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POSITIVE SOLUTIONS TO SINGULAR SEMIPOSITONE SECOND ORDER DYNAMIC SYSTEMS

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Abstract. By employing the Schauder's fixed point theorem, we study the existence of positive solutions for a singular semipositone dynamic system on time scales. New existence results are established, which is in essence different from the known results.

1. Introduction

Boundary value problems for an ordinary differential system arise from many fields in physics, biology and chemistry, which play an important role in both theory and application. A brief discussion of the chemical interpretation of some of the boundary value conditions can be found in Aris [3]. There were many works to be done for a variety of nonlinear boundary value problems [20] and nonlinear ordinary differential systems [10, 11, 16, 19] and the references therein. In literature, most papers only focus on attention to the case where the nonlinearity takes non-negative values (positone problems) and has no any singularities. However, singular semipositone boundary value problems for nonlinear ordinary differential systems has started to study in recent years [4, 7, 9, 13, 14].

The theory of measure chains was introduced and developed by Aulbach and Hilger in 1988. It has been created in order to unify continuous and discrete analysis, and it allows a simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. The study of time scales has led to many important applications, for example, in the study of insect population models, neural networks, heat transfer and epidemic models. We refer the

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reader to the excellent introductory text by Bohner and Peterson [5] as well as their recent research monograph [6]. In recent years, several authors studied positone and semipositone boundary value problems on time scales and we want to mention some papers in literature [1, 2, 17, 18, 21] and the references therein, but there is only a few study on boundary value problems for dynamic systems on time scales [8, 12, 15].

As far as we know, singular semipositone boundary value problems for an second order dynamic system are seldom investigated. This paper attempt to fill part of this gap in the literature.

In this paper, we shall consider the nonlinear singular semipositone dynamic system of m-point boundary value problem (SSS)(1.1),

$$\begin{cases} -[p(t)u_1^{\triangle}(t)]^{\nabla} + q(t)u_1(t) = f_1(t, u_2(t)) + h_1(t), \quad t \in (a, b), \\ -[p(t)u_2^{\triangle}(t)]^{\nabla} + q(t)u_2(t) = f_2(t, u_1(t)) + h_2(t), \quad t \in (a, b), \\ \alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k u_i(\xi_k), \quad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k u_i(\xi_k), \quad i = 1, 2, \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \xi_k, \alpha_k, \beta_k$ (for $k \in \{1, 2, ..., m-2\}$) are complex constants such that $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$ and $\xi_k \in \mathrm{T}\backslash\{a, b\}$, $q : \mathrm{T} \to \mathcal{C}$ is a continuous function, $p : \mathrm{T} \to \mathcal{C}$ is $\nabla-$ differentiable on T_k, $p^{\nabla} : \mathrm{T}_k \to \mathcal{C}$ is continuous, $p(t) \neq 0$ for all $t \in \mathrm{T}$, f_1 , $f_2 : (a, b) \times [0, \infty) \to (0, \infty)$ are continuous and may be singular at $t = a, b$ and h_1 , $h_2 : (a, b) \to (-\infty, \infty)$ are continuous. By an interval (a, b), we mean the intersection of the real interval (a, b) with the given time scale T.

Different from the previous papers, in this paper we not only allow f_1 and f_2 to have finitely many singularities, but also allow the nonlinearity terms h_1 and h_2 to change sign and may tend to negative infinity. Besides these, we give the problem with more general boundary conditions.

We shall organize this paper as follows. In Section 2, we present some lemmas to be used later. In Section 3, we state our results and give their proofs.

2. The Preliminary Lemmas

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the Green's function of the following boundary value problem

$$-[p(t)u^{\triangle}(t)]^{\nabla} + q(t)u(t) = 0, \ \ t \in (a, b),$$
$$\alpha u(a) - \beta u^{[\triangle]}(a) = 0,$$
$$\gamma u(b) + \delta u^{[\triangle]}(b) = 0.$$

The Green's function for the above problem is given by

$$G(t, s) = \frac{1}{d} \begin{cases} \phi_1(s)\phi_2(t), & a \leqslant s \leqslant t \leqslant b, \\ \phi_1(t)\phi_2(s) & a \leqslant t \leqslant s \leqslant b, \end{cases}$$

where ϕ_1, ϕ_2 are the solutions of the linear problems

$[p(t)\phi_1^{\triangle}(t)]^{\nabla} - q(t)\phi_1(t) = 0, \ t \in (a, b),$
$\phi_1(a) = \beta, \ \phi_1^{[\triangle]}(a) = \alpha,$
and
$[p(t)\phi_2^{\triangle}(t)]^{\nabla} - q(t)\phi_2(t) = 0, \ t \in (a, b),$
$\phi_2(b) = \delta, \ \phi_2^{[\triangle]}(b) = -\gamma,$
respectively, $d = -W_t(\phi_1, \phi_2) = p(t)[\phi_1^{\triangle}(t)\phi_2(t) - \phi_1(t)\phi_2^{\triangle}(t)]$.

Let we define

$$\Omega := \begin{vmatrix} -\sum_{k=1}^{m-2}\alpha_k\phi_1(\xi_k) & d - \sum_{k=1}^{m-2}\alpha_k\phi_2(\xi_k) \\ d - \sum_{k=1}^{m-2}\beta_k\phi_1(\xi_k) & -\sum_{k=1}^{m-2}\beta_k\phi_2(\xi_k) \end{vmatrix},$$

and assume that the following conditions are satisfied:
(H_1) $p(t) > 0$, $q(t) \geqslant 0$,
(H_2) $\alpha, \gamma \geqslant 0$, $\beta, \delta > 0$, α_k, $\beta_k \geqslant 0$ for $k \in \{1, 2, ..., m-2\}$,
(H_3) If $q(t) \equiv 0$, then $\alpha + \gamma > 0$,
(H_4) $\Omega < 0$, $d - \sum_{k=1}^{m-2}\alpha_k\phi_2(\xi_k) > 0$, $d - \sum_{k=1}^{m-2}\beta_k\phi_1(\xi_k) > 0$.

Lemma 2.1. [5] *Under the conditions (H_1) and (H_2), the solutions $\phi_1(t)$ and $\phi_2(t)$ posses the following properties:*
$\phi_1(t), \ \ \phi_2(t) \geqslant 0, \quad \phi_1^{[\triangle]}(t) \geqslant 0, \quad \phi_2^{[\triangle]}(t) \leqslant 0, \qquad t \in [a, b].$

Lemma 2.2. [5] *If the conditions $(H_1) - (H_3)$ are hold, then $G(t, s) > 0$ for $t, s \in [a, b]$.*

Lemma 2.3. *Assume that $(H_1) - (H_3)$ hold. Then*
$$g(t)G(s, s) \leqslant G(t, s) \leqslant G(s, s), \quad t, s \in [a, b],$$
where g is defined by
$$g(t) := \min_{t\in[a,b]}\{\frac{\phi_1(t)}{\phi_1(b)}, \frac{\phi_2(t)}{\phi_2(a)}\}.$$

Proof. It follows from Lemma 2.1 that $\phi_1(t)$ is increasing and $\phi_2(t)$ is decreasing on $t \in [a, b]$. Then we have $G(t, s) \leqslant G(s, s)$. Besides this inequality, for all $t, s \in [a, b]$, we have

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{\phi_2(t)}{\phi_2(s)}, & s \leqslant t \\ \dfrac{\phi_1(t)}{\phi_1(s)}, & t \leqslant s \end{cases} \geqslant \begin{cases} \dfrac{\phi_2(t)}{\phi_2(a)}, & s \leqslant t \\ \dfrac{\phi_1(t)}{\phi_1(b)}, & t \leqslant s \end{cases} \geqslant g(t).$$

□

We consider the following boundary value problem

$$-[p(t)u^{\triangle}(t)]^{\nabla} + q(t)u(t) = y(t) \ , \qquad t \in (a, b), \tag{2.1}$$

$$\alpha u(a) - \beta u^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u(\xi_k), \quad \gamma u(b) + \delta u^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u(\xi_k). \tag{2.2}$$

Lemma 2.4. [18] *Let the conditions* $(H_1) - (H_3)$ *be hold. Assume that* $\Omega :\neq 0$. *Then for* $y \in C([a,b])$, *the boundary value problem* $(2.1)-(2.2)$ *has a unique solution*

$$u(t) = \int_a^b G(t,s)y(s)\nabla s + A(y)\phi_1(t) + B(y)\phi_2(t),$$

where

$$A(y) := \frac{1}{\Omega}\begin{vmatrix} \sum_{k=1}^{m-2}\alpha_k \int_a^b G(\xi_k,s)y(s)\nabla s & d - \sum_{k=1}^{m-2}\alpha_k\phi_2(\xi_k) \\ \sum_{k=1}^{m-2}\beta_k \int_a^b G(\xi_k,s)y(s)\nabla s & -\sum_{k=1}^{m-2}\beta_k\phi_2(\xi_k) \end{vmatrix}$$

and

$$B(y) := \frac{1}{\Omega}\begin{vmatrix} -\sum_{k=1}^{m-2}\alpha_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\alpha_k \int_a^b G(\xi_k,s)y(s)\nabla s \\ d - \sum_{k=1}^{m-2}\beta_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\beta_k \int_a^b G(\xi_k,s)y(s)\nabla s \end{vmatrix}.$$

Lemma 2.5. *If* $\int_a^b G(s,s)y(s)\nabla s < \infty$, *then the following inequalities are satisfied:*

$$A(y) \leqslant A\int_a^b G(s,s)y(s)\nabla s, \qquad B(y) \leqslant B\int_a^b G(s,s)y(s)\nabla s,$$

where

$$A = \frac{1}{\Omega}\begin{vmatrix} \sum_{k=1}^{m-2}\alpha_k & d - \sum_{k=1}^{m-2}\alpha_k\phi_2(\xi_k) \\ \sum_{k=1}^{m-2}\beta_k & -\sum_{k=1}^{m-2}\beta_k\phi_2(\xi_k) \end{vmatrix}$$

and

$$B = \frac{1}{\Omega}\begin{vmatrix} -\sum_{k=1}^{m-2}\alpha_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\alpha_k \\ d - \sum_{k=1}^{m-2}\beta_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\beta_k \end{vmatrix}.$$

Proof. It can be easily proven with simple calculation. □

3. Main Result

In this section, we apply Schauder's fixed point theorem to prove the existence of at least one positive solution for the SSS (1.1). For that purpose, let us do some preparations. We shall work in the space $E = C([a,b]) \times C([a,b])$.

In the following, we define the closed, convex set by

$M = \{(u_1,u_2) \in E : r_1 \leqslant u_1(t) \leqslant R_1,\ r_2 \leqslant u_2(t) \leqslant R_2,\ t \in [a,b]\}$,

where R_1, R_2, r_1, r_2 are positive constants to be fixed properly such that $R_1 > r_1$ and $R_2 > r_2$.

We define the functions $\varphi_i : [a,b] \to \mathcal{R}$ by

$$\varphi_i(t) = \int_a^b G(t,s)h_i(s)\nabla s + A(h_i)\phi_1(t) + B(h_i)\phi_2(t), \quad i = 1,2,$$

which is the unique solution of

$$-[p(t)u_i^{\triangle}(t)]^{\nabla} + q(t)u_i(t) = h_i(t), \quad t \in (a,b), \quad i = 1,2,$$
$$\alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u_i(\xi_k), \qquad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u_i(\xi_k).$$

Using Lemma 2.3 and 2.5, we find

$$
\begin{aligned}
|\varphi_i(t)| &\leqslant \int_a^b G(t,s)|h_i(s)|\nabla s + A(|h_i|)\phi_1(t) + B(|h_i|)\phi_2(t) \\
&\leqslant G(t,t)\int_a^b |h_i(s)|\nabla s + A\phi_1(b)\int_a^b G(s,s)|h_i(s)|\nabla s \\
&\quad + B\phi_2(a)\int_a^b G(s,s)|h_i(s)|\nabla s \\
&\leqslant G(t,t)\int_a^b |h_i(s)|\nabla s + A\phi_1(b)\frac{1}{g(t)}\int_a^b G(t,s)|h_i(s)|\nabla s \\
&\quad + B\phi_2(a)\tfrac{1}{g(t)}\int_a^b G(t,s)|h_i(s)|\nabla s \\
&\leqslant G(t,t)f(t)\int_a^b |h_i(s)|\nabla s, \quad i=1,2,
\end{aligned}
$$

where $f(t) = 1 + \dfrac{A}{g(t)}\phi_1(b) + \dfrac{B}{g(t)}\phi_2(a)$.

Now, let us define the operator $F(u_1,u_2) = (F_1u_1, F_2u_2) : E \to E$ by

$$
\begin{aligned}
F_1u_1(t) &:= \int_a^b G(t,s)[f_1(s,u_2(s)) + h_1(s)]\nabla s + A(f_1+h_1)\phi_1(t) \\
&\quad + B(f_1+h_1)\phi_2(t) \\
&= \int_a^b G(t,s)f_1(s,u_2(s))\nabla s + A(f_1)\phi_1(t) + B(f_1)\phi_2(t) + \varphi_1(t)
\end{aligned}
$$

and

$$
\begin{aligned}
F_2u_2(t) &:= \int_a^b G(t,s)[f_2(s,u_1(s)) + h_2(s)]\nabla s + A(f_2+h_2)\phi_1(t) \\
&\quad + B(f_2+h_2)\phi_2(t) \\
&= \int_a^b G(t,s)f_2(s,u_1(s))\nabla s + A(f_2)\phi_1(t) + B(f_2)\phi_2(t) + \varphi_2(t).
\end{aligned}
$$

It is well known that the existence of the solution to the system (1.1) is equivalent to the existence of fixed point of the operator F. So we shall seek a fixed point of F .

Given $v \in L^1(a,b)$, we write $v \succ 0$ if $v \geqslant 0$ for a.e. $t \in [a,b]$ and it is positive in a set of positive measure.

In the rest of the paper, we assume that the following condition is satisfied;
(H_5) For all $u > 0$, a.e. $t \in (a,b)$, there exist $b_i, \hat{b}_i \in L^1(a,b)$ and $0 < \alpha_i < 1$ such that $b_i, \hat{b}_i \succ 0$ and

$$
0 \leqslant \frac{\hat{b}_i(t)}{u^{\alpha_i}} \leqslant f_i(t,u) \leqslant \frac{b_i(t)}{u^{\alpha_i}}, \quad i=1,2.
$$

For convenience, we introduce the following notations

$$\beta_i(t) = \int_a^b G(t,s) b_i(s) \nabla s, \quad \hat{\beta}_i(t) = \int_a^b G(t,s) \hat{b}_i(s) \nabla s, \quad i = 1, 2.$$

Also, for the function $\psi \in L^1[a,b]$, we denote the maximum and minimum by

$$\psi^* = \max_{t\in[a,b]} \frac{\psi(t)}{f(t)} \quad \text{and} \quad \psi_* = \min_{t\in[a,b]} \frac{\psi(t)}{f(t)}, \quad i = 1, 2.$$

THEOREM 3.1. *Assume that the conditions* $(H_1) - (H_5)$ *hold. If* $\varphi_{1*} \geqslant 0$, $\varphi_{2*} \geqslant 0$, *then the SSS* (1.1) *has at least one positive solution.*

PROOF. We use Schauder's fixed point theorem to prove that F has a fixed point in the closed, convex set M. First, we shall show that $F(M) \subseteq M$. Let $(u_1, u_2) \in M$, then by using the definition of $\hat{\beta}_{1*}$, the assumption (H_5) and the nonnegativity of Green's function, $A, B, f_1, \varphi_1, \phi_1, \phi_2$, we obtain

$$\begin{aligned}
F_1 u_1(t) &= \int_a^b G(t,s) f_1(s, u_2(s)) \nabla s + A(f_1)\phi_1(t) + B(f_1)\phi_2(t) + \varphi_1(t) \\
&\geqslant \int_a^b G(t,s) f_1(s, u_2(s)) \nabla s \geqslant \int_a^b G(t,s) \frac{\hat{b}_1(s)}{u_2^{\alpha_1}(s)} \nabla s \geqslant \int_a^b G(t,s) \frac{\hat{b}_1(s)}{R_2^{\alpha_1}} \nabla s \\
&= \frac{1}{R_2^{\alpha_1}} \hat{\beta}_1(t) \geqslant \frac{1}{R_2^{\alpha_1}} \hat{\beta}_{1*} f(t) \geqslant \frac{1}{R_2^{\alpha_1}} \hat{\beta}_{1*}
\end{aligned}$$

and from the definition of φ_1^*, Lemma 2.3 and 2.5, we get

$$\begin{aligned}
F_1 u_1(t) &= \int_a^b G(t,s) f_1(s, u_2(s)) \nabla s + A(f_1)\phi_1(t) + B(f_1)\phi_2(t) + \varphi_1(t) \\
&\leqslant \int_a^b G(t,s) \frac{b_1(s)}{u_2^{\alpha_1}(s)} \nabla s + A(\frac{b_1}{u_2^{\alpha_1}})\phi_1(b) + B(\frac{b_1}{u_2^{\alpha_1}})\phi_2(a) + \varphi_1^* f(t) \\
&\leqslant \int_a^b G(t,s) \frac{b_1(s)}{r_2^{\alpha_1}} \nabla s + A\phi_1(b) \int_a^b G(s,s) \frac{b_1(s)}{r_2^{\alpha_1}} \nabla s \\
&\quad + B\phi_2(a) \int_a^b G(s,s) \frac{b_1(s)}{r_2^{\alpha_1}} \nabla s + \varphi_1^* f(t) \\
&\leqslant \int_a^b G(t,s) \frac{b_1(s)}{r_2^{\alpha_1}} \nabla s + \frac{A\phi_1(b)}{g(t)} \int_a^b G(t,s) \frac{b_1(s)}{r_2^{\alpha_1}} \nabla s \\
&\quad + \frac{B\phi_2(a)}{g(t)} \int_a^b G(t,s) \frac{b_1(s)}{r_2^{\alpha_1}} \nabla s + \varphi_1^* f(t) \\
&= \frac{1}{r_2^{\alpha_1}} \beta_1(t) f(t) + \varphi_1^* f(t) \\
&\leqslant \frac{1}{r_2^{\alpha_1}} \beta_1^* f^2(t) + \varphi_1^* f(t) \\
&\leqslant \Big[\frac{K\beta_1^*}{r_2^{\alpha_1}} + \varphi_1^* \Big] K,
\end{aligned}$$

where noting that $f(t) \leqslant K$ and $K = 1 + \dfrac{A[\phi_1(b)]^2}{\beta} + \dfrac{B[\phi_2(a)]^2}{\delta}$.

Moreover, following the same strategy, we have

$$\begin{aligned}
F_2u_2(t) &= \int_a^b G(t,s)f_2(s,u_1(s))\nabla s + A(f_2)\phi_1(t) + B(f_2)\phi_2(t) + \varphi_2(t)\\
&\geqslant \int_a^b G(t,s)f_2(s,u_1(s))\nabla s \geqslant \int_a^b G(t,s)\frac{\hat{b}_2(s)}{u_1^{\alpha_2}(s)}\nabla s \geqslant \int_a^b G(t,s)\frac{\hat{b}_2(s)}{R_1\alpha_2}\nabla s\\
&= \frac{1}{R_1^{\alpha_2}}\hat{\beta}_2(t) \geqslant \frac{1}{R_1^{\alpha_2}}\hat{\beta}_{2*}f(t) \geqslant \frac{1}{R_1^{\alpha_2}}\hat{\beta}_{2*}
\end{aligned}$$

and similarly

$$\begin{aligned}
F_2u_2(t) &= \int_a^b G(t,s)f_2(s,u_1(s))\nabla s + A(f_2)\phi_1(t) + B(f_2)\phi_2(t) + \varphi_2(t)\\
&\leqslant \int_a^b G(t,s)\frac{b_2(s)}{u_1^{\alpha_2}(s)}\nabla s + A(\frac{b_2}{u_1^{\alpha_2}})\phi_1(b) + B(\frac{b_2}{u_1^{\alpha_2}})\phi_2(a) + \varphi_2^* f(t)\\
&\leqslant \int_a^b G(t,s)\frac{b_2(s)}{r_1^{\alpha_2}}\nabla s + A\phi_1(b)\int_a^b G(s,s)\frac{b_2(s)}{r_1^{\alpha_2}}\nabla s\\
&\quad + B\phi_2(a)\int_a^b G(s,s)\frac{b_2(s)}{r_1^{\alpha_2}}\nabla s + \varphi_2^* f(t)\\
&\leqslant \int_a^b G(t,s)\frac{b_2(s)}{r_1^{\alpha_2}}\nabla s + \frac{A\phi_1(b)}{g(t)}\int_a^b G(t,s)\frac{b_2(s)}{r_1^{\alpha_2}}\nabla s\\
&\quad + \frac{B\phi_2(a)}{g(t)}\int_a^b G(t,s)\frac{b_2(s)}{r_1^{\alpha_2}}\nabla s + \varphi_2^* f(t)\\
&= \frac{1}{r_1^{\alpha_2}}\beta_2(t)f(t) + \varphi_2^* f(t)\\
&\leqslant \frac{1}{r_1^{\alpha_2}}\beta_2^* f^2(t) + \varphi_2^* f(t)\\
&\leqslant \Big[\frac{K\beta_2^*}{r_1^{\alpha_2}} + \varphi_2^*\Big]K.
\end{aligned}$$

Thus, if r_1, r_2, R_1 and R_2 satisfying the following inequalities are chosen

$$\begin{aligned}
&\frac{1}{R_2^{\alpha_1}}\hat{\beta}_{1*} \geqslant r_1, \qquad \Big(\frac{K\beta_1^*}{r_2^{\alpha_1}} + \varphi_1^*\Big)K \leqslant R_1,\\
&\frac{1}{R_1^{\alpha_2}}\hat{\beta}_{2*} \geqslant r_2, \qquad \Big(\frac{K\beta_2^*}{r_1^{\alpha_2}} + \varphi_2^*\Big)K \leqslant R_2,
\end{aligned}$$

then, we get $(F_1u_1, F_2u_2) \in M$.

Note that $\hat{\beta}_{i*}, \beta_{i*} > 0$ and let us take $R_1 = R_2 = \dfrac{R}{K}$, $r_1 = r_2 = \dfrac{K}{R}$, $(R > K)$

$$\hat{\beta}_{1*}\Big(\frac{R}{K}\Big)^{1-\alpha_1} \geqslant 1, \quad K^2\beta_1^*\Big(\frac{R}{K}\Big)^{\alpha_1} + \varphi_1^* K \leqslant \frac{R}{K},$$

$$\hat{\beta}_{2*}\Big(\frac{R}{K}\Big)^{1-\alpha_2} \geqslant 1, \quad K^2\beta_2^*\Big(\frac{R}{K}\Big)^{\alpha_2} + \varphi_2^* K \leqslant \frac{R}{K},$$

from which, these inequalities hold for R big enough because $\alpha_i < 1$. Thus, we obtain that $F(M) \subseteq M$. Moreover, we can easily find that F is continuous and compact. Then, from the Schauder fixed theorem, we can say that F has a fixed point. □

THEOREM 3.2. *Let the conditions $(H_1) - (H_5)$ hold. If $\varphi_1^* \leqslant 0$, $\varphi_2^* \leqslant 0$ and*

$$\varphi_{1*} \geqslant \Big[\frac{\alpha_1\alpha_2\hat{\beta}_{1*}}{(\beta_2^* K^2)^{\alpha_1}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}\Big(1 - \frac{1}{\alpha_1\alpha_2}\Big), \quad \varphi_{2*} \geqslant \Big[\frac{\alpha_1\alpha_2\hat{\beta}_{2*}}{(\beta_1^* K^2)^{\alpha_2}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}\Big(1 - \frac{1}{\alpha_1\alpha_2}\Big),$$

then the SSS (1.1) *has at least one positive solution.*

PROOF. From the proof of the previous theorem, we know that the operator F is continuous and compact. Therefore, it is sufficient to prove that $F(M) \subseteq M$. For that purpose, we find $0 < r_1 < R_1$, $0 < r_2 < R_2$ such that

$$\hat{\beta}_{1*}\frac{1}{R_2^{\alpha_1}} + \varphi_{1*} \geqslant r_1, \quad \frac{K^2\beta_1^*}{r_2^{\alpha_1}} \leqslant R_1, \tag{3.1}$$

$$\hat{\beta}_{2*}\frac{1}{R_1^{\alpha_2}} + \varphi_{2*} \geqslant r_2, \quad \frac{K^2\beta_2^*}{r_1^{\alpha_2}} \leqslant R_2. \tag{3.2}$$

If we fix $R_1 = \dfrac{K^2\beta_1^*}{r_2^{\alpha_1}}$ and $R_2 = \dfrac{K^2\beta_2^*}{r_1^{\alpha_2}}$, then the inequalities of (3.2) hold if r_2 satisfies

$$\frac{\hat{\beta}_{2*}}{(K^2\beta_1^*)^{\alpha_2}} r_2^{\alpha_1\alpha_2} + \varphi_{2*} \geqslant r_2,$$

or equivalently

$$\varphi_{2*} \geqslant \mu(r_2) := r_2 - \frac{\hat{\beta}_{2*}}{(K^2\beta_1^*)^{\alpha_2}} r_2^{\alpha_1\alpha_2}.$$

The function $\mu(r_2)$ possesses a minimum at $\tilde{r}_2 := \Big[\dfrac{1}{\alpha_1\alpha_2}\dfrac{(K^2\beta_1^*)^{\alpha_2}}{\hat{\beta}_{2*}}\Big]^{\frac{1}{\alpha_1\alpha_2-1}}$.

Let us take $r_2 = \tilde{r}_2$, then the first inequality of (3.2) satisfies if

$$\varphi_{2*} \geqslant \mu(\tilde{r}_2) = \Big[\alpha_1\alpha_2\frac{\hat{\beta}_{2*}}{(K^2\beta_1^*)^{\alpha_2}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}\Big(1 - \frac{1}{\alpha_1\alpha_2}\Big).$$

Similarly, the inequalities of (3.1) hold if r_1 satisfies

$$\varphi_{1*} \geqslant \kappa(r_1) := r_1 - \frac{\hat{\beta}_{1*}}{(K^2\beta_2^*)^{\alpha_1}} r_1^{\alpha_1\alpha_2}.$$

The function $\kappa(r_1)$ has a minimum at $\tilde{r}_1 := \Big[\frac{1}{\alpha_1\alpha_2}\frac{(K^2\beta_2^*)^{\alpha_1}}{\hat{\beta}_{1*}}\Big]^{\frac{1}{\alpha_1\alpha_2-1}}$.

Let take $r_1 = \tilde{r}_1$, then first inequality of (3.1) satisfies if

$$\varphi_{1*} \geqslant \Big[\alpha_1\alpha_2\frac{\hat{\beta}_{1*}}{(K^2\beta_2^*)^{\alpha_1}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}\Big(1 - \frac{1}{\alpha_1\alpha_2}\Big).$$

Hence, it remains to prove that $R_1 > r_1 = \tilde{r}_1$ and $R_2 > r_2 = \tilde{r}_2$.

We can easily verify this through the following elementary calculations;

$$R_1 = \frac{\beta_1^* K^2}{\tilde{r}_2^{\alpha_1}} = \frac{\beta_1^* K^2}{(\alpha_1\alpha_2\hat{\beta}_{2*}(K^2\beta_1^*)^{-\alpha_2})^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}} = \frac{(\beta_1^* K^2)^{1+\frac{\alpha_1\alpha_2}{1-\alpha_1\alpha_2}}}{(\alpha_1\alpha_2\hat{\beta}_{2*})^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}}$$

$$= \Big[\frac{\beta_1^* K^2}{(\alpha_1\alpha_2\hat{\beta}_{2*})^{\alpha_1}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}} = \Big[\frac{1}{(\alpha_1\alpha_2)^{\alpha_1}}\frac{\beta_1^* K^2}{(\hat{\beta}_{2*})^{\alpha_1}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}$$

$$> \Big[\alpha_1\alpha_2\tfrac{\hat{\beta}_{1*}}{\beta_2^*)^{\alpha_1}K^{2\alpha_1}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}} = \tilde{r}_1.$$

In a similar way, we obtain $R_2 > \tilde{r}_2$. Thus, it follows from the Schauder fixed theorem, then SSS (1.1) has a positive solution. □

THEOREM 3.3. *Assume that the conditions* $(H_1)-(H_5)$ *are satisfied. If* $\varphi_{1*} \geqslant 0$, $\varphi_2^* \leqslant 0$ *and*

$$\varphi_{2*} \geqslant \acute{r}_2 - \frac{\hat{\beta}_{2*}\acute{r}_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^*\acute{r}_2^{\alpha_1})^{\alpha_2}},$$

then the SSS (1.1) *has at least one positive solution, where* $\acute{r}_2 > 0$ *is a unique solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{2*}\beta_1^*.$$

PROOF. We follow the same strategy as in the previous theorem. So, to prove that $F(M) \subseteq M$, it is sufficient to find $0 < r_1 < R_1$, $0 < r_2 < R_2$ such that

$$\hat{\beta}_{1*}\frac{1}{R_2^{\alpha_1}} \geqslant r_1, \qquad \Big(K\beta_1^*\frac{1}{r_2^{\alpha_1}} + \varphi_1^*\Big)K \leqslant R_1, \tag{3.3}$$

$$\hat{\beta}_{2*}\frac{1}{R_1^{\alpha_2}} + \varphi_{2*} \geqslant r_2, \qquad K^2\beta_2^*\frac{1}{r_1^{\alpha_2}} \leqslant R_2. \tag{3.4}$$

If we fix $R_2 = \dfrac{K^2\beta_2^*}{r_1^{\alpha_2}}$, then the first inequality of (3.3) holds if r_1 satisfies

$$\frac{\hat{\beta}_{1*}}{R_2^{\alpha_1}} = \frac{\hat{\beta}_{1*}}{\left(\frac{K^2\beta_2^*}{r_1^{\alpha_2}}\right)^{\alpha_1}} = \frac{\hat{\beta}_{1*} r_1^{\alpha_1\alpha_2}}{(K^2\beta_2^*)^{\alpha_1}} \geqslant r_1,$$

or equivalently

$$r_1 \leqslant \Big[\frac{\hat{\beta}_{1*}}{(K^2\beta_2^*)^{\alpha_1}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}.$$

If we choose $r_1 > 0$ small enough, then the first inequality of (3.3) holds.

If we fix $R_1 = K^2\beta_1^*\dfrac{1}{r_2^{\alpha_1}} + K\varphi_1^*$, then the first inequality of (3.4) holds providing r_2 satisfies

$$\varphi_{2*} \geqslant r_2 - \frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} = r_2 - \frac{\hat{\beta}_{2*}}{\Big(K^2\beta_1^*\dfrac{1}{r_2^{\alpha_1}} + K\varphi_1^*\Big)^{\alpha_2}} = r_2 - \frac{\hat{\beta}_{2*} r_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2}}$$

or equivalently

$$\varphi_{2*} \geqslant f(r_2) := r_2 - \frac{\hat{\beta}_{2*} r_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2}}.$$

From which, we obtain

$$\begin{aligned} f'(r_2) = 1 - &\frac{\hat{\beta}_{2*} r_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{2\alpha_2}} \Big[\alpha_1\alpha_2 r_2^{(\alpha_1\alpha_2-1)}(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2} \\ &- r_2^{\alpha_1\alpha_2}\alpha_2(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{(\alpha_2-1)} K\varphi_1^* r_2^{(\alpha_1-1)}\Big] \\ &= 1 - \frac{\hat{\beta}_{2*}\alpha_1\alpha_2 r_2^{(\alpha_1\alpha_2-1)}}{(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2}}\Big[1 - \frac{K\varphi_1^* r_2^{\alpha_1}}{K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1}}\Big] \\ &= 1 - \alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^*\frac{1}{r_2^{(1-\alpha_1\alpha_2)}}\frac{1}{(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2+1}} \end{aligned}$$

and we have $f'(0) = -\infty$, $f'(+\infty) = 1$, then there exists $\acute{r}_2$ such that $f'(\acute{r}_2) = 0$,

$$\begin{aligned} f''(r_2) = -[&\alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^*(\alpha_1\alpha_2 - 1)r_2^{\alpha_1\alpha_2-2}(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{-\alpha_2-1} \\ &-\alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^* r_2^{\alpha_1\alpha_2-1}(\alpha_2+1)(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{-\alpha_2-2}K\varphi_1^*\alpha_1 r_2^{\alpha_1-1}] > 0, \end{aligned}$$

Thus, the function $f(r_2)$ possesses a minimum at $\acute{r}_2$. Considering $f'(\acute{r}_2) = 0$, we get

$$1 - \alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^*\acute{r}_2^{\alpha_1\alpha_2-1}(K^2\beta_1^* + K\varphi_1^*\acute{r}_2^{\alpha_1})^{-\alpha_2-1} = 0,$$

or equivalently

$$\acute{r}_2^{1-\alpha_1\alpha_2}(K^2\beta_1^* + K\varphi_1^*\acute{r}_2^{\alpha_1})^{\alpha_2+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{2*}\beta_1^*.$$

If we take $r_2 = \acute{r}_2$, then the first inequality of (3.4) holds. The second inequality of (3.3), (3.4) are satisfied directly by the choice of R_1 and R_2. Moreover, for R_2 big enough and r_1 small enough, we have $\acute{r}_2 < R_2$ and $r_1 < R_1$. Thus, we obtain that the SSS (1.1) has a positive solution. $\square$

THEOREM 3.4. *Let the conditions* $(H_1) - (H_5)$ *hold. If* $\varphi_1^* \leqslant 0$, $\varphi_{2*} \geqslant 0$ *and*

$$\varphi_{1*} \geqslant \acute{r}_1 - \frac{\hat{\beta}_{1*}\acute{r}_1^{\alpha_1\alpha_2}}{(K^2\beta_2^* + K\varphi_2^*\acute{r}_1^{\alpha_2})^{\alpha_1}},$$

then the SSS (1.1) *has at least one positive solution, where* $\acute{r}_1 > 0$ *is a unique solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}(K^2\beta_2^* + K\varphi_2^* r_1^{\alpha_2})^{\alpha_1+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{1*}\beta_2^*.$$

THEOREM 3.5. *Assume that* $(H_1) - (H_5)$ *hold. If* $\varphi_{1*} < 0 < \varphi_1^*$, $\varphi_{2*} < 0 < \varphi_2^*$ *and*

$$\varphi_{1*} \geqslant \tilde{r}_1 - \frac{\hat{\beta}_{1*}\tilde{r}_1^{\alpha_1\alpha_2}}{(K^2\beta_2^* + K\varphi_2^*\tilde{r}_1^{\alpha_2})^{\alpha_1}}, \quad \varphi_{2*} \geqslant \tilde{r}_2 - \frac{\hat{\beta}_{2*}\tilde{r}_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^*\tilde{r}_2^{\alpha_1})^{\alpha_2}},$$

then the SSS (1.1) *has at least one positive solution, where* $\tilde{r}_1 > 0$ *is a unique solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}(K^2\beta_2^* + K\varphi_2^* r_1^{\alpha_2})^{\alpha_1+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{1*}\beta_2^*,$$

and $\tilde{r}_2 > 0$ *is a unique solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{2*}\beta_1^*.$$

PROOF. We follow the same strategy as in the proof of Theorem 3.3. In this case, to prove that $F(M) \subseteq M$, we shall find $r_1 < R_1$, $r_2 < R_2$ such that

$$\hat{\beta}_{1*}\frac{1}{R_2^{\alpha_1}} + \varphi_{1*} \geqslant r_1, \qquad K^2\beta_1^*\frac{1}{r_2^{\alpha_1}} + K\varphi_1^* \leqslant R_1, \tag{3.5}$$

$$\hat{\beta}_{2*}\frac{1}{R_1^{\alpha_2}} + \varphi_{2*} \geqslant r_2, \qquad K^2\beta_2^*\frac{1}{r_1^{\alpha_2}} + K\varphi_2^* \leqslant R_2. \tag{3.6}$$

If we take $R_1 = K^2\beta_1^*\frac{1}{r_2^{\alpha_1}} + K\varphi_1^*$ and $R_2 = K^2\beta_2^*\frac{1}{r_1^{\alpha_2}} + K\varphi_2^*$, then the first inequality of (3.6) holds if r_2 satisfies

$$\varphi_2^* \geqslant h(r_2) := r_2 - \frac{\hat{\beta}_{2*}r_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2}}.$$

By following the same way as in the proof of Theorem 3.3, we find

$$h'(r_2) = 1 - \alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^*\frac{1}{r_2^{(1-\alpha_1\alpha_2)}}\frac{1}{(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2+1}}$$

and we have $h'(0) = -\infty$, $h'(+\infty) = 1$, then there exists $\tilde{r}_2$ such that $h'(\tilde{r}_2) = 0$,

$$h''(r_2) = -[\alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^*(\alpha_1\alpha_2 - 1)r_2^{\alpha_1\alpha_2-2}(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{-\alpha_2-1} + \alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^* r_2^{\alpha_1\alpha_2-1}(-\alpha_2-1)(K^2\beta_1^*+K\varphi_1^* r_2^{\alpha_1})^{-\alpha_2-2}K\varphi_1^*\alpha_1 r_2^{\alpha_1-1}] > 0.$$

Therefore the function $h(r_2)$ has a minimum at $\tilde{r}_2$. Considering $h'(\tilde{r}_2) = 0$, we obtain

$$1 - \alpha_1\alpha_2\hat{\beta}_{2*}K^2\beta_1^*\tilde{r}_2^{\alpha_1\alpha_2-1}(K^2\beta_1^* + K\varphi_1^*\tilde{r}_2^{\alpha_1})^{-\alpha_2-1} = 0,$$

or equivalently

$$\tilde{r}_2^{1-\alpha_1\alpha_2}(K^2\beta_1^* + K\varphi_1^*\tilde{r}_2^{\alpha_1})^{\alpha_2+1} = \alpha_1\alpha_2K^2\hat{\beta}_{2*}\beta_1^*. \tag{3.7}$$

Similarly, we get

$$\tilde{r}_1^{1-\alpha_1\alpha_2}(K^2\beta_2^* + K\varphi_2^*\tilde{r}_1^{\alpha_2})^{\alpha_1+1} = \alpha_1\alpha_2K^2\hat{\beta}_{1*}\beta_2^*. \tag{3.8}$$

If we take $r_1 = \tilde{r}_1$, $r_2 = \tilde{r}_2$, then the first inequality of (3.5) and (3.6) hold. The second inequalities hold directly by the choice of R_1 and R_2.

$$R_1 = \frac{K^2\beta_1^*}{\tilde{r}_2^{\alpha_1}} + K\varphi_1^* = \frac{K^2\beta_1^* + K\varphi_1^*\tilde{r}_2^{\alpha_1}}{\tilde{r}_2^{\alpha_1}} = \frac{(\alpha_1\alpha_2K^2\hat{\beta}_{2*}\beta_1^*)^{\frac{1}{1+\alpha_2}}\tilde{r}_2^{\frac{\alpha_1\alpha_2-1}{1+\alpha_2}}}{\tilde{r}_2^{\alpha_1}} = (\alpha_1\alpha_2K^2\hat{\beta}_{2*}\beta_1^*)^{\frac{1}{1+\alpha_2}}\tilde{r}_2^{-\frac{1+\alpha_1}{1+\alpha_2}}.$$

For R_2, we can find similar equality. According to,

$$R_2 = (\alpha_1\alpha_2K^2\hat{\beta}_{1*}\beta_2^*)^{\frac{1}{1+\alpha_1}}\tilde{r}_1^{-\frac{1+\alpha_2}{1+\alpha_1}}.$$

Now, we prove that $\tilde{r}_1 < R_1$ and $\tilde{r}_2 < R_2$, or equivalently

$$\tilde{r}_1\tilde{r}_2^{\frac{1+\alpha_1}{1+\alpha_2}} < (\alpha_1\alpha_2K^2\hat{\beta}_{2*}\beta_1^*)^{\frac{1}{1+\alpha_2}} \quad \text{and} \quad \tilde{r}_2\tilde{r}_1^{\frac{1+\alpha_2}{1+\alpha_1}} < (\alpha_1\alpha_2K^2\hat{\beta}_{1*}\beta_2^*)^{\frac{1}{1+\alpha_1}}.$$

This implies

$$\tilde{r}_1^{1+\alpha_2}\tilde{r}_2^{1+\alpha_1} < \alpha_1\alpha_2K^2\hat{\beta}_{1*}\beta_2^* \text{ and } \tilde{r}_1^{1+\alpha_2}\tilde{r}_2^{1+\alpha_1} < \alpha_1\alpha_2K^2\hat{\beta}_{2*}\beta_1^*.$$

On the other hand, from (3.7), we obtain

$$\tilde{r}_2^{1-\alpha_1\alpha_2}(K^2\beta_1^*)^{1+\alpha_2} \leqslant \alpha_1\alpha_2K^2\hat{\beta}_{2*}\beta_1^*,$$

from which, we have

$$\tilde{r}_2 \leqslant (\alpha_1\alpha_2(K^2\beta_1^*)^{-\alpha_2}\hat{\beta}_{2*})^{\frac{1}{1-\alpha_1\alpha_2}}. \tag{3.9}$$

Similarly, using (3.8), we get

$$\tilde{r}_1 \leqslant (\alpha_1\alpha_2(K^2\beta_2^*)^{-\alpha_1}\hat{\beta}_{1*})^{\frac{1}{1-\alpha_1\alpha_2}}. \tag{3.10}$$

From (3.9) and (3.10), we find

$$\tilde{r}_1^{1+\alpha_2}\tilde{r}_2^{1+\alpha_1} \leqslant (\alpha_1\alpha_2(K^2\beta_2^*)^{-\alpha_1}\hat{\beta}_{1*})^{\frac{1+\alpha_2}{1-\alpha_1\alpha_2}}(\alpha_1\alpha_2(K^2\beta_1^*)^{-\alpha_2}\hat{\beta}_{2*})^{\frac{1+\alpha_1}{1-\alpha_1\alpha_2}}.$$

Thus, if we can prove

$$\tilde{r}_1^{1+\alpha_2}\tilde{r}_2^{1+\alpha_1} \leqslant (\alpha_1\alpha_2(K^2\beta_2^*)^{-\alpha_1}\hat{\beta}_{1*})^{\frac{1+\alpha_2}{1-\alpha_1\alpha_2}}(\alpha_1\alpha_2(K^2\beta_1^*)^{-\alpha_2}\hat{\beta}_{2*})^{\frac{1+\alpha_1}{1-\alpha_1\alpha_2}} < \alpha_1\alpha_2K^2\hat{\beta}_{2*}\beta_1^*,$$

then, we say

$$\tilde{r}_1^{1+\alpha_2}\tilde{r}_2^{1+\alpha_1} < \alpha_1\alpha_2 K^2\hat{\beta}_{2*}\beta_1^*.$$

In fact, since $K \geqslant 1$ and $\hat{\beta}_{i*} \leqslant \beta_i^*$, $(i = 1, 2)$, we can write

$$\Big(\frac{1}{K^2}\Big)^{\frac{\alpha_1+2\alpha_1\alpha_2+1}{1-\alpha_2}}(\alpha_1\alpha_2)^{\frac{1+\alpha_1+\alpha_2}{1-\alpha_1\alpha_2}}\Big(\frac{\hat{\beta}_{1*}}{\beta_1^*}\Big)^{\frac{1+\alpha_2}{1-\alpha_1\alpha_2}}\Big(\frac{\hat{\beta}_{2*}}{\beta_2^*}\Big)^{\frac{\alpha_1(1+\alpha_2)}{1-\alpha_1\alpha_2}} < 1.$$

Similarly, we get

$$\tilde{r}_1^{1+\alpha_2}\tilde{r}_2^{1+\alpha_1} < \alpha_1\alpha_2 K^2\hat{\beta}_{1*}\beta_2^*,$$

This completes the proof. □

The proofs of the following theorems are similar to that of the previous theorems.

THEOREM 3.6. *Assume that* $(H_1) - (H_5)$ *hold. If* $\varphi_1^* \leqslant 0$, $\varphi_{2*} < 0 < \varphi_2^*$ *and*

$$\varphi_{2*} \geqslant \Big[\alpha_1\alpha_2\frac{\hat{\beta}_{2*}}{(\beta_1^*K^2)^{\alpha_2}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}\Big(1-\frac{1}{\alpha_1\alpha_2}\Big), \quad \varphi_{1*} \geqslant \acute{r}_1-\frac{\hat{\beta}_{1*}\acute{r}_1^{\alpha_1\alpha_2}}{(K^2\beta_2^* + K\varphi_2^*\acute{r}_1^{\alpha_2})^{\alpha_1}},$$

then the SSS (1.1) *has at least one positive solution, where* $\acute{r}_1 > 0$ *is a unique solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}(K^2\beta_2^* + K\varphi_2^* r_1^{\alpha_2})^{\alpha_1+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{1*}\beta_2^*.$$

THEOREM 3.7. *Let the conditions* $(H_1) - (H_5)$ *hold. If* $\varphi_2^* \leqslant 0$, $\varphi_{1*} < 0 < \varphi_1^*$ *and*

$$\varphi_{1*} \geqslant \Big[\alpha_1\alpha_2\frac{\hat{\beta}_{1*}}{(\beta_2^*K^2)^{\alpha_1}}\Big]^{\frac{1}{1-\alpha_1\alpha_2}}\Big(1-\frac{1}{\alpha_1\alpha_2}\Big), \quad \varphi_{2*} \geqslant \acute{r}_2-\frac{\hat{\beta}_{2*}\acute{r}_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^*\acute{r}_2^{\alpha_1})^{\alpha_2}},$$

then the SSS (1.1) *has at least one positive solution, where* $\acute{r}_2 > 0$ *is a unique solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{2*}\beta_1^*.$$

THEOREM 3.8. *Assume that* $(H_1) - (H_5)$ *hold. If* $\varphi_{1*} \geqslant 0$, $\varphi_{2*} < 0 < \varphi_2^*$ *and*

$$\varphi_{2*} \geqslant \grave{r}_2 - \frac{\hat{\beta}_{2*}\grave{r}_2^{\alpha_1\alpha_2}}{(K^2\beta_1^* + K\varphi_1^*\grave{r}_2^{\alpha_1})^{\alpha_2}},$$

then the SSS (1.1) *has at least one positive solution, where* $\grave{r}_2 > 0$ *is a unique solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}(K^2\beta_1^* + K\varphi_1^* r_2^{\alpha_1})^{\alpha_2+1} = \alpha_1\alpha_2 K^2\hat{\beta}_{2*}\beta_1^*.$$

THEOREM 3.9. *Let* $(H_1) - (H_5)$ *hold. If* $\varphi_{2*} \geqslant 0$, $\varphi_{1*} < 0 < \varphi_1^*$ *and*

$$\varphi_{1*} \geqslant \grave{r}_1 - \frac{\hat{\beta}_{1*}\grave{r}_1^{\alpha_1\alpha_2}}{(K^2\beta_2^* + K\varphi_2^*\grave{r}_1^{\alpha_2})^{\alpha_1}},$$

then the SSS (1.1) *has at least one positive solution, where* $\grave{r}_1 > 0$ *is a unique solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}(K^2\beta_2^* + K\varphi_2^* r_1^{\alpha_2})^{\alpha_1+1} = \alpha_1\alpha_2 K^2 \hat{\beta}_{1*}\beta_2^*.$$

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