CONE VALUED MEASURE OF NONCOMPACTNES AND RELATED FIXED POINT THEOREMS

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ABSTRACT. The purpose of this article is to introduce the notion of cone valued measure of noncompactness. Our theoretical approach generalizes recent results for cone measure of noncompactness. For the proofs of our results we use the famous Schauder-Tichonov Theorem as well as Robert Cauty theorem in the new framework.

1. Introduction

In dealing with the existence problems for functional operator equations, the concept of measure of noncompactness is very important tool in nonlinear analysis. This significant concept in mathematical science was defined by many authors in different manners (see [**2, 4, 5, 8, 9, 17, 19**]). For the applications of the measure noncompactness, we refer to [**1, 2, 3, 4, 5, 8, 9, 10, 11, 14, 17, 19**] and the references therein.

In [**14**] authors introduced the concept of cone measure of noncompactness and established some generalizations of very significant and well-known Darbo's fixed point theorem with respect to such defined measure. Their results generalize several fixed point theorems obtained recently by many authors. Furthermore, they presented a very nice application in functional integral equations. For the cone measure of noncompactness authors in [**14**] used ordered Banach space. For more details see [**14**].

In this paper we complement, generalize, improve and enrich the approaches and results obtained in [**14**]. Namely, we take an ordered locally convex (resp. topological vector) space instead of ordered Banach space, and use Schauder-Tychonoff (resp. Robert Cauty) fixed point theorem instead of Schauder fixed point theorem.

First, we recall some basic notions in topological vector space. For details, the reader may refer to [**3, 6, 7, 13, 15, 18, 22, 23, 24**] and the references therein.

DEFINITION 1.1. ([**12**]) A subset K of a topological vector space E is said to be a cone if it satisfies the following conditions:

(1) K is nonempty and closed;
(2) $\lambda x + \mu y \in K$ whenever $x, y \in K$ and $\lambda, \mu \in [0, +\infty)$;
(3) $K \cap (-K) = \{\theta\}$, where θ is the zero vector in E.

Given a cone $K \subset E$, one define a partial order $\preceq$ in E by $x \preceq y$ if and only if $y - x \in K$. Further, the notation $x \prec y$ means that $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \mathrm{int}K$ (the set of interior points of K). Clearly, the pair (E, K) is an ordered topological vector space. A cone K is called a solid cone if $\mathrm{int}K \neq \emptyset$.

For a pair of elements x, y in E such that $x \preceq y$, let $[x, y] = \{z \in E : x \preceq z \preceq y\}$. A subset A of E is said to be ordered convex if $[x, y] \subset A$, for $x, y \in A$ and $x \preceq y$. (E, K) is ordered convex if it has a base of neighborhoods of θ consisting of ordered convex subsets. In this case, the cone K is said to be normal (for more details, see [**3, 6, 8, 13, 22**]). Otherwise, one says that K is non-normal if K is not a normal cone (see [**14**, Example 4]). If E is a normed space, the last condition means that the unit ball is ordered convex, which is equivalent to the condition that there is a number M such that $x, y \in E$ and $\theta \preceq x \preceq y$ imply that $\|x\| \leqslant M \|y\|$. The least constant satisfying the above inequality is called normal constant of K. Obviously, the normal constant is always greater than or equal to 1. In the case when $M = 1$, then the cone is called monotone (see [**13**]). The cone K is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$ for some $y \in E$, then there is $x \in E$ such that $x_n \to x$ $(n \to \infty)$. Equivalently, the cone K is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

The next lemma contains a result on cone in ordered Banach space which is rather classical. It is applied to reduce a lot of results to the setting of ordinary metric spaces.

LEMMA 1.1 ([**16**]). *The following conditions are equivalent for a cone K in Banach space $(E, \|\cdot\|)$:*

(1) *K is normal;*
(2) *for arbitrary sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in E,*
for all n, $x_n \preceq y_n \preceq z_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$ imply $\lim_{n\to\infty} y_n = x$;

(3) *there exists a norm $\|\cdot\|_1$ on E, which is equivalent to $\|\cdot\|$, such that K is monotone w.r.t. $\|\cdot\|_1$.*

EXAMPLE 1.1. ([**15**]) Let $E = C_{\mathbb{R}}^1[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $K = \{x \in E : x(t) \geqslant 0, t \in [0, 1]\}$. This cone is solid but non-normal. In fact, let $x_n(t) = \frac{t^n}{n}$

and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \frac{1}{n} + 1 > 1$. Hence, $\{x_n\}$ does not converge to θ. So by (2) of Lemma 1.1 it follows that K is a non-normal cone.

Now consider $E = C^1_{\mathbb{R}}[0,1]$ endowed with the strongest locally convex topology t^*. Then K is also t^*-solid (because it has the nonempty t^*-interior), but not t^*-normal. Indeed, if it is normal then, the space (E, t^*) will be normed, which is impossible since an infinite-dimensional space with the strongest locally convex topology cannot be metrizable (see, e.g., [**20**]).

Note that the following properties of bounded sets in ordered topological vector space E. If K is a solid cone, then each topologically bounded subset of (E, K) is also ordered bounded, i.e., it is contained in a set of the form $[-c, c]$ for some $c \in \mathrm{int}K$. On the other hand, if the cone K is normal, then each ordered bounded subset of (E, K) is topologically bounded. Hence, if the cone is both solid and normal, then these two properties of subsets of E coincide with each other. Moreover, the proof of the following assertion can be found, e.g., in [**21**].

THEOREM 1.1. *If the underlying cone of an ordered topological vector space is solid and normal, then such topological vector space must be an ordered normed space.*

The following results are useful in the sequel of our paper.

LEMMA 1.2 ([**22**]). *Let (E, K) be an ordered topological vector space. Then e is an interior point of K if and only if $[-e, e]$ is the neighborhood of θ in E. In this case, e is an ordered unit element.*

PROOF. Let e be an interior point of K. Then there exists a circled neighborhood V of θ such that $e + V \subseteq K$, which follows that $V \subseteq (K - e) \cap (e - K)$ because V is circled. It is clear that

$$[-e, e] = (K - e) \cap (e - K).$$

Consequently, $[-e, e]$ is a neighborhood of θ. Conversely, assume that $[-e, e]$ is a neighborhood of θ. We conclude from $e + [-e, e] \subseteq K$ that e is an interior point of K. Finally, e is an ordered unit element because $[-e, e]$ is absorbing. □

LEMMA 1.3. *Let (E, K) be an ordered topological vector space. If the cone K is solid, then it is Hausdorff.*

PROOF. If $e \in \mathrm{int}K$, then $(E, \|\cdot\|_e)$ is a normed space where $\|\cdot\|_e$ is the Minkowski functional of neighborhood of θ. It is clear that the normed topology is weaker than the topology of E. Thus the claim holds. □

DEFINITION 1.2 ([**15, 24**]). Let K be a solid cone in a topological vector space E and $\{u_n\}$ be a sequence in K. Then $\{u_n\}$ is said to be a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geqslant n_0$.

PROPOSITION 1.1. *Let K be a solid cone in a topological vector space E and $\{u_n\}$ be a sequence in K. Then the following conditions are equivalent:*
(1) $\{u_n\}$ is a c-sequence;

(2) *For each $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $u_n \preceq c$ for all $n \geqslant n_0$;*

(3) *For each $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $u_n \prec c$ for all $n \geqslant n_0$;*

(4) *There exists $c \gg \theta$ such that for every $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $u_n \preceq \lambda c$ for all $n \geqslant n_0$;*

(5) *There exists a sequence $\{v_n\}$ such that $v_n \gg \theta$, for any $n \in \mathbb{N}$, $v_n \to \theta$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $u_m \preceq v_n$ for each $m > n_0$.*

For more details on c-sequences, see [**24**].

The following result is a genuine generalization of [**14**, Lemma 8].

LEMMA 1.4. *Let K be a solid cone in a topological vector space E. Let $\{u_n\}$ be a sequence in K. Then*

$$u_n \xrightarrow{E} \theta \text{ as } n \to \infty \text{ implies that } u_n \text{ is a c-sequence.}$$

PROOF. Let $c \gg \theta$ be given. Since $u_n \xrightarrow{E} \theta$ as $n \to \infty$, then by Lemma 1.2, there exists $n_0 \in \mathbb{N}$ such that $u_n \in [-c, c]$ for all $n \geqslant n_0$. This means that $u_n \preceq c$ for all $n \geqslant n_0$. Further, the conclusion holds because of (2) of Proposition 1.1. □

In the sequel, let E be a topological vector space, and denote $\mathcal{L}(E)$, $\mathcal{L}^*(E)$ as the set of linear and continuous operators on E, the set of bounded linear functionals on E, respectively. Let K be a solid cone of E.

Let (E, K) be an ordered topological vector space with zero vector θ. For the subsets X and Y of E, we introduce the following notations:

(i) $\overline{X}$ denotes the closure of X;

(ii) $\mathrm{conv}(X)$ denotes the convex hull of X;

(iii) $P(X)$ denotes the set of nonempty subsets of X;

(iv) $X + Y$ and λX ($\lambda \in \mathbb{R}$) stand for algebraic operations on X and Y;

(v) $\mathcal{B}_E$ denotes the family of all nonempty bounded subsets of E.

We introduce the concept of cone valued measure of noncompactness as follows.

DEFINITION 1.3. Let $\mu : \mathcal{B}_E \to K$ be a mapping. One says that μ is a cone valued measure of noncompactness on E if the following conditions are satisfied:

(1) for every $X \in \mathcal{B}_E$, $\mu(X) = \theta$ implies that X is precompact;

(2) for every pair $(X, Y) \in \mathcal{B}_E \times \mathcal{B}_E$, one has

$$X \subseteq Y \text{ implies } \mu(X) \preceq \mu(Y);$$

(3) for every $X \in \mathcal{B}_E$, one has

$$\mu(\overline{X}) = \mu(X) = \mu(\mathrm{conv}(X));$$

(4) if $\{X_n\}_0^\infty \in \mathcal{B}_E$ is a decreasing sequence (w.r.t. $\subseteq$) of closed sets such that $\{\mu(X_n)\}_0^\infty$ is a c-sequence, then $X_\infty = \bigcap_{n=0}^{\infty} X_n$ is nonempty.

REMARK 1.1. Clearly, each cone measure of noncompactness (see [**14**]) is cone valued measure of noncompactness, while conversely it does not holds. Indeed, put $E = C^1_{\mathbb{R}}[0,1]$ endowed with the strongest locally convex topology t^*. Then $K = \{x \in E : x(t) \geqslant 0 \text{ for all } t \in [0,1]\}$ is t^*-solid, but not t^*-normal. Then each

$\mu : \mathcal{B}_E \to K$ satisfying (1)-(4) is cone valued measure of noncompactness which is not cone measure of noncompactness. Hence, our notion is a genuine generalization of [**14**].

If $A : E \to E$, where A is a continuous linear operator and $AP \subseteq P$, then by (4) of Proposition 1.1, it follows that $\{A(x_n)\}$ is a c-sequence if $\{x_n\}$ is a c-sequence.

The following results are well-known and crucial in the theory of fixed point for nonlinear operators (see [**6**]).

THEOREM 1.2 (Brouwer). *Let K be a nonempty compact convex subset of a finite dimensional normed space, and let T be a continuous mapping of K into itself. Then T has a fixed point in K.*

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorems.

THEOREM 1.3 (Schauder). *Let K be a nonempty closed convex subset of a normed space. Let T be a continuous mapping of K into a compact subset of K. Then T has a fixed point in K.*

THEOREM 1.4 (Darbo). *Let K be a nonempty, bounded, closed, and convex subset of a Banach space E and let T be a continuous mapping of K into itself. Assume that there exists a constant $\lambda \in [0, 1)$ such that $\mu(TX) \leqslant \lambda\mu(X)$ for any $X \in K$. Then T has a fixed point.*

THEOREM 1.5 (Schauder-Tychonoff). *Let K be a nonempty compact convex subset of a locally convex Hausdorf space E, and let T be a continuous mapping of K into itself. Then T has a fixed point in K.*

THEOREM 1.6 (Singbal). *Let E be a locally convex Hausdorff space, K a nonempty closed convex subset of E, and T a continuous mapping of K into a compact subset of K. Then T has a fixed point in K.*

THEOREM 1.7 (Robert Cauty). *Let K be a nonempty compact convex subset of a topological vector Hausdorff space E, and let T be a continuous mapping of K into itself. Then T has a fixed point in K.*

2. Main results

Our first result regarding cone valued measure of noncompactness is the following:

THEOREM 2.1. *Let C be a nonempty, bounded, closed, and convex subset of an ordered topological vector space E with solid cone K. Let $T : C \to C$ be a mapping satisfying the following conditions:*
(i) T is continuous;
(ii) There exist $A \in \mathcal{L}^*(E)$ and a cone valued measure of noncompactness $\mu : \mathcal{B}_{(E,t)} \to K$ such that for all $X \in \mathcal{B}_{(E,t)}$,

$$(2.1) \qquad\qquad \mu(TX) \preceq A(\mu(X)).$$

Then T has at least one fixed point. Moreover, the set of fixed points of T is precompact.

PROOF. Taking $X_0 = C$, $X_{n+1} = \overline{\mathrm{conv}\,(TX_n)}$, for $n = 0, 1, 2, ...$, we get that $X_{n+1} \subseteq X_n$ for $n = 0, 1, 2, ...$ Hence, $\{X_n\}_0^\infty$ is a decreasing sequence of closed and convex sets. Furthermore, by (2.1) it is clear that

$$(2.2) \qquad \mu\,(X_{n+1}) = \mu\left(\overline{\mathrm{conv}\,(TX_n)}\right) = \mu\,(TX_n) \preceq A\,(\mu\,(X_n)),$$

for $n = 0, 1, 2, ...$

It follows immediately from (2.2) that

$$(2.3) \qquad \theta \preceq \mu\,(X_n) \preceq A^n\,(\mu\,(X_0)),\ \text{for } n = 0, 1, 2, ...$$

Since $\{A^n\,(\mu\,(X_0))\}$ is a c-sequence, then (2.3) establishes that $\{\mu\,(X_n)\}$ is also a c-sequence.

On the other hand, by virtue of (4) of Definition 1.3, we infer that $X_\infty = \cap_{n=1}^\infty X_n$ is nonempty, closed and convex. Also,

$$(2.4) \qquad \theta \preceq \mu\,(X_\infty) \preceq \mu\,(X_n),\ \text{for } n = 0, 1, 2, ...$$

Now, (2.4) implies $\theta \preceq \mu\,(X_\infty) \ll c$ for all $c \gg \theta$, that is., $\mu\,(X_\infty) = \theta$. Thus, via (1) of Definition 1.3, X_∞ is precompact. Now that X_∞ is closed, so X_∞ is compact. Further, $TX_\infty \subseteq X_\infty$. Then the continuity of the mapping $T : X_\infty \to X_\infty$ and Theorem 1.6 give us that T has at least one fixed point in X_∞. Finally, since the set of fixed points of T is a nonempty subset of X_∞ and $\mu\,(X_\infty) = \theta$, then by (1) and (2) of Definition 1.3, we deduce that the set of fixed points of T is precompact. $\square$

REMARK 2.1. If Theorem 1.6 may not be true, then we can suppose in all results that E is a locally convex space. In this case we use Shauders-Tihonov theorem.

Next we denote Φ as the set of all functions $\phi : K \backslash \{\theta\} \to (0, \infty)$ satisfying the following condition for every sequence $\{u_n\}$ in $K \backslash \{\theta\}$,

$$\lim_{n \to \infty} \phi\,(u_n) = 0 \text{ implies } \{u_n\} \text{ is a } c\text{-sequence.}$$

Now we have the following results in this new framework:

THEOREM 2.2. *Let C be a nonempty, bounded, closed, and convex subset of an ordered topological vector space E with solid cone K. Let $T : C \to C$ be a mapping satisfying the following conditions:*
(i) *T is continuous;*
(ii) *There exist $\phi \in \Phi$, $k \in (0, 1)$, and a cone valued measure of noncompactness $\mu : \mathcal{B}_E \to K$ such that*

$$X \in P\,(C),\ \mu\,(X), \mu\,(TX) \neq \theta \text{ imply } \phi\,(\mu\,(TX)) \leqslant [\phi\,(\mu\,(X))]^k.$$

Then T has at least one fixed point.

PROOF. Taking $X_0 = C, X_{n+1} = \overline{\mathrm{conv}\,(TX_n)}$, for $n = 0, 1, 2, ...$ we get that $X_{n+1} \subseteq X_n$ for $n = 0, 1, 2, ...$ Hence, $\{X_n\}_0^\infty$ is a decreasing sequence of closed and convex sets. Obviously, $\mu(X_n) \neq \theta$ and $\mu(TX_n) = \mu\left(\overline{\mathrm{conv}\,(TX_n)}\right) = \mu(X_{n+1}) \neq \theta$. Accordingly, by (ii), it establishes that

$$\phi\left(\mu\left(X_{n+1}\right)\right) = \phi\left(\mu\left(TX_n\right)\right) \leqslant \left[\phi\left(\mu\left(X_n\right)\right)\right]^k \leqslant \phi\left(\mu\left(X_n\right)\right).$$

As a result, the sequence $\{\phi\left(\mu\left(X_n\right)\right)\}$ is decreasing and thus $\lim_{n\to\infty} \phi\left(\mu\left(X_n\right)\right) = 0$. This means that $\{\mu\left(X_n\right)\}$ is a c-sequence. The remaining proof is the same as in Theorem 2.1. Consequently, T has at least one fixed point. $\quad\square$

Let Ψ be the set of all functions $\psi : K \to K$ satisfying the following conditions:
(1) ψ is a nondecreasing function w.r.t. the partial order $\preceq$, that is,

$$(u, v) \in K \times K \text{ and } u \preceq v \text{ imply } \psi(u) \preceq \psi(v);$$

(2) for all $u \in K \setminus \{\theta\}$, the sequence $\{\psi^n(u)\} \subset K$ is a c-sequence.

Next we announce the following result:

THEOREM 2.3. *Let C be a nonempty, bounded, closed, and convex subset of an ordered topological vector space E with solid cone K. Let $T : C \to C$ be a mapping satisfying the following conditions:*
(i) *T is continuous;*
(ii) *there exist $\psi \in \Psi$ and a cone valued measure of noncompactness $\mu : \mathcal{B}_E \to K$ such that*

$$\mu(TX) \preceq \psi(\mu(X)), \text{ for all } X \in P(C).$$

Then T has at least one fixed point.

PROOF. Taking $X_0 = C, X_{n+1} = \overline{\mathrm{conv}\,(TX_n)}$, for $n = 0, 1, 2, ...,$ we get that $X_{n+1} \subseteq X_n$ for $n = 0, 1, 2, ...$ Hence, $\{X_n\}_0^\infty$ is a decreasing sequence of closed and convex sets. Accordingly, by (ii), it establishes that

$$\mu(X_{n+1}) = \mu(TX_n) \preceq \psi(\mu(X_n)) \preceq \psi^2(\mu(X_{n-1})) \preceq \cdots \preceq \psi^n(\mu(X_1)).$$

Thus, by the definition of ψ, we have $\{\mu(X_n)\}$ is a c-sequence. The remaining proof is the same as in Theorem 2.1. Consequently, T has at least one fixed point. $\quad\square$

THEOREM 2.4. *Let C be a nonempty, bounded, closed, and convex subset of an ordered topological vector space E with solid cone K and $\mu : \mathcal{B}_E \to K$ be a cone valued measure of noncompactness on E. Let $T : C \to C$ be a mapping satisfying the following conditions:*
(i) *T is continuous;*
(ii) *for any $(u, v) \in K \times K$ with $\theta \preceq u \preceq v$, there exists $0 < k(u, v) < 1$ such that*

$$X \in P(C), \text{ and } u \preceq \mu(X) \preceq v \text{ imply } \mu(TX) \preceq k(u, v)\mu(X);$$

(iii) *K is a regular cone.*
Then T has at least one fixed point.

PROOF. Taking $X_0 = C, X_{n+1} = \overline{\text{conv}\,(TX_n)}$, for $n = 0, 1, 2, ...$ we get that $X_{n+1} \subseteq X_n$ for $n = 0, 1, 2, ...$ Hence, $\{X_n\}_0^\infty$ is a decreasing sequence of closed and convex sets. Accordingly, by (ii), it establishes that

$$\mu(X_{n+1}) = \mu(TX_n) \preceq k(u, v)\mu(X_n) \preceq \mu(X_n).$$

Thus, $\{\mu(X_n)\}$ is a decreasing sequence which is bounded below because of $\theta \preceq \mu(X_n)$. Then by the regularity of the cone K, $\{\mu(X_n)\}$ converges to θ. This implies that $\{\mu(X_n)\}$ is a c-sequence. The remaining proof is the same as in Theorem 2.1. Consequently, T has at least one fixed point.   □

The following result is an Sadowskii fixed point theorem with respect to cone valued measure of noncompactness.

THEOREM 2.5. *Let C be a nonempty, bounded, closed, and convex subset of an ordered topological vector space E with solid cone K and $\mu : \mathcal{B}_E \to K$ be a cone valued measure of noncompactness on E satisfying the following conditions:*
(i) there exists $x_0 \in C$ such that

$$\mu\left(X \cup \{x_0\}\right) = \mu\left(X\right), X \in \mathcal{B}_E;$$

(ii) $T : C \to C$ is continuous; (iii) for every $X \in P\left(C\right)$, it holds

$$\mu\left(X\right) \neq \theta \text{ implies } \mu\left(TX\right) \prec \mu\left(X\right).$$

Then T has at least one fixed point.

PROOF. Let us denote $\mathcal{M}$ as the set of subsets $M \subseteq C$ satisfying the following conditions: $x_0 \in M$, M is closed and convex, and $TM \subseteq M$. Obviously, $\mathcal{M}$ is a nonempty set in view of $C \in \mathcal{M}$. Set $X = \cap_{M \in \mathcal{M}} M$, then X is a nonempty (because $x_0 \in X$), closed, and convex set. Clearly, $TX \subseteq X$. Set $Y = \overline{\text{conv}}\,(TX \cup \{x_0\})$. We claim that $X = Y$. In fact, we have $x_0 \in X$ and $TX \subseteq X$, which yields $Y \subseteq X$. On the other hand, the inclusion $Y \subseteq X$ implies that $TY \subseteq TX \subseteq Y$. Note that $x_0 \in Y$, then $Y \in \mathcal{M}$ and $X \subseteq Y$. This proves our claim. Thus, from (1) and (2) of Definition 1.3, we speculate

$$\mu\left(X\right) = \mu\left(TX \cup \{x_0\}\right) = \mu\left(TX\right).$$

If $\mu\left(X\right) \neq \theta$, then by (iii), it means that

$$\mu\left(X\right) = \mu\left(TX\right) \prec \mu\left(X\right),$$

which is a contradiction. As a consequence, $\mu\left(X\right) = \theta$. Now X is precompact, further, it is compact since it is closed. Finally, by Theorem 1.6, the mapping $T : X \to X$ has at least one fixed point.   □

THEOREM 2.6. *Let C be a nonempty, bounded, closed, and convex subset of an ordered topological vector space E with solid cone K and $\mu : \mathcal{B}_E \to K$ be a cone valued measure of noncompactness on E. Let $T : C \to C$ be a given mapping. Suppose that*
(i) T is continuous;
(ii) there exists $x_0 \in C$ such that, for all $\lambda \in (0, 1)$ and $X \in \mathcal{B}_E$,

$$\mu\left(\lambda TX + (1 - \lambda)\{x_0\}\right) \preceq \lambda\mu\left(TX\right);$$

(iii) $(I - T) C$ is closed, where $I : C \to C$ is the identity mapping;

(iv) one has

$$\mu\left(TX\right) \preceq \mu\left(X\right), X \in \mathcal{B}_E.$$

Then T has at least one fixed point.

PROOF. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\lambda_n \to 1$ as $n \to \infty$. Consider the sequence of mappings $T_n : C \to C$ defined by

$$T_n x = \lambda_n T x + (1 - \lambda_n) x_0, \text{ for all } x \in C, \ n = 0, 1, 2, ...$$

Note that T_n is well-defined since C is convex set. Using the assumptions, for all $X \in \mathcal{B}_E$, and for all $n = 0, 1, 2, ...$, we obtain that

$$\mu\left(T_n X\right) = \mu\left(\lambda_n T X + (1 - \lambda_n) \{x_0\}\right) \preceq \lambda_n \mu\left(TX\right) \preceq \lambda_n \mu\left(X\right).$$

Define a sequence of mappings $A_n : E \to E$ by

$$A_n u = \lambda_n u, \ u \in E, \ n = 0, 1, 2, ...,$$

then $A_n \in \mathcal{L}^*(E)$ $(n = 0, 1, 2, ...)$ and it holds (2.1). So by Theorem 2.1, the mapping T_n has a fixed point $x_n \in C$, that is,

$$T_n x_n = \lambda_n T x_n + (1 - \lambda_n) x_0 = x_n, \ n = 0, 1, 2, ...,$$

which yields that

$$(I - T) x_n = T_n x_n - T x_n = (\lambda_n - 1) T x_n + (1 - \lambda_n) x_0, \ n = 0, 1, 2, ...$$

Since $\{T x_n\}$ is a bounded sequence, we get that $\{(I - T) x_n\}$ is a c-sequence. Since $(I - T) C$ is closed, we deduce that $\theta \in (I - T) C$. As a consequence, there is some $x \in C$ such that $(I - T) x = \theta$, which means that $x \in C$ is a fixed point of T. □

REMARK 2.2. From [25], we know that the finest locally convex topology t^* as well as the finest linear topology t^{**} are equal if $\dim E$ is countable. Yet, if $\dim E$ is uncountable then $t^* < t^{**}$. For instance, let $E = C\left[0, 1\right]$ with the supremum norm and let $K = \{x \in E : x\left(t\right) \geqslant 0, t \in \left[0, 1\right]\}$. Then E is ordered Banach space with solid cone K. It is clear that K is t^{**}-solid, further, $t^* < t^{**}$ because Bair theorem, in other words, E is a set with uncountable dimensions. Hence all previous theorems can be considered for ordered topological spaces which are not even locally convex.

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