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ON FUZZY SUMS AND PRODUCTS OF GENERALIZED FUZZY RIGHT h-IDEALS OF HEMIRINGS

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Abstract. In this paper, we establish that the fuzzy sum of two (λ, μ)-fuzzy right [left] ideals of hemirings R is again a (λ, μ)-fuzzy right [left] ideal of R. It fails for two (λ, μ)-fuzzy right [left] h-ideals of R. Moreover, we investigate that the fuzzy h-sum of two (λ, μ)-fuzzy right [left] ideals is a (λ, μ)-fuzzy right [left] h-ideal of R. We explore that the fuzzy h-intrinsic product of two (λ, μ)-fuzzy right [left] ideals of R is a (λ, μ)-fuzzy right [left] h-ideal of R. Further we investigate that the fuzzy h-intrinsic product of (λ, μ)-fuzzy left and (λ, μ)-fuzzy right ideals of R is a (λ, μ)-fuzzy h-ideal of R.

1. Introduction

Since the introduction of Zadeh's theory of fuzzy sets [**11**], these has been a rapid growth of interest of its application. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, topology and so on. Rosenfeld first introduced the concept of fuzzy groups in his pioneering paper [**8**]. The concept of h-ideals of hemirings was made by Torre in [**3**]. Y.B Jun [**2**] has contributed the concepts of fuzzy ideals and fuzzy h-ideals in hemirings. Investigations of characterizations of hemirings by their h-ideals were studied by Wieslaw A. Dudek, Muhammad Shabir and Rukhshanda Anjum in [**7**]. The notions of (λ, μ)-fuzzy groups [**9**] and (λ, μ)-fuzzy subrings [**10**] were initiated by Yao. Moharaj et al. introduced the concept of (λ, μ)-fuzzy ideals in semirings [**1**]. G.Mohanraj and E.Prabu investigated generalized fuzzy right h-ideals of hemirings by using fuzzy sums and products in [**5**].

In this paper, the fuzzy sum of two (λ, μ)-fuzzy right [left] ideals of hemirings R is again a (λ, μ)-fuzzy right [left] ideal is established but it not so for two

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(λ, μ)-fuzzy right [left] h-ideals of R. Moreover, it is explored that the fuzzy h-sum of two (λ, μ)-fuzzy right [left] ideals is a (λ, μ)-fuzzy right [left] h-ideal of R. And also it is investigated that the fuzzy h-intrinsic product of two (λ, μ)-fuzzy right [left] ideals is a (λ, μ)-fuzzy right [left] h-ideal of R. Further we explore that the fuzzy h-intrinsic product of (λ, μ)-fuzzy left ideal and (λ, μ)-fuzzy right ideal is a (λ, μ)-fuzzy h-ideal of R.

2. Preliminaries

A set $R \neq \emptyset$ together with two associative binary operations "$+$" and "$\cdot$" that satisfy both distributive laws is called a hemiring if "$+$" is commutative and there is an absorbing element $0 \in R$ such that $0+x = x = x+0$ and $0 \cdot x = 0 = x \cdot 0$ for all $x \in R$.

A subset $A \neq \emptyset$ of a hemiring R is called a right [left] ideal of R if (i) $l+m \in A$ for all $l, m \in A$ and (ii) $lz \in A$ $[zl \in A]$ for all $l \in A,\ z \in R$. A right [left] ideal A of R is called a right [left] h-ideal of R if $l, m \in A$ and $x + l + z = m + z$ imply $x \in A$ for $x, z \in R$. For a non-empty subset A of a hemiring R, the h-closure $\overline{A}$ of subset A of R is defined as: $\overline{A} = \{x \in R | x + l + z = m + z,\ l, m \in A \ and\ z \in R\}$.

Recall that a mapping $f : R \rightarrow [0, 1]$ is called a fuzzy set of a hemiring R. For a fuzzy set f of a hemiring R, the level set f_t of fuzzy set f of R is defined by $f_t = \{a \in R | f(a) \geqslant t\}$ for all $t \in [0, 1]$. The fuzzy set "1" is defined as $1 = 1(a) = \chi_R(a)$ for every $a \in R$.

Definition 2.1. *For a fuzzy sets f and g of R,*
i) *the fuzzy sum $f + g$ of f and g is defined as*

$$(f+g)(x) = \bigvee_{x=y+z} [f(y) \wedge g(z)]$$

for $x, y, z \in R$ and
ii) *the fuzzy h-sum $f +_h g$ of f and g is defined as*

$$(f +_h g)(x) = \bigvee_{x+a_1+b_1+z=a_2+b_2+z} [f(a_1) \wedge f(a_2) \wedge g(b_1) \wedge g(b_2)]$$

for $x, a_1, b_1, a_2, b_2, z \in R$.

Definition 2.2. *For a fuzzy set f of a hemiring R, the fuzzy h-closure $\overline{f}$ of f is defined by*

$$(\overline{f})(x) = \bigvee_{x+a+z=b+z} [f(a) \wedge f(b)]$$

for $x, a, b, z \in R$.

Definition 2.3. [4] *Let f and g be two fuzzy sets of R. We write $f \subseteq^{\lambda}_{\mu} g$, if $g(x) \vee \lambda \geqslant f(x) \wedge \mu$ for all $x \in R$ and $0 \leqslant \lambda < \mu \leqslant 1$.*

3. Fuzzy sum of (λ, μ)-fuzzy h-ideals

Now R will always represent a hemiring and $0 \leqslant \lambda < \mu \leqslant 1$ unless otherwise specified.

DEFINITION 3.1. *The fuzzy set f is called a $(\lambda, \mu)-$fuzzy right [left] ideal of R if for all $x, y \in R$*

F1a. $f(x+y) \vee \lambda \geqslant f(x) \wedge f(y) \wedge \mu$

F1b. $f(xy) \vee \lambda \geqslant f(x) \wedge \mu$ $[f(xy) \vee \lambda \geqslant f(y) \wedge \mu]$

DEFINITION 3.2. *The $(\lambda, \mu)-$fuzzy right [left] ideal f is called a $(\lambda, \mu)-$ fuzzy right [left] h-ideal of R if*

F1c. $x + a + z = b + z$ *implies* $f(x) \vee \lambda \geqslant f(a) \wedge f(b) \wedge \mu$ *for* $x, a, b, z \in R$,

THEOREM 3.3. [**5**] *The fuzzy set f of R is a $(\lambda, \mu)-$fuzzy right [left] ideal of R if and only if*

F2a) $f + f \subseteq_{\mu}^{\lambda} f$

F2b) $f \cdot 1 \subseteq_{\mu}^{\lambda} f$ $[1 \cdot f \subseteq_{\mu}^{\lambda} f]$

THEOREM 3.4. [**5**] *A fuzzy set f is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R if and only if*

F2a) $f + f \subseteq_{\mu}^{\lambda} f$

F2b) $f \cdot 1 \subseteq_{\mu}^{\lambda} f$ $[1 \cdot f \subseteq_{\mu}^{\lambda} f]$

F2c) $\overline{f} \subseteq_{\mu}^{\lambda} f$

THEOREM 3.5. *If f and g are (λ, μ)-fuzzy right [left] ideals of R, then so is $f + g$.*

PROOF. Let f and g be (λ, μ)-fuzzy right ideal of R.

$$\begin{aligned}\text{Now, } (f+g)+(f+g) &= f+(g+f)+g \\ &= f+(f+g)+g \\ &\subseteq_{\mu}^{\lambda} f+(g+g) \\ &\subseteq_{\mu}^{\lambda} f+g\end{aligned}$$

Therefore $(f+g)+(f+g) \subseteq_{\mu}^{\lambda} (f+g)$. Now, $x = x_1 + x_2$ implies $xy = x_1y + x_2y$.

$$\begin{aligned}\text{Then, } (f+g)(xy) \vee \lambda &\geqslant \bigvee_{xy=x_1y+x_2y} \Big[f(x_1y) \wedge f(x_2y) \Big] \vee \lambda \\ &\geqslant \bigvee_{x=x_1+x_2} \Big[f(x_1) \wedge \mu \wedge f(x_2) \wedge \mu \Big] \\ &= \bigvee_{x=x_1+x_2} \Big[f(x_1) \wedge f(x_2) \wedge \Big] \wedge \mu \\ &= (f+g)(x) \wedge \mu\end{aligned}$$

$$\begin{aligned} \text{Thus, } (f+g)(x) \vee \lambda &\geqslant \bigvee_{x=ab} \left[(f+g)(a) \wedge \mu \right] \\ &= \left[\bigvee_{x=ab} (f+g)(a) \wedge (1)(b) \right] \wedge \mu \\ &= ((f+g) \cdot 1)(x) \wedge \mu \end{aligned}$$

Therefore by Theorem 3.3, $f+g$ is a (λ, μ)-fuzzy right ideal of R.

Similarly, we prove that $f+g$ is a (λ, μ)-fuzzy left ideal of R if f and g are (λ, μ)-fuzzy left ideal of R. □

REMARK 3.6. *The fuzzy sum of two (λ, μ)-fuzzy right h-ideals of R need not be a (λ, μ)-fuzzy right h-ideal of R by the following Example.*

EXAMPLE 3.7. *Let Z^* be a hemiring of non negative integers. Now, we define a fuzzy sets f and g as follows:*

$$f(x) = \begin{cases} 0.9 & \text{if } x = 8 \\ 0.8 & \text{if } x \text{ is even but } x \neq 8 \\ 0.3 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} 0.9 & \text{if } x = 25 \\ 0.8 & \text{if } x \in < 5 > \smallsetminus \{25\} \\ 0.3 & \text{if } x \in Z^* \smallsetminus \{\{4\} \cup < 5 >\} \\ 0.2 & \text{if } x = 4 \end{cases}$$

Clearly, f and g are $(0.3, 0.8)$-fuzzy right h-ideals of R. Then,

$$(f+g)(x) = \begin{cases} 0.9 & \text{if } x = 33 \\ 0.8 & \text{if } x \in Z^* \smallsetminus \{1, 3, 33\} \\ 0.3 & \text{if } x = 1, 3 \end{cases}$$

Now, $1+4+8 = 5+8$ implies $(f+g)(1) \vee 0.3 = 0.3 \ngeqslant 0.8 = (f+g)(4) \wedge (f+g)(5) \wedge 0.8$. Thus $f+g$ is not a $(0.3, 0.8)$-fuzzy right h-ideal of R.

4. Fuzzy h-sum of (λ, μ)-fuzzy h-ideals

LEMMA 4.1. *If $F1a$ holds for the fuzzy sets f and g, then $(f +_h g)(x+y) \vee \lambda \geqslant (f +_h g)(x) \wedge (f +_h g)(y) \wedge \mu$.*

PROOF. Now,

$$x + a_1 + b_1 + z_1 = a_2 + b_2 + z_1$$

and

$$y + c_1 + d_1 + z_2 = c_2 + d_2 + z_2$$

imply

$$x + y + a_1 + b_1 + c_1 + d_1 + z_1 + z_2 = a_2 + b_2 + c_2 + d_2 + z_1 + z_2.$$

Thus,

$$x + y + (a_1 + c_1) + (b_1 + d_1) + z = (a_2 + c_2) + (b_2 + d_2) + z.$$

Then, by $F1a$, we have

$$\begin{aligned}
(f +_h g)(x + y) \vee \lambda \quad &\geqslant \bigvee_{x+y+(a_1+c_1)+(b_1+d_1)+z=(a_2+c_2)+(b_2+d_2)+z} \Big[f(a_1 + c_1) \wedge \\
&\qquad f(a_2 + c_2) \wedge g(b_1 + d_1) \wedge g(b_2 + d_2) \Big] \vee \lambda \\
&\geqslant \bigvee_{x+a_1+b_1+z_1=a_2+b_2+z_1} \ \bigvee_{y+c_1+d_1+z_2=c_2+d_2+z_2} \Big[f(a_1) \wedge f(c_1) \\
&\qquad \wedge\mu \wedge f(a_2) \wedge f(c_2) \wedge \mu \wedge g(b_1) \wedge g(d_1) \wedge \mu \wedge g(b_2) \wedge g(d_2) \wedge \mu \Big] \\
&= \bigvee_{x+a_1+b_1+z_1=a_2+b_2+z_1} \ \bigvee_{y+c_1+d_1+z_2=c_2+d_2+z_2} \Big\{ \Big[f(a_1) \wedge f(a_2) \wedge \\
&\qquad g(b_1) \wedge g(b_2) \Big] \wedge \Big[f(c_1) \wedge f(c_2) \wedge g(d_1) \wedge g(d_2) \Big] \Big\} \wedge \mu \\
&= \Big\{ \bigvee_{x+a_1+b_1+z_1=a_2+b_2+z_1} \Big[f(a_1) \wedge f(a_2) \wedge g(b_1) \wedge g(b_2) \Big] \wedge \\
&\qquad \bigvee_{y+c_1+d_1+z_2=c_2+d_2+z_2} \Big[f(c_1) \wedge f(c_2) \wedge g(d_1) \wedge g(d_2) \Big] \Big\} \wedge \mu \\
&= (f +_h g)(x) \wedge (f +_h g)(y) \wedge \mu
\end{aligned}$$

Therefore $(f +_h g)(x + y) \vee \lambda \geqslant (f +_h g)(x) \wedge (f +_h g)(y) \wedge \mu$ for all $x, y \in R$. □

LEMMA 4.2. *If $F1a$ holds for the fuzzy sets f and g, then $x + a + z = b + z$ implies $(f +_h g)(x) \vee \lambda \geqslant (f +_h g)(a) \wedge (f +_h g)(b) \wedge \mu$.*

PROOF. Let $a+(a_1+b_1)+z_1 = (a_2+b_2)+z_1$ and $b+(c_1+d_1)+z_2 = (c_2+d_2)+z_2$. Now, $x + a + z_3 = b + z_3$ implies

$$x + a + z_3 + a_1 + b_1 + z_1 + c_1 + d_1 + z_2 \quad = \quad b + z_3 + a_1 + b_1 + z_1 + c_1 + d_1 + z_2$$

Then,

$$\begin{aligned}
x + (a + a_1 + b_1 + z_1) + c_1 + d_1 + z_2 + z_3 \quad &= \quad (b + c_1 + d_1 + z_2) + \\
&\qquad a_1 + b_1 + z_1 + z_3
\end{aligned}$$

implies

$$x + a_2 + b_2 + z_1 + c_1 + d_1 + z_2 + z_3 \quad = \quad c_2 + d_2 + z_2 + a_1 + b_1 + z_1 + z_3.$$

Thus,

$$x+(a_2+c_1)+(b_2+d_1)+z_1+z_2+z_3 \quad = \quad (a_1+c_2)+(b_1+d_2)+z_1+z_2+z_3.$$

Then by $F1a$,

$$\begin{aligned}
(f+_h g)(x)\vee\lambda \quad &\geqslant \bigvee_{x+(a_2+c_1)+(b_2+d_1)+z_1+z_2+z_3=(a_1+c_2)+(b_1+d_2)+z_1+z_2+z_3} \Big[\\
&\qquad f(a_2+c_1)\wedge f(a_1+c_2)\wedge g(b_2+d_1)\wedge g(b_1+d_2)\Big]\vee\lambda \\
&\geqslant \bigvee_{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1}\ \bigvee_{b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}\Big[f(a_2)\wedge f(c_1) \\
&\qquad \wedge\mu\wedge f(a_1)\wedge f(c_2)\wedge\mu\wedge g(b_2)\wedge g(d_1)\wedge\mu\wedge g(b_1)\wedge g(d_2)\wedge\mu\Big] \\
&= \bigvee_{x+(a_1+b_1)+z_1=(a_2+b_2)+z_1}\ \bigvee_{y+(c_1+d_1)+z_2=(c_2+d_2)+z_2}\Big\{\Big[f(a_1)\wedge f(a_2)\wedge \\
&\qquad g(b_1)\wedge g(b_2)\Big]\wedge\Big[f(c_1)\wedge f(c_2)\wedge g(d_1)\wedge g(d_2)\Big]\Big\}\wedge\mu \\
&= \Big\{\bigvee_{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1}\Big[f(a_1)\wedge f(a_2)\wedge g(b_1)\wedge g(b_2)\Big]\wedge \\
&\qquad \bigvee_{b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}\Big[f(c_1)\wedge f(c_2)\wedge g(d_1)\wedge g(d_2)\Big]\Big\}\wedge\mu \\
&= (f+_h g)(a)\wedge(f+_h g)(b)\wedge\mu
\end{aligned}$$

Therefore $x+a+z_3=b+z_3$ implies

$$(f+_h g)(x)\vee\lambda \geqslant (f+_h g)(a)\wedge(f+_h g)(b)\wedge\mu.$$

□

THEOREM 4.3. *The fuzzy h-sum of two (λ,μ)-fuzzy right ideals of R is a (λ,μ)-fuzzy right h-ideal of R.*

PROOF. Let f and g be the (λ,μ)-fuzzy right ideal of R. Then $F1a$ and $F1b$ hold. By Lemma 4.1,

$$(f+_h g)(x+y)\vee\lambda \geqslant (f+_h g)(x)\wedge(f+_h g)(y)\wedge\mu$$

for all $x,y\in R$. Now, $x+a_1+b_1+z=a_2+b_2+z$ implies

$$xy+a_1y+b_1y+z=a_2y+b_2y+z.$$

Then by $F1b$,

$$\begin{aligned}(f+_h g)(xy)\vee\lambda &\geqslant \bigvee_{xy+a_1y+b_1y+z=a_2y+b_2y+z}\Big[f(a_1y)\wedge f(a_2y)\wedge g(b_1y)\wedge \\ &\qquad g(b_2y)\Big]\vee\lambda \\ &\geqslant \bigvee_{x+a_1+b_1+z=a_2+b_2+z} f(a_1)\wedge\mu\wedge f(a_2)\wedge\mu\wedge g(b_1)\wedge\mu \\ &\qquad \wedge g(b_2)\wedge\mu \\ &= \bigvee_{x+a_1+b_1+z=a_2+b_2+z} f(a_1)\wedge f(a_2)\wedge g(b_1)\wedge g(b_2)\wedge\mu \\ &= (f+_h g)(x)\wedge\mu\end{aligned}$$

Therefore $(f+_h g)(xy)\vee\lambda \geqslant (f+_h g)(x)\wedge\mu$ for all $x\in R$. By Lemma 4.2, $x+a+z_3=b+z_3$ implies

$$(f+_h g)(x)\vee\lambda \geqslant (f+_h g)(a)\wedge(f+_h g)(b)\wedge\mu.$$

Hence $(f+_h g)$ is a $(\lambda,\mu)-$fuzzy right h-ideal of R. □

THEOREM 4.4. *If f and g are (λ,μ)-fuzzy left ideals of R, then $f+_h g$ is a (λ,μ)-fuzzy left h-ideal of R.*

PROOF. Let f and g be the (λ,μ)-fuzzy left ideal of R, By Lemma 4.1,

$$(f+_h g)(x+y)\vee\lambda \geqslant (f+_h g)(x)\wedge(f+_h g)(y)\wedge\mu$$

and by Lemma 4.2, $x+a+z_3=b+z_3$ implies

$$(f+_h g)(x)\vee\lambda \geqslant (f+_h g)(a)\wedge(f+_h g)(b)\wedge\mu$$

for all $x,y\in R$. Now, $y+a_1+b_1+z=a_2+b_2+z$ implies $xy+xa_1+xb_1+z=xa_2+xb_2+z$. Then by $F1b$,

$$\begin{aligned}(f+_h g)(xy)\vee\lambda &\geqslant \bigvee_{xy+xa_1+xb_1+z=xa_2+xb_2+z}\Big[f(xa_1)\wedge f(xa_2)\wedge g(xb_1) \\ &\qquad \wedge g(xb_2)\Big]\vee\lambda \\ &\geqslant \bigvee_{y+a_1+b_1+z=a_2+b_2+z} f(a_1)\wedge\mu\wedge f(a_2)\wedge\mu\wedge g(b_1)\wedge\mu \\ &\qquad \wedge g(b_2)\wedge\mu \\ &= \bigvee_{y+a_1+b_1+z=a_2+b_2+z} f(a_1)\wedge f(a_2)\wedge g(b_1)\wedge g(b_2)\wedge\mu \\ &= (f+_h g)(y)\wedge\mu\end{aligned}$$

Therefore $(f+_h g)(xy)\vee\lambda \geqslant (f+_h g)(y)\wedge\mu$ for all $y\in R$. Hence $(f+_h g)$ is a $(\lambda,\mu)-$fuzzy left h-ideal of R. □

COROLLARY 4.5. *The fuzzy h-sum of two $(\lambda,\mu)-$fuzzy right [left] h-ideals of R is again $(\lambda,\mu)-$fuzzy right [left] h-ideal of R.*

PROOF. Straightforward. □

COROLLARY 4.6. *The fuzzy h-sum of two $(\lambda,\mu)-$fuzzy h-ideals of R is again $(\lambda,\mu)-$fuzzy h-ideal of R.*

PROOF. Straightforward. □

COROLLARY 4.7. [**7**] *The fuzzy h-sum of two fuzzy h-ideals of R is again fuzzy h-ideal of R.*

PROOF. By taking $\lambda = 0$ and $\mu = 1$ in Corollary 4.6, we get the proof. □

5. Fuzzy h-intrinsic product of (λ,μ)-fuzzy h-ideal

DEFINITION 5.1. *The h-intrinsic product $f \odot g$ of fuzzy sets f and g of R is defined by*

$$(f \odot g)(x) = \begin{cases} \bigvee\limits_{x+\sum\limits_{i=1}^{m} a_ib_i+z=\sum\limits_{j=1}^{n} a_j^{'}b_j^{'}+z} \left[\bigwedge\limits_{i=1}^{m} [f(a_i)\wedge g(b_i)] \wedge \bigwedge\limits_{j=1}^{n} [f(a_j^{'})\wedge g(b_j^{'})] \right] \\ 0 \textit{ if } x \textit{ cannot be expressible as } x+\sum\limits_{i=1}^{m} a_ib_i + z = \sum\limits_{j=1}^{n} a_j^{'}b_j^{'}+z. \end{cases}$$

LEMMA 5.2. *For the fuzzy sets f and g of R, then $(f\odot g)(x+y) \geqslant (f\odot g)(x)\wedge (f\odot g)(y)$ for all $x, y \in R$.*

PROOF. Now,

$$x+\sum_{i=1}^{m} a_ib_i + z = \sum_{j=1}^{n} a_j^{'}b_j^{'} + z \text{ and } y+\sum_{k=1}^{p} c_kd_k + z = \sum_{l=1}^{q} c_l^{'}d_l^{'} + z$$

imply

$$x+y+\sum_{i=1}^{m} a_ib_i + \sum_{k=1}^{p} c_kd_k + z = \sum_{j=1}^{n} a_j^{'}b_j^{'} + \sum_{l=1}^{q} c_l^{'}d_l^{'} + z.$$

Thus,

$$\begin{aligned}(f\odot g)(x+y) \quad \geqslant \quad & \bigvee_{x+y+\sum\limits_{i=1}^{m} a_ib_i+\sum\limits_{k=1}^{p} c_kd_k+z=\sum\limits_{j=1}^{n} a_j^{'}b_j^{'}+\sum\limits_{l=1}^{q} c_l^{'}d_l^{'}+z} \Bigg[\bigwedge_{i=1}^{m} [f(a_i)\wedge g(b_i)] \wedge \\ & \bigwedge_{k=1}^{p} [f(c_k)\wedge g(d_k)] \wedge \bigwedge_{j=1}^{n} [f(a_j^{'})\wedge g(b_j^{'})] \wedge \bigwedge_{l=1}^{q} [f(c_l^{'})\wedge g(d_l^{'})] \Bigg]\end{aligned}$$

$$= \bigvee_{x+y+\sum_{i=1}^{m} a_i b_i+\sum_{k=1}^{p} c_k d_k+z=\sum_{j=1}^{n} a_j' b_j'+\sum_{l=1}^{q} c_l' d_l'+z} \Bigg[\bigwedge_{i=1}^{m}[f(a_i) \wedge g(b_i)] \wedge \bigwedge_{j=1}^{n}[f(a_j') \wedge g(b_j')] \wedge \bigwedge_{k=1}^{p}[f(c_k) \wedge g(d_k)] \wedge \bigwedge_{l=1}^{q}[f(c_l') \wedge g(d_l')] \Bigg]$$

$$= \bigvee_{x+\sum_{i=1}^{m} a_i b_i+z=\sum_{j=1}^{n} a_j' b_j'+z} \ \bigvee_{y+\sum_{k=1}^{p} c_k d_k+z=\sum_{l=1}^{q} c_l' d_l'+z} \Bigg[\bigwedge_{i=1}^{m}[f(a_i) \wedge g(b_i)] \wedge \bigwedge_{j=1}^{n}[f(a_j') \wedge g(b_j')] \wedge \bigwedge_{k=1}^{p}[f(c_k) \wedge g(d_k)] \wedge \bigwedge_{l=1}^{q}[f(c_l') \wedge g(d_l')] \Bigg]$$

$$= \Bigg(\bigvee_{x+\sum_{i=1}^{m} a_i b_i+z=\sum_{j=1}^{n} a_j' b_j'+z} \bigwedge_{i=1}^{m}[f(a_i) \wedge g(b_i)] \wedge \bigwedge_{j=1}^{n}[f(a_j') \wedge g(b_j')] \Bigg) \wedge \Bigg(\bigvee_{y+\sum_{k=1}^{p} c_k d_k+z=\sum_{l=1}^{q} c_l' d_l'+z} \bigwedge_{k=1}^{p}[f(c_k) \wedge g(d_k)] \wedge \bigwedge_{l=1}^{q}[f(c_l') \wedge g(d_l')] \Bigg)$$

$$= (f \odot g)(x) \wedge (f \odot g)(y)$$

Therefore $(f \odot g)(x+y) \geqslant (f \odot g)(x) \wedge (f \odot g)(y)$ for all $x, y \in R$. □

Lemma 5.3. *If f and g are fuzzy sets of R and $x + a + z_3 = b + z_3$, then $(f \odot g)(x) \geqslant (f \odot g)(a) \wedge (f \odot g)(b)$ for all $x, y \in R$.*

Proof. Now

$$a + \sum_{i=1}^{m} a_i b_i + z_1 = \sum_{j=1}^{n} a_j' b_j' + z_1, \ b + \sum_{k=1}^{p} c_k d_k + z_2 = \sum_{l=1}^{q} c_l' d_l' + z_2 \tag{5.1}$$

and

$$x + a + z_3 = b + z_3 \text{ imply}$$
$$x + a + \left(\sum_{i=1}^{m} a_i b_i + z_1\right) + z_3 = b + \left(\sum_{i=1}^{m} a_i b_i + z_1\right) + z_3.$$

Thus

$$x + \sum_{j=1}^{n} a_j' b_j' + z_1 + z_3 = b + \sum_{i=1}^{m} a_i b_i + z_1 + z_3.$$

Then,

$$x+\sum_{j=1}^{n}a_j^{'}b_j^{'}+\left(\sum_{k=1}^{p}c_kd_k+z_2\right)+z_1+z_3=b+\left(\sum_{k=1}^{p}c_kd_k+z_2\right)+\sum_{i=1}^{m}a_ib_i+z_1+z_3$$

implies

$$x+\sum_{j=1}^{n}a_j^{'}b_j^{'}+\sum_{k=1}^{p}c_kd_k+z_2+z_1+z_3=\left(\sum_{l=1}^{q}c_l^{'}d_l^{'}+z_2\right)+\sum_{i=1}^{m}a_ib_i+z_1+z_3.$$

Therefore,

$$x+\sum_{j=1}^{n}a_j^{'}b_j^{'}+\sum_{k=1}^{p}c_kd_k+z_1+z_2+z_3=\sum_{i=1}^{m}a_ib_i+\sum_{l=1}^{q}c_l^{'}d_l^{'}+z_1+z_2+z_3 \tag{5.2}$$

Now, by Equations 5.1 and 5.2, we have

$$\begin{aligned}(f\odot g)(x) \quad &\geqslant \bigvee_{x+\sum_{j=1}^{n}a_j'b_j'+\sum_{k=1}^{p}c_kd_k+z_1+z_2+z_3=\sum_{i=1}^{m}a_ib_i+\sum_{l=1}^{q}c_l'd_l'+z_1+z_2+z_3}\Bigg[\bigwedge_{j=1}^{n}\\ &[f(a_j^{'})\wedge g(b_j^{'})]\wedge\bigwedge_{k=1}^{p}[f(c_k)\wedge g(d_k)]\wedge\bigwedge_{i=1}^{m}[f(a_i)\wedge\\ &g(b_i)]\wedge\bigwedge_{l=1}^{q}[f(c_l^{'})\wedge g(d_l^{'})]\Bigg]\end{aligned}$$

and

$$\begin{aligned}&= \bigvee_{x+\sum_{j=1}^{n}a_j'b_j'+\sum_{k=1}^{p}c_kd_k+z_1+z_2+z_3=\sum_{i=1}^{m}a_ib_i+\sum_{l=1}^{q}c_l'd_l'+z_1+z_2+z_3}\Bigg[\bigwedge_{i=1}^{m}\\ &[f(a_i)\wedge g(b_i)]\wedge\bigwedge_{j=1}^{n}[f(a_j^{'})\wedge g(b_j^{'})]\wedge\bigwedge_{k=1}^{p}[f(c_k)\wedge g(d_k)]\wedge\\ &\bigwedge_{l=1}^{q}[f(c_l^{'})\wedge g(d_l^{'})]\Bigg]\end{aligned}$$

and

$$\begin{aligned}&= \bigvee_{a+\sum_{i=1}^{m}a_ib_i+z_1=\sum_{j=1}^{n}a_j'b_j'+z_1}\ \bigvee_{b+\sum_{k=1}^{p}c_kd_k+z_2=\sum_{l=1}^{q}c_l'd_l'+z_2}\Bigg[\bigwedge_{i=1}^{m}[f(a_i)\wedge g(b_i)]\\ &\wedge\bigwedge_{j=1}^{n}[f(a_j^{'})\wedge g(b_j^{'})]\wedge\bigwedge_{k=1}^{p}[f(c_k)\wedge g(d_k)]\wedge\bigwedge_{l=1}^{q}[f(c_l^{'})\wedge g(d_l^{'})]\Bigg]\end{aligned}$$

and

$$
\begin{aligned}
&= \left(\bigvee_{a+\sum\limits_{i=1}^{m} a_i b_i + z_1 = \sum\limits_{j=1}^{n} a_j' b_j' + z_1} \bigwedge_{i=1}^{m} [f(a_i) \wedge g(b_i)] \wedge \bigwedge_{j=1}^{n} [f(a_j^{'}) \wedge g(b_j^{'})] \right) \\
&\quad \wedge \left(\bigvee_{b+\sum\limits_{k=1}^{p} c_k d_k + z_2 = \sum\limits_{l=1}^{q} c_l' d_l' + z_2} \bigwedge_{k=1}^{p} [f(c_k) \wedge g(d_k)] \wedge \right. \\
&\quad \left. \bigwedge_{l=1}^{q} [f(c_l^{'}) \wedge g(d_l^{'})] \right) \\
&= (f \odot g)(a) \wedge (f \odot g)(b)
\end{aligned}
$$

Thus $x + a + z_3 = b + z_3$ implies

$$(f \odot g)(x) \geqslant (f \odot g)(a) \wedge (f \odot g)(b).$$

□

LEMMA 5.4. *If f is a fuzzy set and g is a (λ, μ)-fuzzy right ideal of R, then*

$$(f \odot g)(xy) \vee \lambda \geqslant (f \odot g)(x) \wedge \mu.$$

PROOF. Let

$$x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j^{'} b_j^{'} + z.$$

Then

$$xy + \sum_{i=1}^{m} (a_i b_i) y + z = \sum_{j=1}^{n} (a_j^{'} b_j^{'}) y + z.$$

and

$$
\begin{aligned}
(f \odot g)(xy) \vee \lambda &\geqslant \left[\bigvee_{xy+\sum\limits_{i=1}^{m} a_i (b_i y) + z = \sum\limits_{j=1}^{n} a_j' (b_j' y) + z} \bigwedge_{i=1}^{m} [f(a_i) \wedge g(b_i y)] \wedge \right. \\
&\quad \left. \bigwedge_{j=1}^{n} [f(a_j^{'}) \wedge g(b_j^{'} y)] \right] \vee \lambda \\
&= \bigvee_{xy+\sum\limits_{i=1}^{m} a_i (b_i y) + z = \sum\limits_{j=1}^{n} a_j' (b_j' y) + z} \bigwedge_{i=1}^{m} [(f(a_i) \vee \lambda) \wedge (g(b_i y) \vee \lambda)] \wedge
\end{aligned}
$$

$$\bigwedge_{j=1}^{n}[(f(a_j^{'})\vee\lambda)\wedge(g(b_j^{'}y)\vee\lambda)]$$

$$\begin{aligned}
&\geqslant \bigvee_{x+\sum\limits_{i=1}^{m}a_ib_i+z=\sum\limits_{j=1}^{n}a_j^{'}b_j^{'}+z}\bigwedge_{i=1}^{m}[f(a_i)\wedge g(b_i)\wedge\mu]\wedge\bigwedge_{j=1}^{n}[f(a_j^{'})\wedge g(b_j^{'})\wedge\mu]\\
&= \left[\bigvee_{x+\sum\limits_{i=1}^{m}a_ib_i+z=\sum\limits_{j=1}^{n}a_j^{'}b_j^{'}+z}\bigwedge_{i=1}^{m}[f(a_i)\wedge g(b_i)]\wedge\bigwedge_{j=1}^{n}[f(a_j^{'})\wedge g(b_j^{'})]\right]\wedge\mu\\
&= (f\odot g)(x)\wedge\mu
\end{aligned}$$

Therefore $(f\odot g)(xy)\vee\lambda\geqslant(f\odot g)(x)\wedge\mu$. □

LEMMA 5.5. *If g is a fuzzy set and f is a (λ,μ)-fuzzy left ideal of R, then $(f\odot g)(xy)\vee\lambda\geqslant(f\odot g)(y)\wedge\mu$.*

PROOF. Let $y+\sum\limits_{i=1}^{m}c_id_i+z=\sum\limits_{j=1}^{n}c_j^{'}d_j^{'}+z$. Then

$$xy+\sum_{i=1}^{m}x(c_id_i)+z=\sum_{j=1}^{n}x(c_j^{'}b_j^{'})+z.$$

$$\begin{aligned}
(f\odot g)(xy)\vee\lambda &\geqslant \left[\bigvee_{xy+\sum\limits_{i=1}^{m}(xc_i)d_i+z=\sum\limits_{j=1}^{n}(xc_j^{'})d_j^{'}+z}\bigwedge_{i=1}^{m}[f(xc_i)\wedge g(d_i)]\wedge\bigwedge_{j=1}^{n}[f(xc_j^{'})\right.\\
&\qquad \left.\wedge g(d_j^{'})]\right]\vee\lambda\\
&= \bigvee_{xy+\sum\limits_{i=1}^{m}(xc_i)d_i+z=\sum\limits_{j=1}^{n}(xc_j^{'})d_j^{'}+z}\bigwedge_{i=1}^{m}[(f(xc_i)\vee\lambda)\wedge(g(d_i)\vee\lambda)]\wedge\\
&\qquad \bigwedge_{j=1}^{n}[(f(xc_j^{'})\vee\lambda)\wedge(g(d_j^{'})\vee\lambda)]\\
&\geqslant \bigvee_{y+\sum\limits_{i=1}^{m}c_id_i+z=\sum\limits_{j=1}^{n}c_j^{'}d_j^{'}+z}\bigwedge_{i=1}^{m}[f(c_i)\wedge\mu\wedge g(d_i)]\wedge\bigwedge_{j=1}^{n}[f(c_j^{'})\wedge\mu\wedge g(d_j^{'})]\\
&= \left[\bigvee_{y+\sum\limits_{i=1}^{m}c_id_i+z=\sum\limits_{j=1}^{n}c_j^{'}d_j^{'}+z}\bigwedge_{i=1}^{m}[f(c_i)\wedge g(d_i)]\wedge\bigwedge_{j=1}^{n}[f(c_j^{'})\wedge g(d_j^{'})]\right]\wedge\mu\\
&= (f\odot g)(y)\wedge\mu
\end{aligned}$$

Therefore $(f \odot g)(xy) \vee \lambda \geqslant (f \odot g)(y) \wedge \mu$. □

Theorem 5.6. *If f and g are $(\lambda, \mu)-$fuzzy right [left] ideals of R, then $(f \odot g)$ is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R.*

Proof. Let f and g be (λ, μ)-fuzzy right [left] ideals of R. By Lemma 5.2, we have $(f \odot g)(x+y) \vee \lambda \geqslant (f \odot g)(x) \wedge (f \odot g)(y) \wedge \mu$ and by Lemma 5.4, $(f \odot g)(xy) \vee \lambda \geqslant (f \odot g)(x) \wedge \mu$. Now by Lemma 5.5, we have $(f \odot g)(xy) \vee \lambda \geqslant (f \odot g)(y) \wedge \mu$. Then by Lemma 5.3, $x + a + z_3 = b + z_3$ implies $(f \odot g)(x) \geqslant (f \odot g)(a) \wedge (f \odot g)(b)$ for all $x, y \in R$. Therefore $(f \odot g)$ is a (λ, μ)-fuzzy right [left] h-ideal of R. □

Corollary 5.7. *If f and g are $(\lambda, \mu)-$fuzzy right [left] h-ideals of R, then so is $(f \odot g)$.*

Proof. Straightforward. □

Theorem 5.8. *If f and g are $(\lambda, \mu)-$fuzzy left and right ideals respectively, then $f \odot g$ is a $(\lambda, \mu)-$fuzzy h-ideal of R.*

Proof. Let f be a (λ, μ)-fuzzy left ideal of R. Then by Lemma 5.5, $(f \odot g)(xy) \vee \lambda \geqslant (f \odot g)(y) \wedge \mu$.
By Lemma 5.2, we have $(f \odot g)(x + y) \vee \lambda \geqslant (f \odot g)(x) \wedge (f \odot g)(y) \wedge \mu$.
Let g be a (λ, μ)-fuzzy right ideal of R and by Lemma 5.4, $(f \odot g)(xy) \vee \lambda \geqslant (f \odot g)(x) \wedge \mu$.
By Lemma 5.3, $x+a+z_3 = b+z_3$ implies $(f \odot g)(x) \vee \lambda \geqslant (f \odot g)(a) \wedge (f \odot g)(b) \wedge \mu$.
Hence $f \odot g$ is a (λ, μ)-fuzzy h-ideal of R. □

Corollary 5.9. *If f and g are $(\lambda, \mu)-$fuzzy left and right h-ideals respectively, then $f \odot g$ is a $(\lambda, \mu)-$fuzzy h-ideal of R.*

Proof. Straightforward. □

Corollary 5.10. *If f and g are $(\lambda, \mu)-$fuzzy h-ideals of R, then $f \odot g$ is again a $(\lambda, \mu)-$fuzzy h-ideal of R.*

Proof. Straightforward. □

Corollary 5.11. [**7**] *If f and g are fuzzy h-ideals of R, then $f \odot g$ is again a fuzzy h-ideal of R.*

Proof. By taking $\lambda = 0$ and $\mu = 1$ in Corollary 5.10, we get the result. □

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