identify a solid based on the form of its faces.

The results – task 5
The aim of the fifth task was to find out the ability of 4th graders to distinguish between models and non-models of oblongs with the help of printed pictures. Compared to the previous task this task is formulated in the form negation.
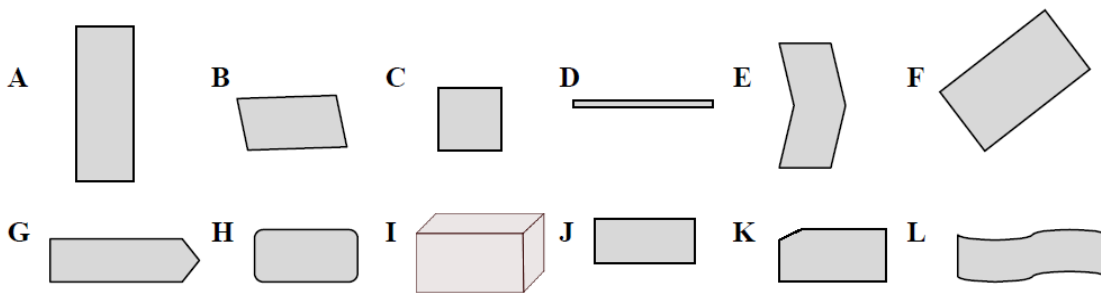*Task 5:* Which shapes are not oblongs? Circle the letters. (see Fig. 19)

Fig. 19. Plane geometrical figures for task 5

From Fig. 20 (the success rate of 4th graders in identification of models of oblongs) it is clear that pupils had problems only with three shapes (B, H, I).
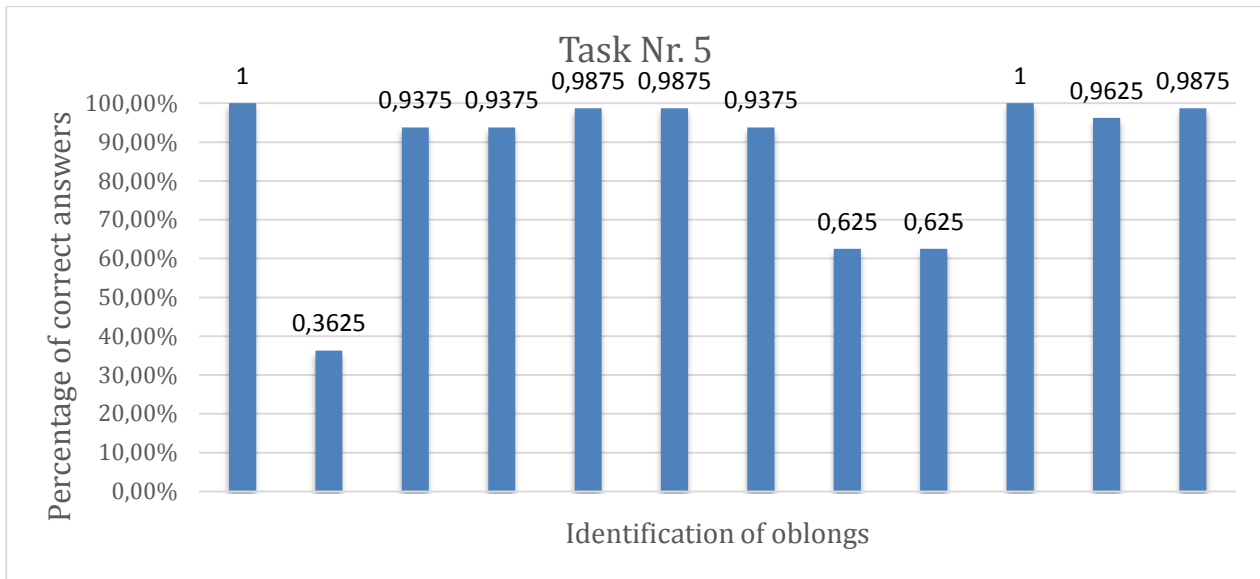
Fig. 20. Percentage of correct answers for task 5

The lowest success rate was for shape B. It is assumed that it is the similar problem as was described for the identification of a square. Younger children often identified this shape as an oblong with a reason that it is long. If we take into consideration the problem of Slovak curriculum, in which the thematic part about right angles is missing from primary education, then this result is not surprising at all.

The items H and I had only 62.5 % success rate. There were shown identical problems as described in the interpretation of previous results. Results for the item I indicate the problem with substitution of planar and spatial shape. This item is cuboid even though its lateral faces are oblongs. For the item H it is clear that the pupils perceive it holistically as an oblong and not as a shape with oblong form - which is in fact not an oblong.

The implicative graph for the items of this task (Fig. 21) illustrates the most important implications between identification of shapes.
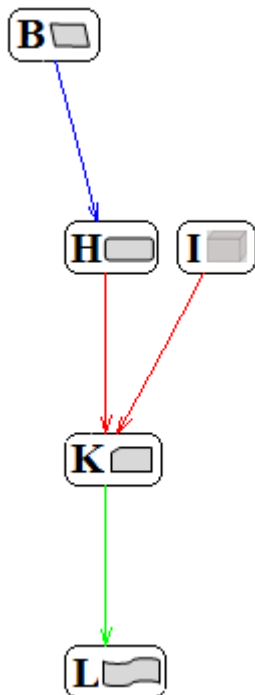

Fig. 21. The implicative graph for the items in task 5 – the strength of implication:
red arrows 90 %, blue arrows 80 %, green arrows 70 %

From the implicative graph follows that the answer to the question concerning item H implicatively determines the answer concerning the item K and the identification of K was easier for pupils than the identification of H and I.

Furthermore, in the graph is seen logical line of shapes B, H, K and L. These shapes are not oblongs, they are non-models of oblongs. The importance of knowledge about properties of oblong as a planar shape, respectively polygon with four vertices, four sides and all internal angles right, was shown again. The internal angles of the shape B are not right, but despite that pupils and younger children tend to identify rhomboid as an oblong. This shape showed to be the most difficult for its identification as a non-model of an oblong.

6. Results and discussion

Considering the aim of the research about the findings of the evidence of the misconceptions in children we publish the most important results and findings from both groups of the sample.

We assumed that the conceptions of pre-school children would correspond with Van Hiele visualization level. This assumption was supported by our results, moreover, some of the children showed signs of thinking in a higher Van Hiele level. Children noticed some of the details that were often crucial for their decision-making. However, we can conclude that pre-school children perceive shapes holistically and they do not notice the deformations of the shapes. The most important findings about conceptions in pre-school children about the elementary planar shapes are as follows:

1. Manipulation of shapes makes it easier for children to identify a shape. Identification of a physical model of the shape is easier for children than identification of a shape from the picture.
2. Slovak names of shapes (square, oblong, triangle and circle) are difficult to pronounce and therefore are often being slurred. This fact was considered during the process of evaluation of the results and the language imperfections were respected.
3. Pre-school children often used their own language and terms when they were about to identify a shape. These terms are mostly logical and understandable among the children.
4. A square is often called a cube. Already in this age the exchange in the terms of planar and spatial shapes is present.
5. In identification of rectangular shapes children often assigned a rhombus and rhomboid to the group of square and oblong models.
6. The easiest shapes in the terms of correct identification for pre-school children are circles.

Overall*, the position and the size of the shapes have influence on the recognition of the shapes*. Identification of models of the shapes corresponds to holistic perception and comparison to the existing prototypes of the shapes.

From the fourth grade students testing we conclude that when children were asked to identify shapes (squares, triangle, circle, oblong), most of the correct answers was present in those shapes that correspond to the prototype of these shapes. This finding also corresponds with van Heel theory about geometrical thinking. However, we expected that the level of geometrical thinking will be higher for fourth grade students of elementary school and they will at least achieve the level of analysis.

After the data analysis we formulated the following findings (categories) about the misconceptions of children and the biggest identification problems:

1. A significant problem is caused by confusion with spatial and planar shapes when the cube is considered to be a square, pyramid considered to be a triangle and cuboid is considered to be an oblong. ( )
2. The greatest difficulties with the identification of models of squares cause squares with the diagonals in vertical and horizontal position. ( )
3. When identifying rectangles (squares, oblongs) the greatest difficulties are caused by shapes with holistic resemblance of rectangles without vertices. ( )
4. During rectangles shape identification (squares, oblongs) pupils often choose the shapes that are without right inner angles and therefore, these are not models of rectangles. ( )
5. The identification of triangles showed that children do not notice relevant parts of the triangle (sides and vertices). They often choose a shape that is not the model of triangle as it does not meet any of the defining attributes of the triangle and identify it as a triangle. ( )

6. For children the elliptical shapes cause the greatest difficulty in identification of circles. (⬭   ◗)

When we compare the results of the pre-school children and results of the fourth grade students from the elementary school, all of the present misconceptions are usually identical. It means, that the initial imaginaries (pre-concepts) are very strong, stable and last longer during life.

These findings show that it is important to present to children the elementary school models of the shapes in different positions and also "counterexamples" (non-models) of the concrete shape. These can be shapes that resemble the prototype but do not meet at least one of the criteria of the shape. A multitude of experiences and the use of terminology have potential to offer to children an environment where they can re-evaluate their preconceptions and create correct mental representations of the abstract concepts from the planar geometry field.

7. Conclusions

The aim of our research activities is to find out the evidence of misconceptions of pre-school and younger school aged children and to find arguments for the need of further analysis in the context of national curricular documents and educational approaches in the sphere of planar geometrical shapes and their properties. Within this goal we tried to identify the most common misconceptions of children in pre-school and younger school age about triangles, squares, oblongs and circles. These misconceptions occur already in pre-school age of children and the question is whether the primary education of mathematics sufficiently contributes to their successive elimination.

Implicative graphs with the graphs of success rates of pupils in identification of models and non-models of elementary planar geometric shapes enabled us to identify those shapes that caused bigger problems to the pupils; respectively pupils have misconceptions about them.

We assume that the cause of these misconceptions can be the fact that the pupils and children have only little experiences with the manipulation with the shapes. It is important to provide not only models but also non-models of these shapes. At the same time, it is very important to discuss the properties of planar shapes, especially those that determine the definition of the shape. It is not sufficient if the pupil knows only the name of the shape. Knowing and naming its elementary properties is a must for the pupil in order to differentiate the shape from other planar shapes. It is useful to focus on models and non-models of individual shapes from real life. Furthermore, it is needed to distinguish between the shape and a form. For examples, if a shape is really a triangle or it just has a triangular form (a traffic sign "Give way" does not have to be a good example for illustration of a model of a triangle but it is a good example of a shape with triangular form).

These recommendations are addressed especially to the teachers in primary education. To reduce the problems mentioned above in the teaching process at the pre-primary and primary education, special attention should be paid in preparation of future teachers (Cọntay and Duatepe Paksu, 2012; Fujita and Jones, 2006; Marchis, 2012), not only from the professional but also from the didactical point of view.

Examined facts are consistent with the standards of performance for primary education in Slovakia which states that the pupil is able to "distinguish between planar geometric shapes: triangle, circle, square and oblong". The problem is that educational materials often show just the samples of models of shapes and very often only in standard positions. At the same time, it is necessary to note that a detailed methodology for this sphere of education is missing in Slovakia. Based on the discovery of misconceptions of pupils about geometric shapes it is possible to prepare appropriate methodical materials for teachers at primary stage of education, include appropriate tasks into textbooks and handbooks of mathematics, as well as to suggest source materials to expand mathematical content of National educational program ISCED1 which would eliminate presence of identified misconceptions (see State Educational Curriculum, 2009).

8. Support and acknowledgement

References

Cǫntay, Duatepe Paksu, 2012 – *Cǫntay, E. G., Duatepe Paksu, A.* (2012). Preservice Mathematics Teachers' Understandings of the Class Inclusion between Kite and Square. *Procedia – Social and Behavioral Sciences*. 55. Vol. 55, 782-788.

Fujita, Jones, 2006 – *Fujita, T., Jones, K.* (2006). Primary Trainee Teachers' Understanding of Basic Geometrical Figures in Scotland. Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education. Prague: PME. Vol. 3, 129-136.

Hejný et al., 2006 – *Hejný, M. et al.* (2006). Creative Teaching in Mathematics. Prague: Charles University.

Marchis, 2012 – *Marchis, I.* (2012). Preservice Primary School Teachers' Elementary Geometry Knowledge. Acta Didactica Napocensia. Volume 5, Number 2, 33-40.

Musser et al., 2001 – *Musser, G. L., Burger, W. F., Peterson, B. E.* (2001). Mathematics for Elementary Teachers. New York: John Wiley

Van Hiele, 1986 – *Van Hiele, P. M.* (1986). Structure and Insight. A theory of Mathematics Education. Orlando: Academic Press Inc.

Van Hiele, 1999 – *Van Hiele, P. M.* (1999). Developing Geometric Thinking through Activities that Begin with Play. Teaching Children Mathematics, Vol. 5, 310–316.

De Villiers, 2010 – De Villiers, M. (2010). Some Reflections on the Van Hiele Theory. Invited plenary from 4th Congress of teachers of mathematics. Zagreb. Retrieved from: http://frink.machighway.com /~dynamicm/vanhiele-reflection.pdf

Usiskin, 1982 – *Usiskin, Z.* (1982). Van Hiele Levels and Achievement in Secondary School Geometry. (Final Report of the Cognitive Development and Achievement in Secondary School Geometry Project.) Chicago, Illinois: University of Chicago.

Žilková, 2013 – Žilková, K. (2013). Teória a prax geometrických manipulácií v primárnom vzdelávaní. Praha: Powerprint.

State Educational Curriculum. 2009 – State Educational Curriculum. (2009). Štátny vzdelávací program Matematika (Vzdelávacia oblasť: Matematika a práca s informáciami) Príloha ISCED 1. Bratislava: Štátny pedagogický ústav. Retrieved from: http://www.statpedu.sk/sites/default/ files/dokumenty/statny-vzdelavaci-program/matematika_isced1.pdf on 10-04-2017.