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A GENERALIZATION OF BI IDEALS IN SEMIRINGS

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Abstract. Bi ideals are the generalization of quasi ideals which are themselves the generalization of the so called one-sided, right and left ideals. In this paper, we define the m-bi ideals as a generalization of the bi ideals. The important properties of the m-bi ideals from the pure algebraic point of view have been described. Moreover, we present the form of the m-bi ideals generated by subsets of the semiring. On the basis of these properties, further characterizations of the semiring will be helpful.

1. Introduction

Vandiver introduced the idea of semirings as a generalization of rings, and having no negative elements in 1934 [**10**]. Their most common example in the daily life is of the non-negative integers which make the semiring under usual addition and multiplication. Since their origin, semrings have been extensively used in the theories of automata, operator algebra, algebras of formal processes, generalized fuzzy computation, optimization and computer science. Recently, they are being used in combinatorial optimization, Baysian networks, belief propagation, algebraic geometry, optimization algebra, dequantization and amoebas.The studies of their sub-structures like the subsemirings and ideals play an important role in their advanced studies and uses. The generalization of the ideals explore the results necessary for the classification of the semirings.

In lieu of their importance, Steinfeld introduced the notion of quasi ideals for semigroups and rings respectively in [**8**] and [**9**]. Iseki [**5**] used this concept for semirings without zero and proved important results on semirings using quasi ideals. Shabir et al [**7**] characterized the semirings by the properties of their quasi-ideals.

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The concept of bi ideals for associative rings were introduced by Lajos and Szasz [**6**]. Quasi ideals are a generalization of left and right ideals. Bi ideals are a generalization of quasi ideals. Munir et al characterized some classes of the semirings e.g. regular and weakly regular semirings using their quasi and bi ideals [**2**]. In this paper, we define another class of such ideals named m-bi ideals as a generalization of bi ideals. We prove important basic results on these ideals.

2. Preliminaries

We present a brief summary of the basic notions and concepts used in semirings that will be of high value for our later pursuits. [**3**] and [**4**] can be referred for the undefined terms.

Definition 2.1. A *semiring* is a nonempty set A together with two binary operation $+$ (Addition) and $\cdot$ (Multiplication) such that

(1) $(A, +)$ is a commutative semigroup,
(2) $(A, \cdot)$ is a semigroup;generally a non-commutative,
(3) the distributive laws hold i.e.,

$$a(b + c) = ab + ac \text{ and } (a + b)c = ac + bc, \text{ for all } a, b, c \in A.$$

We assume that $(A, +, \cdot)$ has an absorbing zero 0, i.e., $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 \cdot a = 0$ hold for all $a \in A$.

Definition 2.2. A nonempty subset S of a semring $(A, +, \cdot)$ is called a *subsemiring* of $(A, +, \cdot)$ if S itself is a semiring under the operations of addition and multiplication of A.

The following theorem characterizes the semirings. We state it without proof.

Theorem 2.3. *Let $(A, +, \cdot)$ be a semiring, then a non-empty subset S of A is a subsemiring of A if and only if*

(1) *$x + y \in S$ for all $x, y \in S$,*
(2) *$xy \in S$ for all $x, y \in S$,*
(3) *$0 \in S$.*

Definition 2.4. Let X be a non-empty subset of a semiring $(A, +, \cdot)$, then the smallest subsemiring of $(A, +, \cdot)$ containing X is called the subsemiring of A generated by X.

Definition 2.5. Let X and Y be two non-empty subsets of a semiring $(A, +, \cdot)$, then the *sum* $X + Y$ respectively *product* XY of X and Y are defined by $X + Y = \{x + y : x \in X \quad \text{and} \quad y \in Y\}$, and $XY = \{\sum_{\text{finite}} x_i y_i : x_i \in X \quad \text{and} \quad y_i \in Y\}$.

Definition 2.6. A nonempty subset E of a semiring $(A, +, \cdot)$, is called a *right(left)* ideal of A if the following conditions are satisfied:

(1) $x + y \in E$ for all $x, y \in E$,
(2) $xa \in E(ax \in E)$ for all $x \in E$ and $a \in A$.

E is called a *two-sided ideal* or simply an *ideal* if it is both a left and a right ideal of A.

THEOREM 2.7. *Let E and F be two ideals of a semiring $(A,+,\cdot)$, then $E+F=\{i+j : i \in E, j \in F\}$ is the smallest ideal containing both E and F.*

PROOF. Let $x=i+j$, $y=i^{'}+j^{'}$, where $i,i^{'} \in E$, $j,j^{'} \in F$, be two elements of $E+F$, then $x+y=(i+j)+(i^{'}+j^{'})=(i+i^{'})+(j+j^{'}) \in E+F$. $xa=(i+j)a=ia+ja \in E+F$, and $ax=a(i+j)=ai+aj \in E+F$. That is $E+F$ is an ideal of $(A,+,\cdot)$. Since $i=i+0 \in E+F$ for all $i \in E$ and $j=0+j \in E+F$ for all $j \in F$. So $E \subset E+F$ and $F \subset E+F$. That is $E+F$ contains both E and F.

Lastly, if S is another ideal of A containing both E and F, then $i+j \in S$ for all $i \in E$ and $j \in F$. So, $E+F \subset S$. Thus, $E+F$ is the smallest ideal of A containing both E and F. $\square$

DEFINITION 2.8. Let X be a nonempty subset of a semiring $(A,+,\cdot)$, then right / left ideal generated by X is the smallest right / left ideal which contains X i.e., it is the intersection of all right / left ideals which contains X. If X is finite set, then the right (left or two-sided) ideal generated by X is called the finitely generated right (left or two-sided) ideal respectively.

THEOREM 2.9. *Let X be a nonempty subset of a semiring $(A,+,\cdot)$, then*

(1) *The right ideal generated by X is N_0X+XA,*
(2) *The left ideal generated by X is N_0X+AX,*
(3) *The two-sided ideal generated by X is $N_0X+AX+XA+XAX$, where N_0 is the set of whole numbers.*

PROOF. (1) Let

$$x=\sum_{\text{finite}} n_jx_j+\sum_{\text{finite}} x_ia_i, \quad y=\sum_{\text{finite}} n_j^{'}x_j^{'}+\sum_{\text{finite}} x_i^{'}a_i^{'},$$

where $n_j,n_j^{'} \in N_0$, $x_j,x_j^{'} \in X$, $a_i,a_i^{'} \in A$, be any two elements of N_0X+XA. Then

$$x+y=\sum_{\text{finite}}(n_jx_j+n_j^{'}x_j^{'})+\sum_{\text{finite}}(x_ia_i+x_i^{'}x_i^{'}) \in N_0X+XA.$$

Now if $a \in A$, then

$$xa=\sum_{\text{finite}}(n_ix_i+x_ia_i)a=\sum_{\text{finite}} 0.x_i+\sum_{\text{finite}}(n_ix_ia+x_ia_ia) \in N_0X+XA,$$

i.e., N_0X+XA is a right ideal of $(A,+,\cdot)$. Now $x=\sum_{\text{finite}} 1.x_i+\sum_{\text{finite}} x_i.0 \in N_0X+XA$ i.e., $X \subset N_0X+XA$.

Suppose that $S \neq \{0\}$ be another right ideal of $(A,+,\cdot)$ containing X. Then an element of the form $\sum_{\text{finite}} n_i.x_i+\sum_{\text{finite}} x_i.a_i, \quad x_i \in X, \quad a_i \in A$ belongs to S because S is the right ideal containing X. Thus $N_0X+XA \subset S$. Therefore, N_0X+XA is the smallest right ideal containing X.

(2) Analogously.

(3) Analogously. $\square$

COROLLARY 2.10. *If the semiring $(A,+,\cdot)$ contains the multiplicative identity, then*

(1) *The right ideal generated by* X *is* XA,
(2) *The left ideal generated by* X *is* AX,
(3) *The two-sided ideal generated by* X *is* AXA.

Proof. Since $1 \in A$,
(1) therefore $N_0X \subset XA$, and $N_0X + XA = XA$.
(2) Similar.
(3) Now N_0X, AX, and XA are contained in AXA, so two-sided ideal generated by X is AXA.

□

Corollary 2.11. *If* $(X, +)$ *is a subsemigroup with zero of* $(A, +)$, *then*
(1) *The right ideal generated by* X *is* $X + XA$,
(2) *The left ideal generated by* X *is* $X + AX$,
(3) *The two-sided ideal generated by* X *is* $X + AX + XA + AXA$.

Proof. If $(X, +)$ is a *submonoid* of $(A, +)$, then $N_0X = X$, so
(1) The right ideal generated by X is $X + XA$.
(2) Similar.
(3) The two-sided ideal generated by X is $X + AX + XA + AXA$.

□

Definition 2.12. Let $(A, +, \cdot)$ be a semiring. A *quasi ideal* Q of A is a subsemigroup $(Q, +)$ of A such that $AQ \cap QA \subseteq Q$ [**5**].

Each quasi ideal of a semiring A is its subsemiring. Every one-sided ideal of A is a quasi ideal of A. Since intersection of any family of quasi ideals of A is a quasi ideal of A [**7**], so intersection of a right ideal R and a left ideal L of a semiring A is a quasi ideal of A.

The sum and the product of quasi ideals both are not quasi ideals [**7**].

Definition 2.13. Let $(A, +, \cdot)$ be a semiring. A *bi ideal* B is a subsemiring of A such that $BAB \subseteq B$.

Every quasi ideal of a semiring A is a bi ideal. The product of two quasi ideals of a semiring A is a bi ideal of A. Bi ideal may not be a quasi ideal. The product RL of a left ideal L and right ideal R of A is a bi ideal of A, but not a quasi ideal[**7**].

The product TB and BT of an arbitrary subset T and bi ideal B of a semiring A are bi ideals of A. So the product of two bi ideals of a semiring is a bi ideal. Thus the intersection of a family of bi ideals of a semiring A is a bi ideal of A.

Definition 2.14. A subsemiring Q of a semiring A is called an (m, n)-*quasi ideal* of A if $A^mQ \cap QA^n \subseteq Q$ where m and n are positive integers [**1**].

It is to be noted that a quasi ideal Q of a semiring A is a $(1, 1)$-quasi ideal of A. If A is a semiring having an identity, then all (m, n)-quasi ideals of A are quasi ideals of A for all $m, n \in N$. Moreover, an (m, n)-quasi ideal of A is a (k, l)-quasi ideal of A for all $k \geqslant m$ and $l \geqslant n$. Any (m, n)-quasi ideal of a semiring S needs not be a quasi ideal of A.

Definition 2.15. For a semiring A, and a positive integer m, we have $A^m = AAA...A$(m-times).

Now $A^2 = AA \subseteq A$; as A is a semiring. Therefore, $A^3 = AAA \subseteq A^2 \subseteq A$, i.e., $A^3 \subseteq A^2$, and $A^3 \subseteq A$. So, we conclude that $A^l \subseteq A^m$ for all positive integers l and m, such that $l \geqslant m$. Consequently $A^m \subseteq A$, for all m.

3. m-Bi Ideals

In this section, we define the notion of m-bi ideals, and discuss their important properties.

Definition 3.1. Let $(A, +, \cdot)$ be a semiring. An *m-bi ideal* B of A is a subsemiring of A such that $BA^mB \subseteq B$ where m is a positive integer, not necessarily 1, called *bipotency* of the *bi ideal* B.

$BA^mB \subseteq B$ is called the *bipotency condition*. It is to be noted that a bi ideal B of a semiring A is a 1-bi ideal of A (bi ideal of bipotency 1). All the so-called 1-bi ideals are the simply the bi ideals, whereas those with bipotency $m > 1$ are to specified with the value of m.

Proposition 3.2. *For every $m \geqslant 1$, every bi-ideal is an m-bi ideal.*

Proof. If B is a bi ideal of A, then $BAB \subseteq B$ can be written as $BA^1B \subseteq B$ employing that B is a bi ideal with bipotency $m = 1$. $\square$

The converse of the above result is not true as is evident from the following example.

Example 3.3. *Let*

$$S = \left\{ \begin{bmatrix} 0 & u & v & w \\ 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} : u, v, w, x, y \text{ are any positive real numbers} \right\},$$

and

$$A = S^0 = S \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then $(A, +, \cdot)$ is a semiring under the usual operations of addition $+$ and multiplication $\cdot$ of matrices.

Let

$$B = \left\{ \begin{bmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} : u, z \text{ are any positive real numbers} \right\} \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then B is 2-bi ideal of A as $BA^2B \subseteq B$, and $BAB \not\subseteq B$.

The following two examples characterize the m-bi ideals in classes of *idempotent* and *nilpotent* matrices.

EXAMPLE 3.4. *Let S be the set of all idempotent matrices of idempotency m, then S forms a semiring under the usual addition and multiplication of matrices. In this case, every bi ideal B of S forms its m-bi ideal; as $BS^mB = BSB \subseteq B$ implies $BS^mB \subseteq B$.*

EXAMPLE 3.5. *Let S be the set of all nilpotent matrices of nilpotency m, then S forms a semiring under the usual addition and multiplication of matrices as described before. In this case, every subsemiring B of S forms its m-bi ideal; as $BS^mB = B0B \subseteq B$ implies $BS^mB \subseteq B$, 0 is the zero matrix.*

The left ideal L and the right ideal R of the semiring A are the bi ideals or the 1-bi ideals. Every ideal of A is a 1-bi ideal of A.

PROPOSITION 3.6. *The product of any two m-bi ideals of a semiring A, with identity e, is m-bi ideal.*

PROOF. Let B_1 and B_2 be two bi ideals of a semiring A with bipotencies m_1 and m_2 respectively, that is, $B_1A^{m_1}B_1 \subseteq B_1$ and $B_2A^{m_1}B_2 \subseteq B_2$, m_1 and m_2 are any positive integers. Then B_1B_2 is obviously closed under addition by the Definition 2.5. Now we have,

$$(B_1B_2)^2 = (B_1B_2)(B_1B_2) = (B_1AB_1)B_2 = (B_1Ae...eB_1)B_2 \subseteq$$
$$(B_1AA...AB_1)B_2 \subseteq (B_1A^mB_1)B_2 \subseteq B_1B_2.$$

That is, $(B_1B_2)^2 \subseteq B_1B_2$. So, B_1B_2 is closed under multiplication. B_1B_2 is a subsemiring of A. Moreover,

$$B_1B_2(A^{\max(m_1,m_2)})B_1B_2 \subseteq B_1AA^{\max(m_1,m_2)}B_1B_2 =$$
$$B_1A^{1+\max(m_1,m_2)}B_1B_2 \subseteq B_1A^{m_1}B_1B_2 \subseteq B_1B_2.$$

We used the result $A^{1+\max(m_1,m_2)} \subseteq A^{m_1}$ as is evident by Definition 2.15. So,

$$B_1B_2(A^{\max(m_1,m_2)})B_1B_2 \subseteq B_1B_2$$

. Thus, B_1B_2 is an m-bi ideal of A with bipotency $\max(m_1, m_2)$. □

PROPOSITION 3.7. *Let T be an arbitrary subset of a semiring A with identity e, and B be an m- bi ideal of A, m not necessarily* 1. *Then the product BT is also m-bi ideal of A.*

PROOF. It is straightforward to show that BT as defined by the Definition 2.5 is closed under addition. Next,

$$(BT)^2 = (BT)(BT) =$$
$$(BTB)T \subseteq (BAB) \subseteq BAe...eB \subseteq BAA...AB \subseteq (BA^mB)T \subseteq BT.$$

So, $BT^2 \subseteq BT$ making it a subsemiring of A. Moreover,

$$BT(A^m)BT \subseteq BAA^mBT \subseteq BA^{1+m}BT \subseteq BA^mBT \subseteq BT.$$

Therefore BT is an m-bi ideal of A. □

Similarly, we can show that TB is also an m-bi ideal of A.

PROPOSITION 3.8. *The intersection of a family of bi ideals of semiring A with bipotencies $m_1, m_2, ...,$ is also a bi ideal with bipotency $max\{m_1, m_2, ...\}$.*

PROOF. Let $\{B_\lambda : \lambda \in \wedge\}$ be a family of m-bi ideals of semiring A. Then $B = \bigcap\limits_{\lambda \in \wedge} B_\lambda$, being the intersection of subsemirings of A is a subsemiring of A. Since $B_\lambda A^{m_\lambda} B_\lambda \subseteq B_\lambda \quad \forall \quad \lambda \in \wedge$, and $B \subseteq B_\lambda \quad \forall \quad \lambda \in \wedge$, therefore

$$BA^{\max\{m_\lambda : \lambda \in \wedge\}} B \subseteq B_\lambda A^{m_\lambda} B_\lambda \subseteq B_\lambda \quad \forall \quad \lambda \in \wedge.$$

That is, $BA^{\max\{m_\lambda : \lambda \in \wedge\}} B \subseteq B_\lambda \quad \forall \quad \lambda \in \wedge$. This gives $BA^{\max\{m_\lambda : \lambda \in \wedge\}} B \subseteq \bigcap\limits_{\lambda \in \wedge} B_\lambda = B$. So, $BA^{\max\{m_\lambda : \lambda \in \wedge\}} B \subseteq B$. Thus B is an m-bi ideal with *bipotency* $\max\{m_1, m_2, ...\}$. □

Sum of two m-bi ideals of a semiring is not an m-bi ideals.

EXAMPLE 3.9. *Let*

$$A = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,\ b,\ c,\ d \text{ are non-negative integers} \right\}.$$

Then A is a semiring under usual addition and multiplication of matrices.

Let

$$B_1 = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \text{ is a non-negative integers} \right\},$$

and

$$B_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} : y \text{ is a non-negative integers} \right\},$$

then B_1 and B_2 are 1-bi ideals of A. But $B = B_1 + B_2$, is not a bi ideal of A. Indeed, in this case,

$$B = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x \text{ and } y \text{ are non-negative integers} \right\},$$

and so, $BAB \not\subseteq B$.

PROPOSITION 3.10. *Every (m, m)-quasi ideal Q of a semiring A is an m-bi ideal of A.*

PROOF. Consider

$$QA^mQ \subseteq QA^mA = QA^{m+1} \subseteq QA^m,$$

and so $QA^mQ \subseteq QA^m$. Similarly, $QA^mQ \subseteq A^mQ$. Combining these two, we have

$$QA^mQ \subseteq QA^m \cap A^mQ \subseteq Q$$

. Thus $QA^mQ \subseteq Q$. That is, Q is an m-bi ideal. □

PROPOSITION 3.11. *Product of two (m,n)-quasi ideals Q_1 and Q_2 of A is an m-bi ideal of A.*

PROOF. Since

$$Q_1Q_2 = \left\{ \sum_{\text{finite}} m_i n_i : m_i \in Q_1, n_i \in Q_2, i \in \wedge \right\},$$

this means that Q_1Q_2 is closed under addition. Since every quasi ideal is bi ideal,

$$(Q_1Q_2)(Q_1Q_2) \subseteq Q_1(Q_2AQ_2) \subseteq Q_1Q_2.$$

That is, $(Q_1Q_2)^2 \subseteq Q_1Q_2$. So Q_1Q_2 is closed under multiplication. Clearly

$$(Q_1Q_2)A^{\max\{m_1,m_2,n_1,n_2\}}(Q_1Q_2) \subseteq Q_1Q_2A^{\max\{m_1,m_2,n_1,n_2\}}(AQ_2) \subseteq$$
$$Q_1(Q_2A^{\max\{m_1,m_2,n_1,n_2\}+1}Q_2) \subseteq Q_1Q_2.$$

So, Q_1Q_2 is Bi-ideal of A. $\square$

DEFINITION 3.12. Let A be a semiring. A subsemiring L of A is called an m-left ideal of A if $A^mL \subseteq L$ where m is a positive integer. An n-right ideal of S is defined analogously where n is a positive integer [**1**].

PROPOSITION 3.13. *An m-left/ n-right ideal of semiring A is an m-bi ideal.*

PROOF. Let L be the m-left ideal of A, then $LA^mL \subseteq LL \subseteq L$. This gives that L is m-bi ideal of A. The proof for m-right ideal is analogous. $\square$

THEOREM 3.14. *Let A be a semiring.*
(1) *Let L_i be an m-left ideal of A for all $i \in I$. Then $\bigcap_{i\in I} L_i$ is an m-left ideal of A.*
(2) *Let R_i be an n-right ideal of A for all $i \in I$. Then $\bigcap_{i\in I} R_i$ is an n-right ideal of A.*

PROOF. As Proposition 3.8. $\square$

The following theorem shows that the intersection of an m-left ideal and an n-right ideal of a semiring A is its t-bi ideal, where $t = \max(m,n)$.

THEOREM 3.15. *Let L and R be an m-left ideal and an n-right ideal of a semiring A. Then $L \cap R$ is an t-bi ideal of A, where $t = max(m,n)$.*

PROOF. Since $0 \in L \cap R$, by Lemma 3.1, we have $L \cap R$ is a subsemiring of A. Next, since L and R are also m-bi and n-bi ideals of A, their intersection becomes $max(m,n)$-bi ideals from the result 3.8. Alternatively,

$$L \cap R(A^{max\{m,n\}})L \cap R \subseteq LA^{max\{m,n\}}L \subseteq A^{max\{m,n\}+1}L \subseteq A^mL \subseteq L.$$

Similarly, we can show that $L \cap R(A^{max\{m,n\}})L \cap R \subseteq R$. Consequently,

$$L \cap RA^{max\{m,n\}}L \cap R \subseteq L \cap R$$

$\square$

3.1. Finitely Generated m-bi Ideals. Let S be a subset of a semiring A and

$$\tau = \{B : B \quad \text{is an } m\text{-bi ideal of A containing S }\}.$$

Therefore τ is nonempty because $A \in \tau$. Let $< S >_m = \bigcap_{B \in \tau} B$. Clearly, $< S >_m$ is nonempty because $0 \in < S >_m$. Since the intersection of m-bi ideals is an m-bi ideal, so $< S >_m$ is an m-bi ideal of A. Moreover, $< S >_m$ is the smallest m-bi ideal of A containing S. The m-bi ideal $< S >_{(m)}$ is called the m-bi ideal of A generated by S. It is clear that $< \phi >_m = < 0 >_m = \{0\}$. An m-bi ideal is called principal if it is generated as an m-bi ideal by a single element.

Theorem 3.16. *Let S be a nonempty subset of a semiring A. Then the m-bi ideal generated by S is $< S >_m = \sum_{i=1}^{m} N_0 S^i + SA^m S$*

Proof. We need to show that $< S >_m = \sum_{i=1}^{m} N_0 S^i + SA^m S$ is the smallest m-bi ideal of A containing S. Let $a, b \in < S >_m$. Therefore,

$$a = \sum_{\text{finite}} (n_j x_{j_1} \cdots x_{j_m}) + \sum_{\text{finite}} (s_j a_{j_1} \cdots a_{j_m} t_j), \text{ and}$$
$$b = \sum_{\text{finite}} (n_j^{'} x_{j_1}^{'} \cdots x_{j_m}^{'}) + \sum_{\text{finite}} (s_j^{'} a_{j_1}^{'} \cdots a_{j_m}^{'} t_j^{'}),$$

where $n_j, n_j^{'} \in N_0$,

$$x_{j_1}, \cdots, x_{j_m}, s_j, t_j, x_{j_1}^{'}, ..., x_{j_m}^{'}, s_j^{'}, t_j^{'} \in S,\ a_{j_1}, \cdots, a_{j_m}, a_{j_1}^{'}, \cdots, a_{j_m}^{'} \in A,$$

be any two elements of $< S >_m$.

$$\begin{aligned}
& a + b \\
= & \sum_{\text{finite}} (n_j x_{j_1} \cdots x_{j_m}) + \sum_{\text{finite}} (s_j a_{j_1} \cdots a_{j_m} t_j) + \sum_{\text{finite}} (n_j^{'} x_{j_1}^{'} \cdots x_{j_m}^{'}) + \\
& \sum_{\text{finite}} (s_j^{'} a_{j_1}^{'} \cdots a_{j_m}^{'} t_j^{'}) \\
= & \sum_{\text{finite}} (n_j x_{j_1} \cdots x_{j_m}) + \sum_{\text{finite}} (n_j^{'} x_{j_1}^{'} \cdots x_{j_m}^{'}) + \sum_{\text{finite}} (s_j a_{j_1} \cdots a_{j_m} t_j) + \\
& \sum_{\text{finite}} (s_j^{'} a_{j_1}^{'} \cdots a_{j_m}^{'} t_j^{'}) \\
= & \sum_{\text{finite}} \big((n_j x_{j_1} \cdots x_{j_m}) + (n_j^{'} x_{j_1}^{'} \cdots x_{j_m}^{'}) \big) + \\
& \sum_{\text{finite}} \big((s_j a_{j_1} \cdots a_{j_m} t_j) + (s_j^{'} a_{j_1}^{'} \cdots a_{j_m}^{'} t_j^{'}) \big) \in \sum_{i=1}^{m} N_0 S^i + SA^m S.
\end{aligned}$$

$$
\begin{aligned}
& ab \\
=& \Big(\sum_{\text{finite}} (n_j x_{j_1} \cdots x_{j_m}) + \\
& \sum_{\text{finite}} (s_j a_{j_1} \cdots a_{j_m} t_j) \Big) \Big(\sum_{\text{finite}} (n_j^{'} x_{j_1}^{'} \cdots x_{j_m}^{'}) + \sum_{\text{finite}} (s_j^{'} a_{j_1}^{'} \cdots a_{j_m}^{'} t_j^{'}) \Big) \\
=& \sum_{\text{finite}} (n_j x_{j_1} \cdots x_{j_m}) \sum_{\text{finite}} (n_j^{'} x_{j_1}^{'} \cdots x_{j_m}^{'}) + \sum_{\text{finite}} (n_j x_{j_1} \cdots x_{j_m}) \sum_{\text{finite}} (s_j^{'} a_{j_1}^{'} \cdots a_{j_m}^{'} t_j^{'}) \\
+& \sum_{\text{finite}} (s_j a_{j_1} \cdots a_{j_m} t_j) \sum_{\text{finite}} (n_j^{'} x_{j_1}^{'} \cdots x_{j_m}^{'}) + \\
& \sum_{\text{finite}} (s_j a_{j_1} \cdots a_{j_m} t_j) \sum_{\text{finite}} (s_j^{'} a_{j_1}^{'} \cdots a_{j_m}^{'} t_j^{'}) \\
\in& \sum_{i=1}^{m} N_0 S^i + S A^m S,
\end{aligned}
$$

as the first two terms belong to $\sum_{i=1}^{m} N_0 S^i$, and the second two terms belong to SA^mS. So $< S >_m$ is a subsemiring of A. Next, we need to show that $< S >_m$ $A^m < S >_m \subseteq < S >_m$. Consider

$$
\begin{aligned}
< S >_m A^m < S >_m &= \Big(\sum_{i=1}^{m} N_0 S^i + S A^m S \Big) A^m \Big(\sum_{i=1}^{m} N_0 S^i + S A^m S \Big) \\
&= \Big(\sum_{i=1}^{m} N_0 S^i \Big) A^m \Big(\sum_{i=1}^{m} N_0 S^i \Big) + \Big(\sum_{i=1}^{m} N_0 S^i \Big) A^m (S A^m S) \\
&+ (S A^m S) A^m \Big(\sum_{i=1}^{m} N_0 S^i \Big) + (S A^m S) A^m (S A^m S)
\end{aligned}
$$

The first two terms in the above expression belong to $\sum_{i=1}^{m} N_0 S^i$ by the definition of the finite sums, and the second two terms belong to SA^mS. Therefore $< S >_m$ $A^m < S >_m \subseteq < S >_m$. That is, $< S >_m$ $A^m < S >_m$ is an m-bi ideal containing S.That is $BA^mB \subseteq B$. To show that $< S >_m$ is the smallest m-bi ideal of A containing S, let B' be any other m-bi ideal of A containing S. Then $N_0S^i \subseteq B'$ for all $i \in N$ and $SA^mS \subseteq B'A^mB' \subseteq B'$. Therefore $< S >_m = \sum_{i=1}^{m} N_0 S^i + SA^mS \subseteq B'$. Hence, $< S >_m$ is the smallest m-bi ideal of A containing S. □

COROLLARY 3.17. *If the semiring $(A, +, \cdot)$ contains the multiplicative identity* 1, *then the m-bi ideal generated by the nonempty set S is $< S >_m = SA^mS$.*

PROOF. If A contains 1, then $\sum_{i=1}^{m} N_0 S^i \subseteq SA^mS$. So, $< S >_m = SA^mS$. □

COROLLARY 3.18. *If S is a subsemigroup with zero of $(A, +)$, then the m-bi ideal generated by S is $< S >_m = S + SA^mS$.*

PROOF. If S is a subsemigroup with *zero* of $(A,+)$, then $\sum_{i=1}^{m} N_0 S^i \subseteq S$. In this case, the m-bi ideal generated by S is $< S >_m = S + SA^m S$. □

4. Conclusions

We introduced the notion of m-bi ideal in semirings as a generalization of their bi ideals. We have studied some of their basic properties and characterized some of their properties using their m-bi ideals. We also presented the forms of the m-bi ideals of a semiring generated by a subset of the semiring. In the future, we want to characterize some more classes of the semirings like regular semirings, intra regular and weakly regular semirings using their m-bi ideals. Moreover, some other classes of the m-bi ideals like prime m-bi ideals, maximal and minimal m-bi ideals, principal m-bi ideals will be studied. Their studies with regard to the semiring homomorphisms and factor semirings will be explored.

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References

[1] R.Chinram. A note on (m, n)-quasi-ideals in semirings. *Int. J. Pure Appl. Math.*, **49** (1)(2008), 45-52.

[2] M. Munir and M.Habib. Characterizing Semirings using Their Quasi and Bi-Ideals. *Proceedings of the Pakistan Academy of Sciences, A. Physical and Computational Sciences*, **53**(4)(2016), 469-475.

[3] J. S. Golan, *The theory of semirings with applications in mathematics and theoretical computer science.* Addison-Wesley Longman Ltd, (1992).

[4] J.M. Howie, *An introduction to semigroup theory.* New York-London, (1976).

[5] K. Iski. it Quasiideals in semirings without zero. *Proc. Japan Acad.*, **34**(2)(1958), 79-81.

[6] S. Lajos, F.A. Szasz. On the bi-ideals in associative rings. *Proc. Japan Acad.*, **49**(6)(1970), 505-507. 16, 317-321.

[7] M.Shabir, A. Ali and S. Batool. A Note on Quasi-Ideals in Semirings. *Southeast Asian Bull. Math.*, **27**(5)(2004) , 923-928.

[8] O. Steinfeld. Ueber die quasiideale von halbgruppen, I. *Publ. Math. Debrecen*, **4**(1956), 262-275.

[9] O. Steinfeld. *Quasi-ideals in rings and semigroups*, Akadmiai Kiad Budapest, **10**, (1978).

[10] H.S. Vandiver. Note on a simple type of algebra in which the cancellation law of addition does not hold. *Bull. Amer. Math. Soc.*, **40**(12)(1934), 914-920.

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