

***C*-CLASS FUNCTION ON SOME COMMON FIXED POINT THEOREMS FOR WEAKLY SUB-SEQUENTLY CONTINUOUS MAPPINGS IN MANGER SPACES**

Said Beloul and Arslan Hojet Ansari

ABSTRACT. The aim of this paper is to prove some common fixed point theorems for two weakly subsequentially continuous and compatible of type (E) pairs of self mappings in Menger spaces, two examples are given to illustrate our results.

1. Introduction

Menger introduced the notion of probabilistic metric spaces (shortly, PM-spaces), which is a generalization of metric spaces. This notion based in idea to use distribution functions instead of non- negative real numbers as values of the metric. The concept of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. Since the work of Schweizer and Sklar [27], many authors have some results in probabilistic metric spaces due its importance in probabilistic functional analysis. Recently the study of fixed point or common fixed point in PM-spaces has a part by many authors in their researches.

Jungck [17] introduced the notion of compatible maps, the same author Jungck and Rhoades [18] weakened the concept of compatibility to the weak compatibility. Recently Al-Thagafi and Shahzad [2] gave a generalization, which is called the occasional weak compatibility property, this notion is weaker than the weak compatibility due to Jungck and Rhoades [18]. Dorić et al. [11] mentioned that the condition of occasionally weak compatibility reduces to weak compatibility, in the case where the two mappings have a unique point of coincidence (or a unique

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common fixed point). In 2009 Bouhadjera and Godet Tobie [10] introduced the concepts of subcompatibility and subsequential continuity which are more general than the occasional weak compatibility and the reciprocal continuity due to Pant [24] respectively, later Imdad et al. [16] improved the results in paper [10], by using subcompatibility with reciprocal continuity or subsequential continuity with compatibility. Many authors proved some results concerning common fixed point in Menger spaces as in papers [1, 5, 6, 7, 15, 13].

2. Preliminaries

DEFINITION 2.1. A mapping $\Delta : [0, 1] \times [0, 1] \times [0, 1]$ is a t-norm (or a triangular norm) if it satisfies the following conditions:

- (1) $\Delta(a, 1) = a$, for all $a \in [0, 1]$,
- (2) $\Delta(a, b) = \Delta(b, a)$,
- (3) $\Delta(a, b) \leq \Delta(c, d)$ for all $a \leq c$ and $b \leq d$,
- (4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

EXAMPLE 2.1. Let (X, d) be a metric space, define $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$, then Δ is a t-norm. Also $\Delta(a, b) = ab$ and $\Delta(a, b) = \max\{0, a + b - 1\}$ are t-norms.

DEFINITION 2.2. A real valued mapping $F : \mathbb{R} \rightarrow \mathbb{R}_+$ is called a distribution function, if it is non decreasing and left-continuous with:

$$\inf_{x \in \mathbb{R}} F(x) = 0, \quad \sup_{x \in \mathbb{R}} F(x) = 1.$$

We denote by \mathfrak{F} set of all distribution functions, and denote by H the Heaviside distribution function defined by:

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

DEFINITION 2.3. Let X be a non empty set, an order pair (X, F) is called a probabilistic metric space if F is a mapping from $X \times X$ into $\{g \in \mathfrak{F} : g(0) = 0\}$ and satisfying the following conditions:

- (1) $F_{xy} = H$ if and only if $x = y$,
- (2) $F_{xy} = F_{yx}$ for all $x, y \in X$,
- (3) if $F_{xy}(t) = 1$ and $F_{yz}(s) = 1$, then $F_{xz}(t + s) = 1$ for all $x, y, z \in X$ and $t, s \geq 0$.

If F satisfies only (1) and (2), the pair (X, F) is called a probabilistic semi metric space.

DEFINITION 2.4. A triplet (X, F, Δ) is called to be a Menger space if (X, F) is a probabilistic metric space and Δ is a t-norm such for all $x, y \in X$ and $t, s \geq 0$ the following inequality holds:

$$F_{xz}(t + s) \geq \Delta(F_{xy}, F_{yz}).$$

DEFINITION 2.5. If (X, d) is a metric space, by taking $F_{xy} = H(t - d(x, y))$, it becomes (X, F) probabilistic metric space, so every metric space can be realized as a probabilistic metric space.

DEFINITION 2.6. Let (X, F, Δ) be a Menger space with a continuous t-norm.

- (i) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if and only if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such $F_{x_n x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.
- (ii) A sequence $\{x_n\}$ in X is called to a Cauchy one, if and only if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such $F_{x_n x_m}(\varepsilon) > 1 - \lambda$ for all $n, m \geq N$.
- (iii) A Menger space is called to be complete if every Cauchy sequence in it, is convergent.

DEFINITION 2.7. A pair (A, S) of self mappings from a Menger space (X, F, Δ) into itself is compatible if and only if

$$\lim_{n \rightarrow \infty} F_{ASx_n, SAx_n} = 1,$$

for all $t \geq 0$, whenever $\{x_n\}$ is a sequence in X such

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$.

DEFINITION 2.8. Two self mappings A, S of a Menger space (X, F, Δ) into itself are called to be weakly compatible if and only if they commute at their coincidence points, i.e if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Kumar et al. [20] generalized the reciprocal continuity concept due to Pant [24] in the setting of Menger space as follows:

DEFINITION 2.9. Two self mappings A and S of a Menger space (X, F, Δ) are called reciprocally continuous if

$$\lim_{n \rightarrow \infty} ASx_n = Az \text{ and } \lim_{n \rightarrow \infty} SAx_n = Sz,$$

whenever $\{x_n\}$ is a sequence in X such

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} SAx_n = z,$$

for some $z \in X$.

Bouhadjera and Ghodet Tobie [10] introduced the concept of subsequential continuity in metric spaces, in the setting of Menger spaces it becomes:

DEFINITION 2.10. Let (X, F, Δ) be a Menger space, the pair of self mappings (A, S) is said to be subsequentially continuous, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$, for some $z \in X$ and $\lim_{n \rightarrow \infty} ASx_n = Az$.

Motivated by the above definition, the first author gave the following definition:

DEFINITION 2.11. ([9]) The pair (A, S) is said to be weakly subsequentially continuous (wsc) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z, \text{ for some } z \in X$$

and

$$\lim_{n \rightarrow \infty} ASx_n = Az, \text{ or } \lim_{n \rightarrow \infty} SAx_n = Sz.$$

The pair (A, S) is said to be A -subsequentially continuous (S -subsequentially continuous), if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$, $\lim_{n \rightarrow \infty} SAx_n = Sz$.

EXAMPLE 2.2. Let $X = [0, \infty)$ and let a continuous t -norm: $\Delta(x, y) = \frac{t}{t+|x-y|}$ for all $t > 0$, define A, S as follows:

$$Ax = \begin{cases} 2+x, & 0 \leq x \leq 2 \\ 0, & x > 2 \end{cases}, \quad Sx = \begin{cases} 2-x, & 0 \leq x \leq 2 \\ \frac{x}{2}, & x > 2 \end{cases}$$

Clearly that A and S are discontinuous at 1.

Consider a sequence $\{x_n\}$ such that for each $n \geq 1 : x_n = \frac{1}{n}$, it is clear that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = 2$, also we have:

$$\lim_{n \rightarrow \infty} ASx_n = \lim_{n \rightarrow \infty} A\left(2 - \frac{1}{n}\right) = A(1) = 1,$$

then (A, S) is A -subsequentially continuous, i.e., it is wsc.

Singh et al. [28, 29] introduced the notion of compatibility of type (E) in metric spaces, in the setting of the Menger spaces, it becomes:

DEFINITION 2.12. Self maps A and S of a Menger space (X, F, Δ) are said to be compatible of type (E), if

$$\lim_{n \rightarrow \infty} S^2x_n = \lim_{n \rightarrow \infty} SAx_n = Az \text{ and } \lim_{n \rightarrow \infty} A^2x_n = \lim_{n \rightarrow \infty} ASx_n = Sz,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z,$$

for some $z \in X$.

DEFINITION 2.13. Two self maps A and S of a Menger space (X, M, Δ) into itself are said to be A -compatible of type (E), if

$$\lim_{n \rightarrow \infty} A^2x_n = \lim_{n \rightarrow \infty} ASx_n = Sz,$$

for some $z \in X$. The pair $\{A, S\}$ is said to be S -compatible of type (E), if

$$\lim_{n \rightarrow \infty} S^2x_n = \lim_{n \rightarrow \infty} SAx_n = Az,$$

for some $z \in X$.

Notice that if A and S are compatible of type (E), then they are A -compatible and S -compatible of type (E), but the converse is not true.

EXAMPLE 2.3. Let $X = [0, \infty)$ with the continuous t-norm $\Delta(x, y) = \frac{t}{t+|x-y|}$ for all $t \geq 0$, define A, S as follows:

$$Ax = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1 \\ \frac{x}{2}, & x > 1 \end{cases} \quad Sx = \begin{cases} 2-x, & 0 \leq x \leq 1 \\ 2x-1 & x > 1 \end{cases}$$

Consider a sequence $\{x_n\}$ which defined by: $x_n = 1 - \frac{1}{n}$, for all $n \geq 1$, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} Ax_n &= \lim_{n \rightarrow \infty} Sx_n = 1, \\ \lim_{n \rightarrow \infty} SAx_n &= \lim_{n \rightarrow \infty} S(1 - \frac{1}{2n}) = A(1) = 1, \\ \lim_{n \rightarrow \infty} S^2x_n &= \lim_{n \rightarrow \infty} S(1 + \frac{1}{n}) = A(1) \end{aligned}$$

then the pair (A, S) is S -compatible of type (E), but never compatible of type (E) since:

$$\lim_{n \rightarrow \infty} ASx_n = \lim_{n \rightarrow \infty} S(\frac{1}{2} + \frac{1}{2(n+1)}) = \frac{1}{2} \neq S(2).$$

The aim of this paper is to prove the existence and the uniqueness of common fixed point for two pairs of self-mappings in Menger metric space, which satisfying implicit relation by using the weak subsequential continuity with compatibility of type (E), to illustrate our results we give an examples.

In 2014, Ansari [16] introduced the concept of C -class functions. By using this concept we can generalize many fixed point theorems in the literature.

DEFINITION 2.14. ([4]) A mapping $f : [0, \infty)^2 \rightarrow \mathbb{R}$ is called C -class function if it is continuous and satisfies following axioms:

- (1) $f(s, t) \leq s$;
- (2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

Note for some F we have that $f(0, 0) = 0$.

We denote C -class functions as \mathcal{C} .

EXAMPLE 2.4. ([4]) The following functions $F : [0, \infty)^2 \rightarrow \mathbb{R}$ are elements of \mathcal{C} , for all $s, t \in [0, \infty)$:

- (1) $f(s, t) = s - t$, $f(s, t) = s \Rightarrow t = 0$;
- (2) $f(s, t) = ks$, $0 < k < 1$, $f(s, t) = s \Rightarrow s = 0$;
- (3) $F(s, t) = \frac{s}{(1+t)^r}$; $r \in (0, \infty)$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
- (4) $F(s, t) = \log(t + a^s)/(1 + t)$, $a > 1$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
- (5) $f(s, t) = \ln(1 + a^s)/2$, $a > e$, $f(s, t) = s \Rightarrow s = 0$;
- (6) $F(s, t) = (s + l)^{1/(1+t)^r} - l$, $l > 1$, $r \in (0, \infty)$, $F(s, t) = s \Rightarrow t = 0$;
- (7) $F(s, t) = s \log_{t+a} a$, $a > 1$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
- (8) $F(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s, t) = s \Rightarrow t = 0$;
- (9) $F(s, t) = s\beta(s)$, $\beta : [0, \infty) \rightarrow [0, 1)$ is continuous function, $F(s, t) = s \Rightarrow s = 0$;

$$(10) F(s, t) = s - \frac{t}{k+t}, F(s, t) = s \Rightarrow t = 0;$$

$$(11) F(s, t) = s - \varphi(s), F(s, t) = s \Rightarrow s = 0,$$

here $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$.

Let Φ be a set of all continuous functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following conditions:

(φ_1) φ is continuous.

(φ_2) $\varphi(t) > 0$ if $t > 0$ and $\varphi(0) = 0$.

LEMMA 2.1. ([12]) *Let (X, F, Δ) be a Menger space. For each $\lambda \in (0, 1]$, define a function $d_\lambda X \times X \rightarrow \mathbb{R}_+$ by $d_\lambda(x, y) = \{t > 0, F_{x,y}(t) > 1 - \lambda\}$. Then the following statements hold:*

(1) $d_\lambda(x, y) < t$ if and only if $F_{x,y}(t) > 1 - \lambda$.

(2) $d_\lambda(x, y) = 0$ if and only if $x = y$.

(3) $d_\lambda(x, y) = d_\lambda(y, x)$, for all $x, y \in X$.

LEMMA 2.2. *Let (X, F, Δ) be a Menger space. If there exists a constant $f \in \mathcal{C}$ such that $F_{x,y}(f(t, \varphi(t))) \geq F_{x,y}(t)$, for all $t > 0$ and fixed $x, y \in X$, then $x = y$.*

PROOF. Suppose

$$F_{x,y}(f(t, \varphi(t))) \geq F_{x,y}(t),$$

for all $t > 0$ and putting $a = d(x, y) = \inf\{t > 0, F_{x,y}(t) > 1 - \lambda\}$. We have

$$F_{x,y}(a) \geq F(f(a, \varphi(a))) \geq F_{x,y}(a) > 1 - \lambda,$$

which implies $f(a, \varphi(a)) = a$, from condition on f we get

$$a = 0, \text{ or, } \varphi(a) = 0,$$

hence $a = 0$ and so from lemma 2.1 $x = y$. □

3. Main results

THEOREM 3.1. *Let (X, F, Δ) be a Menger space and let A, B, S be four mappings on X . If the two pairs (A, S) and (B, T) are weakly subsequentially continuous (wsc) and compatible of type (E), then (A, S) and (B, T) has a coincidence point. Further if there exist $\varphi \in \Phi, f \in \mathcal{C}$ such for all $x, y \in X$ and each $t > 0$, we have:*

$$(3.1) \quad \begin{aligned} & F_{Sx, Ty}(f(t, \varphi(t))) \\ & \geq \min\{F_{Ax, By}(t), F_{Ax, Sx}(t), F_{By, Ty}(t), F_{Ax, Ty}(t), F_{By, Sx}(t)\} \geq 0, \end{aligned}$$

then A, B, S and T have a unique common fixed point in X .

PROOF. Since (A, S) is wsc, there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$ and $\lim_{n \rightarrow \infty} ASx_n = Az, \lim_{n \rightarrow \infty} SAx_n = Sz$, the compatibility of type (E) of (A, S) implies that

$$\lim_{n \rightarrow \infty} ASx_n = \lim_{n \rightarrow \infty} A^2x_n = Sz$$

and

$$\lim_{n \rightarrow \infty} SAx_n = \lim_{n \rightarrow \infty} S^2x_n = Az,$$

then $Az = Sz$ and z is a coincidence point for A and S . Similarly for B and T , since (B, T) is wsc (suppose that it is B -subsequentially continuous) there exists a sequence $\{y_n\}$ such

$$\lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = w$$

for some $w \in X$ and

$$\lim_{n \rightarrow \infty} BTy_n = Bw,$$

also the pair (B, T) is compatible of type (E) implies that

$$\lim_{n \rightarrow \infty} BTy_n = \lim_{n \rightarrow \infty} B^2y_n = Tw$$

$$\lim_{n \rightarrow \infty} TBy_n = \lim_{n \rightarrow \infty} T^2y_n = Bw,$$

so we have $Bw = Tw$.

We claim $Az = Bw$, if not by using (3.1) we get:

$$F_{Sz, Tw}(f(t, \varphi(t))) \geq \min\{F_{Az, Bw}(t), F_{Az, Sz}(t), F_{Bw, Tw}(t), F_{Az, Tw}(t), F_{Bw, Sz}(t)\}$$

since $Az = Sz$ and $Bw = Tw$, we get:

$$F_{Az, Bw}(f(t, \varphi(t))) \geq \min\{F_{Az, Bw}(t), 1, 1, F_{Az, Bw}(t), F_{Az, Bw}(t)\} = F_{Az, Bw}(t),$$

from lemma 2.1, we obtain $Az = Bw$

Now we prove $z = Az$, if not by using (3.1) we get:

$$F_{Sx_n, Tw}(f(t, \varphi(t)))$$

$$\geq \min\{F_{Ax_n, Bw}(t), F_{Ax_n(t), Sx_n}(t), F_{Bw, Tw}(t), F_{Ax_n, Tw}(t), F_{Bw, Sx_n}(t)\},$$

letting $n \rightarrow \infty$ we get:

$$F_{z, Tw}(f(t, \varphi(t))) \geq \{F_{z, Bw}(t), 1, 1, F_{z, Tw}(t), F_{Bw, z}(t)\},$$

since $Az = Bw = Tw$, we get:

$$F_{z, Az}(f(t, \varphi(t))) \geq \min\{F_{z, Az}(t), 1, 1, F_{z, Az}(t), F_{z, Az}(t)\} = F_{z, Az}(t).$$

Hence $z = Az = Sz$. Next we shall prove $z = t$, if not by using (3.1) we get:

$$F_{Sx_n, Ty_n}(f(t, \varphi(t)))$$

$$\geq \min\{F_{Ax_n, By_n}(t), F_{Ax_n, Sx_n}(t), F_{By_n, Ty_n}(t), F_{Ax_n, Ty_n}(t), F_{By_n, Sx_n}(t)\},$$

letting $n \rightarrow \infty$ we get:

$$F_{z, w}(f(t, \varphi(t))) \geq \min\{F_{z, w}(t), 1, 1, F_{z, w}(t), F_{w, z}(t)\} = F_{z, w}(t).$$

Hence z is a fixed point for A, B, S and T .

For the uniqueness, if q is another fixed point q , by using (3.1) we get:

$$\begin{aligned} F_{Sz, Tq}(f(t, \varphi(t))) &\geq \min\{F_{Az, Bq}(t), F_{Az, Sq}(t), F_{(Bq, Tq)}(t), F_{Az, Tq}(t), F_{Bq, Sz}(t)\} \\ &= F_{z, q}(t). \end{aligned}$$

Hence $z = q$, and z is unique. □

Taking $f(t, \varphi(t)) = kt$, where $k \in (0, 1)$ we get the following corollary:

COROLLARY 3.1. *Let (X, F, Δ) be a Menger space and let A, B, S and T be self mappings on X satisfying*

$$F_{Sx, Ty}(k) \geq \min\{F_{Ax, By}(t), F_{Ax, Sx}(t), F_{By, Ty}(t), F_{Ax, Ty}(t), F_{By, Sx}(t)\}.$$

Further, if the two pairs (A, S) and (B, T) are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point.

If we combine Theorem 3.1 with example (2) we get the following corollary:

COROLLARY 3.2. *Let (X, F, Δ) be a Menger space and let A, B, S and T be four self mappings on X . Suppose that there exist an upper semi continuous function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such $\phi(0) = 0$, $\phi(t) < t$ for each $t > 0$ and for all $x, y \in X$ we have:*

$$F_{Sx, Ty}(\phi(t)) \geq \min\{F_{Ax, By}(t), F_{Ax, Sx}(t), F_{By, Ty}(t), F_{Ax, Ty}(t), F_{By, Sx}(t)\},$$

if the pairs $(A, S), (B, T)$ are weakly subsequentially continuous and compatible of type (E), then A, B, S and T have a unique common fixed point.

THEOREM 3.2. *Let (X, F, Δ) be a Menger space and let A, B, S and T be self mappings on X , if*

- (1) *the pair (A, S) is weakly subsequentially continuous and compatible of type (E),*
 - (2) *the pair (B, T) is weakly subsequentially continuous and compatible of type (E).*
- Hence (A, S) and (B, T) has a coincidence point. Moreover the maps A, B, S and T have a unique common fixed point provided there exist $\varphi \in \Phi_u, f \in \mathbb{C}$ such for all $x, y \in X$ and $t > 0$ we have:*

(3.2)

$$F_{Sx, Ty}(f(t, \varphi(t))) \geq \phi(\min\{F_{Ax, Ay}(t), F_{Ax, Sx}(t), F_{Ay, Sy}(t), F_{Ax, Sy}(t), F_{Ay, Sx}(t)\}),$$

where $\phi : [0, 1] \rightarrow [0, 1]$ is a lower semi continuous function such $\phi(t) > t$ for each $t \in (0, 1)$ with $\phi(0) = 0$ and $\phi(1) = 1$.

PROOF. Since for all $x, y \in X$ and $t > 0$ we have $\phi(t) > t$, then result of Theorem 3.3 is a consequence of the result of Theorem 3.1. \square

REMARK 3.1. Theorem 3.1 and Theorem 3.2 remain true if we replace the weakly subsequentially continuity and compatibility of type (E) by one of the following conditions:

- (1) *S, T -subsequentially continuity and S, T -compatibility of type (E),*
- (2) *subsequentially continuity and A, B -compatibility of type (E),*
- (3) *subsequentially continuity and S, T -compatibility of type (E),*
- (4) *subsequentially continuity and compatibility of type (E).*

THEOREM 3.3. *Let (X, F, Δ) be a Menger space and let A, B, S and T be self mappings on X , if*

- (1) *the pair (A, S) is A -subsequentially continuous and A -compatible of type (E),*
- (2) *the pair (B, T) is B -subsequentially continuous and B -compatible of type (E).*

Hence (A, S) and (B, T) has a coincidence point. Moreover the maps A, B, S and T have a unique common fixed point provided the maps satisfy (3.1) or (3.2).

EXAMPLE 3.1. Let (X, F, Δ) be a Menger metric space such $X = [0, \infty)$, $\Delta(x, y) = \min(x, y)$ and

$$F_{x,y} = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & t = 0 \end{cases}$$

define mappings A and S as follows:

$$Ax = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \quad Sx = \begin{cases} \frac{x+1}{2} & 0 \leq x \leq 1 \\ \frac{1}{4}, & x > 1 \end{cases},$$

We consider a sequence $\{x_n\}$ which defined for each $n \geq 1$ by: $x_n = 1 - \frac{1}{n}$, clearly that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = 1$, also we have:

$$\lim_{n \rightarrow \infty} ASx_n = A(1) = S(1) = 1$$

$$\lim_{n \rightarrow \infty} A^2x_n = S(1) = 1,$$

then (A, S) is A -subsequentially continuous and A -compatible of type (E).

For the inequality (3.1) we have the following cases:

(1) For $x, y \in [0, 1]$, we have

$$d(Sx, Sy) = \frac{1}{2}|x - y| \leq |x - y| = d(Ax, Ay),$$

which implies that for $k = \frac{2}{3}$, we have:

$$F_{Sx, Sy}(\frac{2}{3}t) \geq F_{Sx, Sy}(t) \geq F_{Ax, Ay}(t)$$

(2) For $x \in [0, 1]$ and $1 < y \leq 2$, we have

$$d(Sx, Sy) = \frac{1}{4}|2x - 1| \leq \frac{3}{4} = d(Ay, Sy),$$

so for $k = \frac{2}{3}$ we have:

$$F_{Sx, Sy}(\frac{2}{3}t) \geq F_{Sx, Sy}(t) \geq F_{Ay, Sy}(t),$$

(3) For $x \in (1, \infty)$ and $y \in [0, 1]$, we have

$$d(Sx, Sy) = \frac{1}{4}|2y - 1| \leq \frac{3}{4} = d(Ax, Sx),$$

which implies that for $k = \frac{2}{3}$ we have:

$$F_{Sx, Sy}(\frac{2}{3}t) \geq F_{Sx, Sy}(t) \geq F_{Ax, Sx}(t)$$

(4) For $x, y \in (1, \infty)$, it is obviously, because $F_{Sx, Ty}(\frac{2}{3}t) = 1$.

Consequently, all hypotheses of Corollary 3.2 are satisfied, with $f(t, \varphi(t)) = \frac{2}{3}t$ and the point 1 is the unique common fixed for A and S .

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DEPARTMENT OF MATHEMATICS, EL-OUED UNIVERSITY, EL-OUED, ALGERIA
E-mail address: beloulsaid@gmail.com

DEPARTMENT OF MATHEMATICS, KARAJ BRANCH, ISLAMIC AZAD UNIVERSITY, KARAJ, IRAN.
E-mail address: amiranalismath3@gmail.com