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APPROACHING SIMULTANEOUS FREDHOLM INTEGRAL EQUATIONS USING COMMON FIXED POINT THEOREMS IN COMPLEX METRIC SPACES

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Abstract. The aim of this paper is to discuss the existence and uniqueness of a common solution for the following system of linear Fredholm integral equations (of the second kind):

$$u(t) = f_i(t) + \beta \int_a^b K_i(t,s)F_i(u(s))ds, \quad t, s \in [a,b],$$

where $\beta \in \mathbb{R}$, f_i, K_i and F_i are given continuous functions, $i = 1, 2$ while u is unknown function to be determined. To establish this, we prove a common fixed point theorem for two self-mappings defined on a complex metric space. Moreover, we prove coincidence and common fixed point theorems for two weakly compatible self-mappings defined on a complex metric space.

1. Introduction

The theory of integral equations is a very important as well as applicable branch of the mathematical analysis and have several applications to real world problems. Many problems which arise in several domains of mathematics, engineering and physical sciences, lead to mathematical models expressible in the form of linear integral equations. There exist numerous advanced and efficient methods, which are often utilized to find the solution of linear integral equations. One of the powerful tools for obtaining the solutions of such equations is the use of fixed point theoretic results.

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Banach contraction principle is the most fundamental and natural result of metric fixed point theory. This principle was formulated by Banach in his Ph.D thesis in 1922 wherein it was originally proved in function spaces and was also utilized to establish the existence of a solution of an integral equation. In recent years, due to the enormous potential of utility and usefulness, this principle has been generalized and improved in several directions. One of the main direction is accomplished by proving this principle in different types of spaces. With similar quest, in 2011 Azam et al. [**1**] introduced the concept of complex valued metric space and proved some fixed point theorems using a rational contraction condition. Since then, many authors have studied the existence and uniqueness results on fixed point, coincidence point and common fixed point for self-mappings satisfying different contraction conditions especially of rational type contractions.

The purpose of this paper is three-fold:

(i) to prove a common fixed point theorem involving two self-mappings defined on a complex valued metric space;

(ii) to prove a coincidence and a common fixed point theorem for two weakly compatible self-mappings defined on a complex valued metric space;

(iii) to study the existence and uniqueness of a common solution for the following system of linear Fredholm integral equations of the second kind:

$$u(t) = f_1(t) + \beta \int_a^b K_1(t,s)F_1(u(s))ds,$$

$$u(t) = f_2(t) + \beta \int_a^b K_2(t,s)F_2(u(s))ds,$$

where $t, s \in [a,b]$, $\beta \in \mathbb{R}$, f_1, f_2, K_1, K_2, F_1 and F_2 are given continuous functions, and u is unknown function to be determined.

2. Preliminaries

For the sake of completeness, we collect some basic notions, definitions and auxiliary results from the existing literature.

Definition 2.1. ([**1**]) Let $w_1, w_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$w_1 \precsim w_2 \iff Re(w_1) \leqslant Re(w_2) \text{ and } Im(w_1) \leqslant Im(w_2),$$

that is, $w_1 \precsim w_2$, if one of the following conditions holds:

(i) $Re(w_1) = Re(w_2),\ Im(w_1) = Im(w_2)$,

(ii) $Re(w_1) < Re(w_2),\ Im(w_1) = Im(w_2)$,

(iii) $Re(w_1) = Re(w_2),\ Im(w_1) < Im(w_2)$,

(iv) $Re(w_1) < Re(w_2),\ Im(w_1) < Im(w_2)$.

Especially, we write $w_1 = w_2$ if (i) is satisfied and we write $w_1 \precnsim w_2$ if $w_1 \neq w_2$ and one of (ii), (iii) and (iv) holds while $w_1 \prec w_2$ if only (iv) is satisfied.

In the sequel, $\mathbb{C}_+ = \{w \in \mathbb{C} : 0 \precsim w\}$. Also, by writing $\succsim$, we refer to the dual relation of $\precsim$.

REMARK 2.1. For all $w_1, w_2, w_3 \in \mathbb{C}$, we have the following:

(*i*) $r_1, r_2 \in \mathbb{R}$ with $r_1 \leqslant r_2$ and $0 \precsim w_1 \Longrightarrow r_1 w_1 \precsim r_2 w_1$,

(*ii*) $0 \precsim w_1 \precnsim w_2 \Longrightarrow |w_1| < |w_2|$,

(*iii*) $w_1 \precsim w_2, w_2 \prec w_3 \Longrightarrow w_1 \prec w_3$.

DEFINITION 2.2. ([**1**]) Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{C}_+$ is said to be a complex valued metric if it satisfies the following conditions:

(*i*) $d(u, v) = 0$ if and only if $u = v$ for all $u, v \in X$,

(*ii*) $d(u, v) = d(v, u)$ for all $u, v \in X$,

(*iii*) $d(u, v) \precsim d(u, z) + d(z, v)$ for all $u, v, z \in X$.

The pair (X, d) is called a complex valued metric space (in short, complex metric space).

DEFINITION 2.3. ([**1**]) Let (X, d) be a complex metric space and K a subset of X. Then

(*i*) a point u in X is said to be an interior point of K, if there exists $c \in \mathbb{C}$ such that $0 \prec c$ and $N(u, c) = \{v \in X : d(u, v) \prec c\} \subseteq K$. Further, if every element of K is an interior point, then K is called an open set,

(*ii*) a point u in X is said to be a limit point of K, if for every $c \in \mathbb{C}$ with $0 \prec c$, we have $N(u, c) \cap (K \setminus \{u\}) \neq \phi$. Further, if K contains all its limit points, then it is called a closed set.

(*iii*) the family $\Xi = \{N(u, c) : u \in X, 0 \prec c \in \mathbb{C}\}$ forms a sub-basis of a Hausdorff topology τ on X.

EXAMPLE 2.1. Let $X = C\big([a, b], \mathbb{R}\big)$ where $a, b \in \mathbb{R}$ with $0 < a \leqslant b$. Define a mapping $d : X \times X \to \mathbb{C}$ as follows:

$$d(u, v) = \max_{t \in [a,b]} ||u(t) - v(t)||e^i.$$

Then (X, d) is a complex metric space.

DEFINITION 2.4. ([**1**]) Let (X, d) be a complex metric space, $u \in X$ and $\{u_n\}$ a sequence in X. Then

(*i*) $\{u_n\}$ converges to u, if for every $c \in \mathbb{C}$ with $0 \prec c$, there exists an $N_0 \in \mathbb{N}$ such that

$$d(u_n, u) \prec c \ \ \forall n > N_0.$$

Symbolically, this denoted by $\lim_{n\to\infty} u_n = u$ or $u_n \to u$, as $n \to \infty$,

(*ii*) $\{u_n\}$ is said to be a Cauchy sequence if for every $0 \prec c \in \mathbb{C}$ there exists an $N_0 \in \mathbb{N}$ such that

$$d(u_n, u_{n+m}) \prec c \ \ \forall n > N_0,$$

where $m \in \mathbb{N}$,

(*iii*) (X,d) is said to be complete complex metric space (in short: complete) if every Cauchy sequence in X converges to some element in X.

LEMMA 2.1 ([**1**]). *Let (X,d) be a complex metric space and $\{u_n\}$ a sequence in X. Then $\{u_n\}$ is a Cauchy sequence if and only if $|d(u_n, u_{n+m})| \longrightarrow 0$ as $n \longrightarrow \infty$, where $n, m \in \mathbb{N}$.*

DEFINITION 2.5. ([**8**]) Let (T,Q) be a pair of self-mappings on a complex metric space (X,d) and $u, v \in X$. If

(*i*) $Tu = u$, then u is said to be a fixed point of T,

(*ii*) $Tu = Qu$, then u is said to be a coincidence point of T and Q,

(*iii*) $Tu = Qu = v$, then v is called a point of coincidence of T and Q,

(*iv*) $Tu = Qu = u$, then u is said to be a common fixed point of T and Q.

DEFINITION 2.6. ([**5**]) A pair of self-mappings (T,Q) defined on a nonempty set X is said to be weakly compatible if T and Q commute at their coincidence points, i.e., $TQu = QTu$ whenever $Tu = Qu,\ u \in X$.

DEFINITION 2.7. ([**4**]) Two finite families $\{T_i\}_{i=1}^m$ and $\{Q_j\}_{j=1}^n$, $m, n \in \mathbb{N}$, of self-mappings defined on a nonempty set are said to be pairwise commuting if:

(*i*) $T_iT_j = T_jT_i$, $i, j \in \{1, 2, ..., m\}$,

(*ii*) $Q_iQ_j = Q_jQ_i$, $i, j \in \{1, 2, ..., n\}$,

(*iii*) $T_iQ_j = Q_jT_i$, $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$.

LEMMA 2.2 ([**3**]). *Let X be a nonempty set and $T : X \to X$. Then there exists a subset $A \subseteq X$ such that $T(A) = T(X)$ and $T : A \to X$ is one-to-one.*

3. Main Results

Throughout this work, Γ stands for the class of all functions $\gamma : \mathbb{C}_+ \to [0,1)$ which satisfy the following condition:

$$\gamma(u_n) \to 1 \implies |u_n| \to 0,$$

for any sequence $\{u_n\}$ in $\mathbb{C}_+$.

The following functions are in Γ (see [**8**]):

(*i*) $\gamma(u) = \lambda$, where $\lambda \in [0,1)$;

(*i*) $\gamma(u) = \frac{1}{1+\lambda|u|}$, where $\lambda \in [0,\infty)$.

LEMMA 3.1. *Let (X,d) be a complex metric space and $\{u_n\}$ a sequence in X. If $\{u_n\}$ is not Cauchy in X, then there exist $\epsilon_0 > 0$ and two subsequences $\{u_{n(k)}\}$ and $\{u_{m(k)}\}$ of $\{u_n\}$ such that*

$$k < m(k) < n(k) < m(k+1),\ |d(u_{m(k)}, u_{n(k)})| \geqslant \epsilon_0 \text{ and } \quad |d(u_{m(k)}, u_{n(k)-1})| < \epsilon_0.$$

PROOF. We know that $\{u_n\}$ is a Cauchy sequence if, and only if, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|d(u_n, u_m)| \leqslant \epsilon$ for all $n, m \geqslant N$. If such condition does not hold, then there exists $\epsilon_0 > 0$ such that for all $N \in \mathbb{N}$ there exist $n, m \geqslant N$ with $|d(u_n, u_m)| \geqslant \epsilon_0$.

Let $N = 2$. Then there exist $n_1, m_1 \geqslant 2$ such that $|d(u_{n_1}, u_{m_1})| \geqslant \epsilon_0$. Let $m(1) = \min\{n_1, m_1\}$ and consider

$$|d(u_{m(1)}, u_{m(1)+1})|,\ |d(u_{m(1)}, u_{m(1)+2})|, \dots , |d(u_{m(1)}, u_{\max\{n_1,m_1\}})|.$$

Since

$$|d(u_{m(1)}, u_{\max\{n_1,m_1\}})| = |d(u_{n_1}, u_{m_1})| \geqslant \epsilon_0,$$

between the previous numbers there exists a first nonnegative integer

$$n(1) \in \{m(1) + 1, m(1) + 2, ..., \max\{n_1, m_1\}\}$$

such that $|d(u_{m(1)}, u_{n(1)})| \geqslant \epsilon_0$ and $|d(u_{m(1)}, u_i)| < \epsilon_0$ for all $i \in \{m(1)+1, m(1)+2, ..., n(1) - 1\}$. In particular, $|d(u_{m(1)}, u_{n(1)-1})| < \epsilon_0$.

Next, let $N = n(1) + 1$. Then there exist $n_2, m_2 \geqslant n(1) + 1$ such that $|d(u_{n_2}, u_{m_2})| \geqslant \epsilon_0$. Let $m(2) = \min\{n_2, m_2\}$ and consider

$$|d(u_{m(2)}, u_{m(2)+1})|,\ |d(u_{m(2)}, u_{m(2)+2})|, \dots , |d(u_{m(2)}, u_{\max\{n_2,m_2\}})|.$$

Since $|d(u_{m(2)}, u_{\max\{n_2,m_2\}})| = |d(u_{n_2}, u_{m_2})| \geqslant \epsilon_0$, between the previous numbers there exists a first nonnegative integer $n(2) \in \{m(2)+1, m(2)+2, ..., \max\{n_2, m_2\}\}$ such that $|d(u_{m(2)}, u_{n(2)})| \geqslant \epsilon_0$ and $|d(u_{m(2)}, u_i)| < \epsilon_0$ for all $i \in \{m(2)+1, m(2)+2, ..., n(2) - 1\}$. In particular, $|d(u_{m(2)}, u_{n(2)-1})| < \epsilon_0$.

Repeating this process, we can find two subsequences $\{u_{m(k)}\}$ and $\{u_{n(k)}\}$ such that, for all $k \in \mathbb{N}$

$$k < m(k) < n(k) < m(k + 1),\ |d(u_{m(k)}, u_{n(k)})| \geqslant \epsilon_0,\ |d(u_{m(k)}, u_{n(k)-1})| < \epsilon_0.$$

□

Now, we prove our main result as follows:

Theorem 3.1. *Let (T, Q) be a pair of self-mappings defined on a complex metric space (X, d) and $\mu_1, \mu_2 : \mathbb{C}_+ \to [0, 1)$ given mappings such that $\mu_1 + \mu_2 \in \Gamma$. Assume that $TX \cup QX$ is complete subspace of X and for all $u, v \in X$,*

$$d(Tu, Qv) \precsim \mu_1\big(d(u, v)\big)d(u, v) + \mu_2\big(d(u, v)\big)\frac{d(Tu, u)d(Qv, v)}{1 + d(Tu, Qv)}. \tag{3.1}$$

Then the pair (T, Q) has a unique common fixed point.

Proof. Let u_0 be an arbitrary element of X. Construct a sequence $\{u_n\}$ in $TX \cup QX$ as follows:

$$u_{2n+1} = Tu_{2n}, \quad u_{2n+2} = Qu_{2n+1}, \quad n = 0, 1, 2, ... \tag{3.2}$$

Now, we assert that $\{u_n\}$ is a Cauchy sequence. Using (3.1) and (3.2) we have (for all $n \geqslant 0$)

$$\begin{aligned} d(u_{2n+1}, u_{2n+2}) &= d(Tu_{2n}, Qu_{2n+1}) \\ &\precsim \mu_1(d(u_{2n}, u_{2n+1}))d(u_{2n}, u_{2n+1}) \\ &\quad +\mu_2(d(u_{2n}, u_{2n+1}))\frac{d(Tu_{2n}, u_{2n})d(Qu_{2n+1}, u_{2n+1})}{1 + d(Tu_{2n}, Qu_{2n+1})} \\ &= \mu_1(d(u_{2n}, u_{2n+1}))d(u_{2n}, u_{2n+1}) \\ &\quad +\mu_2(d(u_{2n}, u_{2n+1}))\frac{d(u_{2n+1}, u_{2n})d(u_{2n+2}, u_{2n+1})}{1 + d(u_{2n+1}, u_{2n+2})}, \end{aligned}$$

yielding thereby

$$\begin{aligned} |d(u_{2n+1}, u_{2n+2})| &\leqslant \mu_1(d(u_{2n}, u_{2n+1}))|d(u_{2n}, u_{2n+1})| \\ &\quad +\mu_2(d(u_{2n}, u_{2n+1}))\Big|\frac{d(u_{2n+1}, u_{2n})d(u_{2n+2}, u_{2n+1})}{1 + d(u_{2n+1}, u_{2n+2})}\Big| \\ &\leqslant \big(\mu_1(d(u_{2n}, u_{2n+1})) + \mu_2(d(u_{2n}, u_{2n+1}))\big)||d(u_{2n+1}, u_{2n})|, \end{aligned}$$

so that

$$|d(u_{2n+1}, u_{2n+2})| \leqslant (\mu_1 + \mu_2)(d(u_{2n}, u_{2n+1}))|d(u_{2n}, u_{2n+1})|. \tag{3.3}$$

Similarly, one can show that

$$|d(u_{2n+2}, u_{2n+3})| \leqslant (\mu_1 + \mu_2)(d(u_{2n+2}, u_{2n+1}))|d(u_{2n+2}, u_{2n+1})|. \tag{3.4}$$

In view of (3.3) and (3.4), we have

(3.5)
$$|d(u_n, u_{n+1})| \leqslant (\mu_1 + \mu_2)(d(u_{n-1}, u_n))|d(u_{n-1}, u_n)| < |d(u_{n-1}, u_n)| \text{ for all } n \in \mathbb{N}.$$

Hence, $\{d(u_n, u_{n+1})\}$ is non-decreasing sequence of non-negative real numbers. Thus, $\{d(u_n, u_{n+1})\}$ converges to some $r \geqslant 0$. We claim that $r = 0$. Otherwise, letting $n \to \infty$ in (3.5), we obtain $|(\mu_1 + \mu_2)(d(u_{n-1}, u_n))| \to 1$ which in turn implies that $\{|d(u_{n-1}, u_n))|\}$ tends to 0 (as $(\mu_1 + \mu_2) \in \Gamma$), a contradiction. Thus, we have

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{3.6}$$

Next, we prove that $\{u_n\}$ is a Cauchy sequence. According to (3.6) it is enough to show that $\{u_{2n}\}$ is Cauchy. For this, let us assume the contrary, i.e., $\{u_{2n}\}$ is not a Cauchy sequence. In view of Lemma 3.1, there exist $\epsilon_0 > 0$ and two subsequences $\{u_{2n(k)}\}$ and $\{u_{2m(k)}\}$ of $\{u_{2n}\}$ such that $k < m(k) < n(k) < m(k + 1)$,

$$|d(u_{2n(k)}, u_{2m(k)})| \geqslant \epsilon_0 \text{ and } |d(u_{2n(k)}, u_{2m(k)-2}) < \epsilon_0 \ \ \forall k \in \mathbb{N}_0. \tag{3.7}$$

Using (3.7) and triangular inequality, we have

$$\begin{aligned} \epsilon_0 &\leqslant |d(u_{2n(k)}, u_{2m(k)})| \\ &\leqslant |d(u_{2n(k)}, u_{2m(k)-2})| + |d(u_{2m(k)-2}, u_{2m(k)-1})| + |d(u_{2m(k)-1}, u_{2m(k)})| \\ &< \epsilon_0 + |d(u_{2m(k)-2}, u_{2m(k)-1})| + |d(u_{2m(k)-1}, u_{2m(k)})|. \end{aligned}$$

Letting $k \to \infty$ and using (3.6), we have

$$\lim_{k\to\infty} |d(u_{2n(k)}, u_{2m(k)})| = \epsilon_0. \tag{3.8}$$

Further, we have

$$\begin{aligned}|d(u_{2n(k)}, u_{2m(k)})| &\leqslant |d(u_{2n(k)}, u_{2m(k)+1})| + |d(u_{2m(k)+1}, u_{2m(k)})| \\ &\leqslant |d(u_{2n(k)}, u_{2m(k)})| + |d(u_{2m(k)}, u_{2m(k)+1})| + \\ &\qquad |d(u_{2m(k)+1}, u_{2m(k)})|.\end{aligned}$$

Letting $k \to \infty$ and using (3.8), we have

$$\lim_{k\to\infty} |d(u_{2n(k)}, u_{2m(k)+1})| = \epsilon_0. \tag{3.9}$$

Now, consider

$$\begin{aligned}d(u_{2n(k)}, u_{2m(k)+1}) &\precsim d(u_{2n(k)}, u_{2n(k)+1}) + d(u_{2n(k)+1}, u_{2m(k)+2}) + \\ &\qquad d(u_{2m(k)+2}, u_{2m(k)+1}) \\ &= d(u_{2n(k)}, u_{2n(k)+1}) + d(Tu_{2n(k)}, Qu_{2m(k)+1}) + \\ &\qquad d(u_{2m(k)+2}, u_{2m(k)+1}) \\ &\precsim d(u_{2n(k)}, u_{2n(k)+1}) + d(u_{2m(k)+2}, u_{2m(k)+1}) \\ &\quad +\mu_1(d(u_{2n(k)}, u_{2m(k)+1}))d(u_{2n(k)}, u_{2m(k)+1}) \\ &\quad +\mu_2(d(u_{2n(k)}, u_{2m(k)+1}))\tfrac{d(Tu_{2n(k)}, u_{2n(k)})d(Qu_{2m(k)+1}, u_{2m(k)+1})}{1+d(Tu_{2n(k)}, Qu_{2m(k)+1})} \\ &\precsim d(u_{2n(k)}, u_{2n(k)+1}) + d(u_{2m(k)+2}, u_{2m(k)+1}) \\ &\quad +\mu_1(d(u_{2n(k)}, u_{2m(k)+1}))d(u_{2n(k)}, u_{2m(k)+1}) \\ &\quad +\mu_2(d(u_{2n(k)}, u_{2m(k)+1}))\tfrac{d(u_{2n(k)+1}, u_{2n(k)})d(u_{2m(k)+2}, u_{2m(k)+1})}{1+d(u_{2n(k)+1}, u_{2m(k)+2})}.\end{aligned}$$

This implies that

$$\begin{aligned}|d(u_{2n(k)}, u_{2m(k)+1})| \;\leqslant\;& |d(u_{2n(k)}, u_{2n(k)+1})| + |d(u_{2m(k)+2}, u_{2m(k)+1})| \\ & +\mu_1(d(u_{2n(k)}, u_{2m(k)+1}))|d(u_{2n(k)}, u_{2m(k)+1})| \\ & +\mu_2(d(u_{2n(k)}, u_{2m(k)+1})) \\ & \left|\frac{d(u_{2n(k)+1}, u_{2n(k)})d(u_{2m(k)+2}, u_{2m(k)+1})}{1 + d(u_{2n(k)+1}, u_{2m(k)+2})}\right| \\ \leqslant\;& |d(u_{2n(k)}, u_{2n(k)+1})| + |d(u_{2m(k)+2}, u_{2m(k)+1})| \\ & +(\mu_1+\mu_2)(d(u_{2n(k)}, u_{2m(k)+1}))|d(u_{2n(k)}, u_{2m(k)+1})| \\ & +\mu_2(d(u_{2n(k)}, u_{2m(k)+1})) \\ & \left|\frac{d(u_{2n(k)+1}, u_{2n(k)})d(u_{2m(k)+2}, u_{2m(k)+1})}{1 + d(u_{2n(k)+1}, u_{2m(k)+2})}\right| \\ \leqslant\;& |d(u_{2n(k)}, u_{2n(k)+1})| + |d(u_{2m(k)+2}, u_{2m(k)+1})| \\ & +|d(u_{2n(k)}, u_{2m(k)+1})| + \\ & \left|\frac{d(u_{2n(k)+1}, u_{2n(k)})d(u_{2m(k)+2}, u_{2m(k)+1})}{1 + d(u_{2n(k)+1}, u_{2m(k)+2})}\right|.\end{aligned}$$

Letting $k \to \infty$, we have

$$\epsilon_0 \leqslant \big(\lim_{k\to\infty}(\mu_1+\mu_2)(d(u_{2n(k)}, u_{2m(k)+1}))\big)\epsilon_0 \leqslant \epsilon_0,$$

which implies that

$$\lim_{k\to\infty}(\mu_1+\mu_2)(d(u_{2n(k)}, u_{2m(k)+1})) = 1.$$

Therefore, $\lim_{k\to\infty}|d(u_{2n(k)}, u_{2m(k)+1})| = 0$ (as $(\mu_1+\mu_2) \in \Gamma$), which in view of (3.9) contradicts $\epsilon_0 > 0$. Thus, we conclude that $\{u_n\}$ is a Cauchy sequence. By the completeness of $TX \cup QX$, there exists $u \in TX \cup QX$ such that $\{u_n\}$ converges to u.

Next, we claim that $Tu = u$. If $Tu \neq u$, then $|d(u, Tu)| > 0$. Consider

$$\begin{aligned}
d(u,Tu) &\precsim d(u,u_{2n+2}) + d(u_{2n+2},Tu)\\
&= d(u,u_{2n+2}) + d(Tu,Qu_{2n+1})\\
&\precsim d(u,u_{2n+2}) + \mu_1(d(u,u_{2n+1}))d(u,u_{2n+1})\\
&\quad +\mu_2(d(u,u_{2n+1}))\frac{d(Tu,u)d(Qu_{2n+1},u_{2n+1})}{1+d(Tu,Qu_{2n+1})}\\
&= d(u,u_{2n+2}) + \mu_1(d(u,u_{2n+1}))d(u,u_{2n+1})\\
&\quad +\mu_2(d(u,u_{2n+1}))\frac{d(Tu,u)d(u_{2n+2},u_{2n+1})}{1+d(Tu,u_{2n+2})},
\end{aligned}$$

yielding thereby

$$\begin{aligned}
|d(u,Tu)| &\leqslant |d(u,u_{2n+2})| + |\mu_1(d(u,u_{2n+1}))||d(u,u_{2n+1})|\\
&\quad +|\mu_2(d(u,u_{2n+1}))|\Big|\frac{d(Tu,u)d(u_{2n+2},u_{2n+1})}{1+d(Tu,u_{2n+2})}\Big|\\
&\leqslant |d(u,u_{2n+2})| + |d(u,u_{2n+1})| + \Big|\frac{d(Tu,u)d(u_{2n+2},u_{2n+1})}{1+d(Tu,u_{2n+2})}\Big|.
\end{aligned}$$

Letting $n \to \infty$, we have $|d(u,Tu)| \leqslant 0$, a contradiction. Thus, we conclude that $Tu = u$. Similarly, one can prove that $Qu = u$. Hence, $Tu = Qu = u$, i.e., u is a common fixed point of the pair (T, Q).
Finally, we prove that u is unique. Assume that u' is another common fixed point of the pair (T, Q). On using (3.10) with $u = u$ and $v = u'$, we have

$$\begin{aligned}
d(u,u') &= d(Tu,Qu')\\
&\precsim \mu_1(d(u,u'))d(u,u') + \mu_2(d(u,u'))\frac{d(Tu,u)d(Qu',u')}{1+d(Tu,Qu')}\\
&= \mu_1(d(u,u'))d(u,u').
\end{aligned}$$

This implies that

$$|d(u,u')| \leqslant \mu_1(d(u,u'))|d(u,u')| < |d(u,u')|.$$

a contradiction. Hence, u is a unique common fixed point of the pair (T, Q). □

REMARK 3.1. Conclusions of Theorems 3.1 remain true if the completeness of $TX \cup QX$ is replaced by the completeness of X.

On setting $T = Q$ in Theorem 3.1 we get the following fixed point result:

Corollary 3.1. *Let T be a self-mapping defined on a complex metric space (X, d) and $\mu_1, \mu_2 : \mathbb{C}_+ \to [0, 1)$ given mappings such that $\mu_1 + \mu_2 \in \Gamma$. Assume that TX (or X) is complete space and for all $u, v \in X$,*

$$d(Tu, Tv) \precsim \mu_1\big(d(u, v)\big)d(u, v) + \mu_2\big(d(u, v)\big)\frac{d(Tu, u)d(Tv, v)}{1 + d(Tu, Tv)}. \tag{3.10}$$

Then T has a unique fixed point.

Applying Corollary 3.1, we prove the existence and uniqueness of a point of coincidence and a common fixed point for two weakly compatible self-mappings defined on a complex metric space as under.

Theorem 3.2. *Let (T, Q) be a pair of self-mappings on a complex metric space (X, d) and $\mu_1, \mu_2 : \mathbb{C}_+ \to [0, 1)$ be given mappings such that $\mu_1 + \mu_2 \in \Gamma$. Assume that for all $u, v \in X$,*

$$d(Tu, Tv) \precsim \mu_1\big(d(Qu, Qv)\big)d(Qu, Qv) + \mu_2\big(d(Qu, Qv)\big)\frac{d(Tu, Qu)d(Tv, Qv)}{1 + d(Tu, Tv)}. \tag{3.11}$$

If $TX \subseteq QX$ and either TX (or QX) is complete space, then the pair (T, Q) has a unique point of coincidence in X. Moreover, if (T, Q) is weakly compatible pair, then it has a unique common fixed point.

Proof. On using Lemma 2.2, there exists $A \subseteq X$ such that $Q(A) = Q(X)$ and $Q : A \to X$ is one-to-one. Define a mapping $M : Q(A) \to Q(A)$ by $M(Qu) = Tu$ for all $Qu \in Q(A)$. As Q is one-to-one on A and $TX \subseteq QX$, M is well defined. We observe that $M \circ Q = T$, hence, on using (3.11), we have

$$\begin{aligned} d(M(Qu), M(Qv)) \;&\precsim\; \mu_1\big(d(Qu, Qv)\big)d(Qu, Qv) \\ &\quad +\mu_2\big(d(Qu, Qv)\big)\frac{d(M(Qu), Qu)d(M(Qv), Qv)}{1 + d(M(Qu), M(Qv))}, \end{aligned} \tag{3.12}$$

for all $Qu, Qv \in Q(A)$. Since either $M(QX) = M \circ Q(X) = T(X)$ or $Q(A) = Q(X)$ is complete and (3.12) holds, then by Corollary 3.1 M has a unique fixed point in $Q(A) = Q(X)$. Therefor, there exists a unique $u \in Q(A) = Q(X)$ such that $Mu = u$. Since $u \in Q(A) = Q(X)$ and Q is one-to-one on A, then there exists a unique $u' \in X$ such that $u = Qu'$. So that $M(Qu') = Qu'$, that is, $Tu' = Qu' = u$. Thus, the pair (T, Q) has a unique point of coincidence.
Next, assume that (T, Q) is weakly compatible pair. As $u = Qu' = Tu'$, we have

$$Tu = TQu' = QTu' = Qu,$$

proving that u is a coincidence point of the pair (T, Q). We assert that $Tu = u$. If not, then on setting $u = u$ and $v = u'$ in (3.11), we have

$$d(Tu, Tu') \precsim \mu_1\big(d(Qu, Qu')\big)d(Qu, Qu') + \mu_2\big(d(Qu, Qu')\big)\frac{d(Tu, Qu)d(Tu', Qu')}{1 + d(Tu, Tu')},$$

yielding thereby

$$d(Tu, u) \precsim \mu_1\big(d(Tu, u)\big)d(Tu, u),$$

which implies that

$$\begin{aligned} |d(Tu,u)| &\leqslant |\mu_1\big(d(Tu,u)\big)||d(Tu,u)| \\ &< |d(Tu,u)|, \end{aligned}$$

a contradiction. Hence $u = Tu$. Since $Tu = Qu$, therefore u is a common fixed point of the pair (T, Q).
Finally, the uniqueness of the common fixed point u of (T, Q) is a direct consequence of the uniqueness of the point of coincidence of (T, Q). This completes the proof. □

As a consequence of Theorem 3.1, we have the following result for two finite families of self-mappings defined on a complex valued metric space.

THEOREM 3.3. *Let $\{T_i\}_1^m$ and $\{Q_j\}_1^n$ be two finite pairwise commuting families of self-mappings defined on a complex metric space (X, d). Let $T = T_1T_2...T_m$ and $Q = Q_1Q_2...Q_n$ satisfying inequality (3.1) with μ_1 and μ_2 as in Theorem 3.1. If $TX \cup QX$ is complete subspace of X, then the component maps of the families $\{T_i\}_1^m$ and $\{Q_j\}_1^n$ have a unique common fixed point.*

PROOF. In view of Theorem 3.1, we conclude that the pair (T, Q) has a unique common fixed point u in X. Now, we prove that u is also a common fixed point of the component maps of the families $\{T_i\}_1^m$ and $\{Q_j\}_1^n$. Due to the componentwise commutativity of the families $\{T_i\}_1^m$ and $\{Q_j\}_1^n$, we have

$$\begin{aligned} T_iu &= T_iQu = T_iQ_1Q_2...Q_nu = Q_1T_iQ_2...Q_nu \\ &= Q_1Q_2T_i...Q_nu = ... = Q_1Q_2...T_iQ_nu \\ &= Q_1Q_2...Q_nT_iu = QT_iu, \text{ for each } 1 \leqslant i \leqslant m. \end{aligned}$$

Similarly $T_iu = TT_iu$ (for each $1 \leqslant i \leqslant m$), showing that T_iu (for each i) is a common fixed point of the pair (T, Q). Since u is unique common fixed point of (T, Q), we get that $T_iu = u$ (for each i). Using similar arguments, one can prove that $Q_ju = u$ (for each $1 \leqslant j \leqslant n$). Proving that u is also a common fixed point of the component maps of the families $\{T_i\}_1^m$ and $\{Q_j\}_1^n$.
Finally, if u' is another common fixed point of the component maps of the families $\{T_i\}_1^m$ and $\{Q_j\}_1^n$ then one can prove that u' is also a common fixed point of the pair (T, Q), which contradicts the fact that (T, Q) has a unique common fixed point. Hence, the component maps of the families $\{T_i\}_1^m$ and $\{Q_j\}_1^n$ have a unique common fixed point. This completes the proof. □

By setting $T_1 = T_2 = ... = T_m = T$ and $Q_1 = Q_2 = ... = Q_n = Q$ in Theorem 3.3, we derive the following corollary:

COROLLARY 3.2. *Let (T, Q) be a pair of self-mappings on a complex metric space (X, d) and $\mu_1, \mu_2 : \mathbb{C}_+ \to [0, 1)$ given mappings such that $\mu_1 + \mu_2 \in \Gamma$. Assume that for all $u, v \in X$,*

$$d(T^mu, Q^nv) \precsim \mu_1\big(d(u,v)\big)d(u,v) + \mu_2\big(d(u,v)\big)\frac{d(T^mu,v)d(Q^nv,u)}{1 + d(T^mu,Q^nv)}. \tag{3.13}$$

If either $T^mX \cup Q^nX$ or X is complete space, then the pair (T, Q) has a unique common fixed point.

Setting $\mu_2(w) \equiv 0 \ \ \forall w \in \mathbb{C}_+$ and $m = n$ in Corollary 3.2, we deduce the following corollary which generalizes Corollary 2.8 of Rouzkard and Imdad [**6**].

COROLLARY 3.3. *Let T be a self-mapping on a complex metric space (X, d) and $\mu \in \Gamma$. Assume that for all $u, v \in X$,*

$$d(T^m u, T^m v) \precsim \mu\big(d(u,v)\big)d(u,v). \tag{3.14}$$

If either $T^m X$ or X is complete space, then T has a unique fixed point.

Now, we furnish an example to exhibit the utility of Corollary 3.3.

EXAMPLE 3.1. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}_+$ as follows:

$$d(w_1, w_2) = |Re(w_1) - Re(w_2)| + i|Im(w_1) - Im(w_2)| \quad \forall w_1, w_2 \in \mathbb{C}.$$

Then $(\mathbb{C}, d)$ is a complex valued metric space. Define $T : \mathbb{C} \to \mathbb{C}$ by

$$T(w) = \begin{cases} 0, & if\ Re(w), Im(w) \in \mathbb{Q}; \\ w, & if\ Re(w), Im(w) \in \mathbb{Q}^c; \\ 5+i, & if\ Re(w) \in \mathbb{Q}, Im(w) \in \mathbb{Q}^c; \\ 5, & if\ Re(w) \in \mathbb{Q}^c, Im(w) \in \mathbb{Q}. \end{cases}$$

Now, for $u = \frac{1}{\sqrt{7}}$ and $v = 1$, we have

$$d\Big(T\big(\frac{1}{\sqrt{7}}\big), T(0)\Big) = d(5,0) = 5 \not\precsim \frac{1}{\sqrt{7}}\mu\big(d(u,v)\big) = \mu\big(d(u,v)\big)d\big(\frac{1}{\sqrt{7}}, 0\big),$$

for any mapping $\mu : \mathbb{C}_+ \to [0,1)$. Notice that $T^2 w = 0$ for all $w \in \mathbb{C}$ and if we define $\mu : \mathbb{C}_+ \to [0,1)$ by $\mu(w) = \frac{1}{1+|w|}$ for all $w \in \mathbb{C}_+$, then we have

$$0 = d(T^2 u, T^2 v) \precsim \mu\big(d(u,v)\big)d(u,v) = \frac{1}{1+|d(u,v)|}d(u,v),$$

for all $u, v \in \mathbb{C}$. Hence, Corollary 3.3 ensures the existence and uniqueness of a fixed point of T (namely $w = 0$).

4. Fredholm Integral Equations

In this section, we apply Theorem 3.1 to prove the existence and uniqueness of a common solution of the following system of linear Fredholm integral equations of the second kind:

$$\begin{aligned} u(t) &= f_1(t) + \beta \textstyle\int_a^b K_1(t,s)F_1(u(s))ds, \\ u(t) &= f_2(t) + \beta \textstyle\int_a^b K_2(t,s)F_2(u(s))ds, \end{aligned} \tag{4.1}$$

where $t, s \in [a,b]$, $\beta \in \mathbb{R}$, f_1, f_2, K_1, K_2, F_1 and F_2 are given continuous functions, and u is unknown function to be determined.

For simplification, we use the following symbols:

$$\begin{aligned}
\mathcal{F}_i\big(u(t)\big) &= \int_a^b K_i(t,s)F_i(u(s))ds,\\
\Gamma_{uv}(t) &= ||u(t)-v(t)||e^i,\\
\Lambda_{uv}(t) &= ||f_1(t)+\mathcal{F}_1\big(u(t)\big)-u(t)||e^i,\\
\Upsilon_{uv}(t) &= ||f_2(t)+\mathcal{F}_2\big(v(t)\big)-v(t)||e^i,\\
\Omega_{uv}(t) &= ||f_1(t)+\mathcal{F}_1\big(u(t)\big)-f_2(t)-\mathcal{F}_2\big(v(t)\big)||e^i
\end{aligned}$$

and $X = C([a,b],\mathbb{R})$ is the space of all real valued continuous functions defined on $[a,b]$.

Define two mappings on X as follows:

$$Tu(t) = f_1(t)+\mathcal{F}_1\big(u(t)\big) = f_1(t)+\int_a^b K_1(t,s)F_1(u(s))ds, \tag{4.2}$$

$$Qu(t) = f_2(t)+\mathcal{F}_2\big(u(t)\big) = f_2(t)+\int_a^b K_2(t,s)F_2(u(s))ds. \tag{4.3}$$

Observe that the system (4.1) of linear Fredholm integral equations of the second kind has a unique common solution if and only if the the pair (T,Q) given in (4.2) and (4.3) has a unique common fixed point.

THEOREM 4.1. *The system (4.1) of linear Fredholm integral equations of the second kind has a unique common solution if*

(i) there exist two mappings $\mu_1,\mu_2 : \mathbb{C}_+ \to [0,1)$ such that $\mu_1+\mu_2 \in \Gamma$,

(ii) for all $u,v \in X$ and $t \in [a,b]$

$$\Omega_{uv}(t) \precsim \mu_1\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\Gamma_{uv}(t)+\mu_2\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\frac{\Lambda_{uv}(t)\Upsilon_{uv}(t)}{1+\max_{t\in[a,b]}\Omega_{uv}(t)}.$$

PROOF. Define a mapping $d : X\times X \to \mathbb{C}_+$ by

$$d(u,v) = \max_{t\in[a,b]}||u(t)-v(t)||e^i.$$

Then (X,d) forms a complete complex valued metric space.

Now, from assumption (ii) (for all $u,v\in X$ and $t\in[a,b]$), we have

$$\begin{aligned}
\Omega_{uv}(t) &\precsim \mu_1\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\Gamma_{uv}(t)+\mu_2\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\frac{\Lambda_{uv}(t)\Upsilon_{uv}(t)}{1+\max_{t\in[a,b]}\Omega_{uv}(t)}\\
&\precsim \mu_1\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\max_{t\in[a,b]}\Gamma_{uv}(t)\\
&\quad +\mu_2\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\frac{\max_{t\in[a,b]}\Lambda_{uv}(t)\max_{t\in[a,b]}\Upsilon_{uv}(t)}{1+\max_{t\in[a,b]}\Omega_{uv}(t)},
\end{aligned}$$

which implies that

$$\begin{aligned}\max_{t\in[a,b]}\Omega_{uv}(t) \;\precsim\; & \mu_1\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\max_{t\in[a,b]}\Gamma_{uv}(t)\\ & +\mu_2\big(\max_{t\in[a,b]}\Gamma_{uv}(t)\big)\frac{\max_{t\in[a,b]}\Lambda_{uv}(t)\max_{t\in[a,b]}\Upsilon_{uv}(t)}{1+\max_{t\in[a,b]}\Omega_{uv}(t)},\end{aligned}$$

implying thereby

$$d(Tu,Qv)\precsim \mu_1\big(d(u,v)\big)d(u,v)+\mu_2\big(d(u,v)\big)\frac{d(Tu,u)d(Qv,v)}{1+d(Tu,Qv)}.$$

Thus, all the conditions of Theorem 3.1 are satisfied so that the pair (T,Q) has a unique common fixed point in X. Hence, the system (4.1) of linear Fredholm integral equations of the second kind has a unique common solution. □

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