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MILDLY $\mathcal{I}_g$-ω-CLOSED SETS

O. Nethaji, I. Rajasekaran, and O. Ravi

Abstract. In this paper, another generalized class of $\tau^\star$ called mildly $\mathcal{I}_g$-ω-closed sets is introduced and the notion of mildly $\mathcal{I}_g$-ω-open sets in ideal topological spaces is introduced and studied. The relationships of mildly $\mathcal{I}_g$-ω-closed sets with various other sets are investigated.

1. Introduction

The first step of generalizing closed sets (briefly, g-closed sets) was done by Levine in 1970 [**13**]. He defined a subset S of a topological space (X,τ) to be g-closed if its closure is contained in every open superset of S. As the weak form of g-closed sets, the notion of weakly g-closed sets was introduced and studied by Sundaram and Nagaveni [**17**]. Sundaram and Pushpalatha [**18**] introduced and studied the notion of strongly g-closed sets, which are weaker than closed sets and stronger than g-closed sets. Park and Park [**16**] introduced and studied the notion of mildly g-closed sets, which is properly placed between the class of strongly g-closed sets and the class of weakly g-closed sets. Moreover, the relations with other notions directly or indirectly connected with g-closed sets were investigated by them. In 1999, Dontchev et al. [**6**] studied the notion of generalized closed sets in ideal topological spaces called $\mathcal{I}_g$-closed sets. In 2008, Navaneethakrishnan and Paulraj Joseph studied some characterizations of normal spaces via $\mathcal{I}_g$-open sets [**14**]. In 2013, Ekici and Ozen [**8**] introduced a generalized class of $\tau^\star$ in ideal topological spaces.

The notion of ω-open sets in topological spaces introduced by Hdeib [**9**] has been studied in recent years by a good number of researchers like Noiri et al [**15**], Al-Omari and Noorani [**3, 4**] and Khalid Y. Al-Zoubi [**11**].

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The main aim of this paper is to introduce another generalized class of $\tau^\star$ called mildly $\mathcal{I}_g$-ω-open sets in ideal topological spaces. Moreover, this generalized class of $\tau^\star$ generalize $\mathcal{I}_g$-ω-open sets and mildly $\mathcal{I}_g$-ω-open sets. The relationships of mildly $\mathcal{I}_g$-ω-closed sets with various other sets are discussed.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{Q}$, $(\mathbb{R}-\mathbb{Q})$, $(\mathbb{R}-\mathbb{Q})_-$ and $(\mathbb{R}-\mathbb{Q})_+$) denotes the set of real numbers (resp. the set of rational numbers, the set of irrational numbers, the set of negative irrational numbers and the set of positive irrational numbers).

In this paper, (X,τ) represents a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of a subset G of a topological space (X,τ) will be denoted by $cl(G)$ and $int(G)$, respectively.

Definition 2.1. *A subset G of a topological space (X,τ) is said to be*

1. *g-closed* [**13**] *if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is open in X;*
2. *g-open* [**13**] *if $X \backslash G$ is g-closed;*
3. *weakly g-closed* [**17**] *if $cl(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is open in X;*
4. *strongly g-closed* [**18**] *if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is g-open in X.*

Definition 2.2. [**20**] *In a topological space (X,τ), a point p in X is called a condensation point of a subset H if for each open set U containing p, $U \cap H$ is uncountable.*

Definition 2.3. [**9**] *A subset H of a topological space (X,τ) is called ω-closed if it contains all its condensation points.*

The complement of an ω-closed set is called ω-open.

It is well known that a subset W of a topological space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X,τ_ω) are denoted by int_ω and cl_ω respectively.

Lemma 2.1. [**9**] *Let H be a subset of a topological space (X,τ). Then*

1. *H is ω-closed in X if and only if $H = cl_\omega(H)$.*
2. *$cl_\omega(X\backslash H) = X\backslash int_\omega(H)$.*
3. *$cl_\omega(H)$ is ω-closed in X.*
4. *$x \in cl_\omega(H)$ if and only if $H \cap G \neq \phi$ for each ω-open set G containing x.*
5. *$cl_\omega(H) \subseteq cl(H)$.*
6. *$int(H) \subseteq int_\omega(H)$.*

Lemma 2.2. [**11**] *If A is an ω-open subset of a space (X,τ), then $A-C$ is ω-open for every countable subset C of X.*

DEFINITION 2.4. *A subset H of a topological space (X,τ) is called generalized ω-closed (briefly, $g\omega$-closed)* [**11**] *if $cl_\omega(H) \subseteq U$ whenever $H \subseteq U$ and U is open in X.*

The complement of a $g\omega$-closed set is called $g\omega$-open.

REMARK 2.1. [**4**] *For a subset of a topological space (X,τ), the following relations hold:*

$$\begin{array}{ccc} \textbf{\textit{open}} & \longrightarrow & g\textbf{\textit{-open}} \\ \downarrow & & \downarrow \\ \omega\textbf{\textit{-open}} & \longrightarrow & g\omega\textbf{\textit{-open}} \end{array}$$

None of the above implications is reversible.

An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$ and
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$ [**12**].

Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X, if $\mathbb{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathbb{P}(X) \to \mathbb{P}(X)$, called a local function [**12**] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^\star(\mathcal{I},\tau) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I},\tau)$ [**19**]. We will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal topological space. On the other hand, $(A, \tau_A, \mathcal{I}_A)$ where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap J : J \in \mathcal{I}\}$ is an ideal topological subspace for the ideal topological space $(X,\tau,\mathcal{I})$ and $A \subseteq X$ [**10**]. For a subset $A \subseteq X$, $cl^\star(A)$ and $int^\star(A)$ will, respectively, denote the closure and the interior of A in $(X,\tau^\star)$.

PROPOSITION 2.1. [**1**] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and H a subset of X. If $\mathcal{I} = \{\phi\}$(resp. $\mathbb{P}(X)$), then $H^\star = cl(H)$(resp. ϕ).*

LEMMA 2.3. [**10**] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B subsets of X. Then the following properties hold:*

(1) $A \subseteq B \Rightarrow A^\star \subseteq B^\star$,
(2) $A^\star = cl(A^\star) \subseteq cl(A)$,
(3) $(A^\star)^\star \subseteq A^\star$,
(4) $(A \cup B)^\star = A^\star \cup B^\star$,
(5) $(A \cap B)^\star \subseteq A^\star \cap B^\star$.

DEFINITION 2.5. *A subset G of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be*

(1) *$\mathcal{I}_g$-closed* [**6, 14**] *if $G^\star \subseteq H$ whenever $G \subseteq H$ and H is open in $(X,\tau,\mathcal{I})$.*
(2) *$pre^\star_{\mathcal{I}}$-open* [**7**] *if $G \subseteq int^\star(cl(G))$.*
(3) *$pre^\star_{\mathcal{I}}$-closed* [**7**] *if $X \backslash G$ is $pre^\star_{\mathcal{I}}$-open (or) $cl^\star(int(G)) \subseteq G$.*
(4) *$\mathcal{I}_g$-ω-closed* [**5**] *if $G^\star \subseteq H$ whenever $G \subseteq H$ and H is ω-open in $(X,\tau,\mathcal{I})$.*
(5) *$\mathcal{I}$-R closed* [**2**] *if $G = cl^\star(int(G))$.*
(6) *$\star$-closed* [**10**] *if $G = cl^\star(G)$ or $G^\star \subseteq G$.*

REMARK 2.2. [**8**] *In any ideal topological space, every $\mathcal{I}$-R closed set is $\star$-closed but not conversely.*

3. Other generalized classes of $\tau^\star$

DEFINITION 3.1. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G of X is said to be*

(1) *weakly $\mathcal{I}_g$-ω-closed if $(int(G))^\star \subseteq H$ whenever $G \subseteq H$ and H is ω-open in X;*
(2) *mildly $\mathcal{I}_g$-ω-closed if $(int(G))^\star \subseteq H$ whenever $G \subseteq H$ and H is $g\omega$-open in X;*
(3) *strongly $\mathcal{I}_g$-ω-closed if $G^\star \subseteq H$ whenever $G \subseteq H$ and H is $g\omega$-open in X.*

EXAMPLE 3.1. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and $\mathcal{I} = \{\phi\}$,*

(1) *For $G = \mathbb{R} - \mathbb{Q}$, if H is any ω-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $(int(G))^\star = \phi^\star = \phi \subseteq H$ and hence G is weakly $\mathcal{I}_g$-ω-closed in X.*
(2) *$K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being ω-open whereas $(int(\mathbb{Q}))^\star = \mathbb{Q}^\star = cl(\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{Q}$ which implies $K = \mathbb{Q}$ is not weakly $\mathcal{I}_g$-ω-closed in X.*
(3) *For $G = \mathbb{R} - \mathbb{Q}$, if H is any $g\omega$-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $(int(G))^\star = \phi^\star = \phi \subseteq H$ and hence G is mildly $\mathcal{I}_g$-ω-closed in X.*
(4) *$K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being $g\omega$-open whereas $(int(\mathbb{Q}))^\star = \mathbb{Q}^\star = cl(\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{Q}$ which implies $K = \mathbb{Q}$ is not mildly $\mathcal{I}_g$-ω-closed in X.*
(5) *For $G = \mathbb{R} - \mathbb{Q}$, if H is any $g\omega$-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $G^\star = (\mathbb{R} - \mathbb{Q})^\star = cl(\mathbb{R} - \mathbb{Q}) = \mathbb{R} - \mathbb{Q} = G \subseteq H$ and hence G is strongly $\mathcal{I}_g$-ω-closed in X.*
(6) *$K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being $g\omega$-open whereas $K^\star = \mathbb{Q}^\star = cl(\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{Q} = K$ which implies $K = \mathbb{Q}$ is not strongly $\mathcal{I}_g$-ω-closed in X.*

DEFINITION 3.2. *A subset G in an ideal topological space $(X, \tau, \mathcal{I})$ is said to be mildly $\mathcal{I}_g$-ω-open (resp. strongly $\mathcal{I}_g$-ω-open, weakly $\mathcal{I}_g$-ω-open) if $X \backslash G$ is mildly $\mathcal{I}_g$-ω-closed (resp. strongly $\mathcal{I}_g$-ω-closed, weakly $\mathcal{I}_g$-ω-closed).*

THEOREM 3.1. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is mildly $\mathcal{I}_g$-ω-closed $\Leftrightarrow (int(G))^\star \subseteq G$.*

PROOF. $\Rightarrow$ If $(int(G))^\star \nsubseteq G$, there exists $x \in X$ such that $x \in (int(G))^\star - G$. Then $x \in (int(G))^\star - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open being ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open. But $(int(G))^\star \nsubseteq X - \{x\}$ since $x \in (int(G))^\star$. This implies that G is not mildly $\mathcal{I}_g$-ω-closed which proves the necessary part.

$\Leftarrow$ Let $(int(G))^\star \subseteq G$ and H be any $g\omega$-open subset such that $G \subseteq H$. Then $(int(G))^\star \subseteq G \subseteq H$. This implies that G is mildly $\mathcal{I}_g$-ω-closed which proves the sufficiency part. $\square$

THEOREM 3.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is weakly $\mathcal{I}_g$-ω-closed $\Leftrightarrow (int(G))^\star \subseteq G$.*

Proof. $\Rightarrow$ If $(int(G))^\star \nsubseteq G$, there exists $x \in X$ such that $x \in (int(G))^\star - G$. Then $x \in (int(G))^\star - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. But $(int(G))^\star \nsubseteq X - \{x\}$ since $x \in (int(G))^\star$. This implies that G is not weakly $\mathcal{I}_g$-ω-closed which proves the necessary part.

$\Leftarrow$ Let $(int(G))^\star \subseteq G$ and H be any ω-open set such that $G \subseteq H$. Then $(int(G))^\star \subseteq G \subseteq H$. This implies that G is weakly $\mathcal{I}_g$-ω-closed which proves the sufficiency part. □

Theorem 3.3. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is mildly $\mathcal{I}_g$-ω-closed $\Leftrightarrow$ G is weakly $\mathcal{I}_g$-ω-closed.*

Proof. Proof follows by Theorem 3.1 and Theorem 3.2. □

Theorem 3.4. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is strongly $\mathcal{I}_g$-ω-closed $\Leftrightarrow$ $G^\star \subseteq G$.*

Proof. $\Rightarrow$ If $G^\star \nsubseteq G$, there exists $x \in X$ such that $x \in G^\star - G$. Then $x \in G^\star - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open being ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open. But $G^\star \nsubseteq X - \{x\}$ since $x \in G^\star$. This implies that G is not strongly $\mathcal{I}_g$-ω-closed which proves the necessary part.

$\Leftarrow$ Let $G^\star \subseteq G$ and H be any $g\omega$-open set such that $G \subseteq H$. Then $G^\star \subseteq G \subseteq H$. This implies that G is strongly $\mathcal{I}_g$-ω-closed which proves the sufficient part. □

Theorem 3.5. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is $\mathcal{I}_g$-ω-closed $\Leftrightarrow$ $G^\star \subseteq G$.*

Proof. $\Rightarrow$ If $G^\star \nsubseteq G$, there exists $x \in X$ such that $x \in G^\star - G$. Then $x \in G^\star - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. But $G^\star \nsubseteq X - \{x\}$ since $x \in G^\star$. This implies that G is not $\mathcal{I}_g$-ω-closed which proves the necessary part.

$\Leftarrow$ Let $G^\star \subseteq G$ and H be any ω-open set such that $G \subseteq H$. Then $G^\star \subseteq G \subseteq H$. This implies that G is $\mathcal{I}_g$-ω-closed which proves the sufficient part. □

Theorem 3.6. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is strongly $\mathcal{I}_g$-ω-closed $\Leftrightarrow$ G is $\mathcal{I}_g$-ω-closed.*

Proof. Proof follows from Theorem 3.4 and Theorem 3.5. □

Proposition 3.1. *In an ideal topological space $(X, \tau, \mathcal{I})$,*

(1) *Every $\mathcal{I}_g$-ω-closed set is weakly $\mathcal{I}_g$-ω-closed.*
(2) *Every strongly $\mathcal{I}_g$-ω-closed set is mildly $\mathcal{I}_g$-ω-closed.*

Proof. Obvious. □

Remark 3.1. *The converses of Proposition 3.1 are not true in general as shown in the following Example.*

Example 3.2. *Consider $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \{1\}\}$ and $\mathcal{I} = \{\phi\}$. Let $G = \mathbb{R} - \mathbb{Q}$.*

(1) *Let H be any ω-open subset of $\mathbb{R}$ such that $G = \mathbb{R} - \mathbb{Q} \subseteq H$. Now $(int(G))^\star = \phi^\star = \phi \subseteq H$. Hence $G = \mathbb{R} - \mathbb{Q}$ is weakly $\mathcal{I}_g$-ω-closed. Also $G \subseteq G$ and G is ω-open. But $G^\star = cl(G) = \mathbb{R}\backslash\{1\} \nsubseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is not $\mathcal{I}_g$-ω-closed.*
(2) *Let H be any $g\omega$-open subset of $\mathbb{R}$ such that $G = \mathbb{R} - \mathbb{Q} \subseteq H$. Now $(int(G))^\star = \phi^\star = \phi \subseteq H$. Hence $G = \mathbb{R} - \mathbb{Q}$ is mildly $\mathcal{I}_g$-ω-closed. But $G \subseteq G$, G being $g\omega$-open whereas $G^\star = cl(G) = \mathbb{R}\backslash\{1\} \nsubseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is not strongly $\mathcal{I}_g$-ω-closed.*

REMARK 3.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, the following relations hold for a subset G of X.*

$$\begin{array}{ccc} \text{strongly } \mathcal{I}_g\text{-}\omega\text{-closed} & \longleftrightarrow & \mathcal{I}_g\text{-}\omega\text{-closed} \\ \downarrow & & \downarrow \\ \text{mildly } \mathcal{I}_g\text{-}\omega\text{-closed} & \longleftrightarrow & \text{weakly } \mathcal{I}_g\text{-}\omega\text{-closed} \end{array}$$

Where $A \longleftrightarrow B$ means A implies and is implied by B and $A \rightarrow B$ means A implies B but not conversely.

THEOREM 3.7. *In an ideal topological space $(X, \tau, \mathcal{I})$, for a subset G of X, the following properties are equivalent.*

(1) *G is mildly $\mathcal{I}_g$-ω-closed;*
(2) *$(int(G))^\star \backslash G = \phi$;*
(3) *$cl^\star(int(G)) \backslash G = \phi$;*
(4) *$cl^\star(int(G)) \subseteq G$;*
(5) *G is $pre^\star_{\mathcal{I}}$-closed.*

PROOF. (1) $\Leftrightarrow$ (2) G is mildly $\mathcal{I}_g$-ω-closed $\Leftrightarrow$ $(int(G))^\star \subseteq G$ by Theorem 3.1 $\Leftrightarrow$ $(int(G))^\star \backslash G = \phi$.

(2) $\Leftrightarrow$ (3) $(int(G))^\star \backslash G = \phi \Leftrightarrow cl^\star(int(G)) \backslash G = ((int(G))^\star \cup int(G)) \backslash G = [(int(G))^\star \backslash G] \cup [int(G) \backslash G] = int(G) \backslash G = \phi$.

(3) $\Leftrightarrow$ (4) $cl^\star(int(G)) \backslash G = \phi \Leftrightarrow cl^\star(int(G)) \subseteq G$.

(4) $\Leftrightarrow$ (5) $cl^\star(int(G)) \subseteq G \Leftrightarrow G$ is $pre^\star_{\mathcal{I}}$-closed by (3) of Definition 2.5. $\square$

THEOREM 3.8. *In an ideal topological space $(X, \tau, \mathcal{I})$, if G is mildly $\mathcal{I}_g$-ω-closed, then $G \cup (X - (int(G))^\star)$ is mildly $\mathcal{I}_g$-ω-closed.*

PROOF. Since G is mildly $\mathcal{I}_g$-ω-closed, $(int(G))^\star \subseteq G$ by Theorem 3.1. Then $X - G \subseteq X - (int(G))^\star$ and $G \cup (X - G) \subseteq G \cup (X - (int(G))^\star)$. Thus $X \subseteq G \cup (X - (int(G))^\star)$ and so $G \cup (X - (int(G))^\star) = X$. Hence $G \cup (X - (int(G))^\star)$ is mildly $\mathcal{I}_g$-ω-closed. $\square$

THEOREM 3.9. *In an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *G is a $\star$-closed and an open set,*
(2) *G is a $\mathcal{I}$-R closed and an open set,*
(3) *G is a mildly $\mathcal{I}_g$-ω-closed and an open set.*

PROOF. (1) $\Rightarrow$ (2): Since G is $\star$-closed and open, $G = cl^\star(G)$ and $G = int(G)$. Thus $G = cl^\star(int(G))$ and $G = int(G)$. Hence G is $\mathcal{I}$-R closed and open.

(2) $\Rightarrow$ (3): Since G is $\mathcal{I}$-R closed and open, $G = cl^\star(int(G)) = (int(G))^\star \cup int(G) = (int(G))^\star \cup G$. Thus $(int(G))^\star \subseteq G$. By Theorem 3.1, G is mildly $\mathcal{I}_g$-ω-closed and open.

(3) $\Rightarrow$ (1): Since G is mildly $\mathcal{I}_g$-ω-closed, $(int(G))^\star \subseteq G$ by Theorem 3.1. Again G is open implies $G^\star = (int(G))^\star \subseteq G$. Thus G is $\star$-closed and open. □

4. Further properties

THEOREM 4.1. *In an ideal topological space $(X, \tau, \mathcal{I})$, if G is mildly $\mathcal{I}_g$-ω-closed and H is a subset such that $G \subseteq H \subseteq cl^\star(int(G))$, then H is mildly $\mathcal{I}_g$-ω-closed.*

PROOF. Since G is mildly $\mathcal{I}_g$-ω-closed, $cl^\star(int(G)) \subseteq G$ by (4) of Theorem 3.7. Thus by assumption, $G \subseteq H \subseteq cl^\star(int(G)) \subseteq G$. Then $G = H$ and so H is mildly $\mathcal{I}_g$-ω-closed. □

COROLLARY 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If G is a mildly $\mathcal{I}_g$-ω-closed set and an open set, then $cl^\star(G)$ is mildly $\mathcal{I}_g$-ω-closed.*

PROOF. Since G is open in X, $G \subseteq cl^\star(G) = cl^\star(int(G))$. G is mildly $\mathcal{I}_g$-ω-closed implies $cl^\star(G)$ is mildly $\mathcal{I}_g$-ω-closed by Theorem 4.1. □

THEOREM 4.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, a nowhere dense subset is mildly $\mathcal{I}_g$-ω-closed.*

PROOF. If G is a nowhere dense subset in X then $int(cl(G)) = \phi$. Since $int(G) \subseteq int(cl(G))$, $int(G) = \phi$. Hence $(int(G))^\star = \phi^\star = \phi \subseteq G$. Thus, G is mildly $\mathcal{I}_g$-ω-closed in $(X, \tau, \mathcal{I})$ by Theorem 3.1. □

REMARK 4.1. *The converse of Theorem 4.2 is not true in general as shown in the following Example.*

EXAMPLE 4.1. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $\mathcal{I} = \mathbb{P}(\mathbb{R})$, for $G = \mathbb{R} - \mathbb{Q}$, $(int(G))^\star = G^\star = \phi \subseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is mildly $\mathcal{I}_g$-ω-closed by Theorem 3.1. On the other hand, $int(cl(G)) = int(\mathbb{R}) = \mathbb{R} \neq \phi$ and thus $G = \mathbb{R} - \mathbb{Q}$ is not nowhere dense in X.*

REMARK 4.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, the intersection of two mildly $\mathcal{I}_g$-ω-closed subsets is mildly $\mathcal{I}_g$-ω-closed.*

PROOF. Let A and B be mildly $\mathcal{I}_g$-ω-closed subsets in $(X, \tau, \mathcal{I})$. Then $(int(A))^\star \subseteq A$ and $(int(B))^\star \subseteq B$ by Theorem 3.1. Also $[int(A \cap B)]^\star \subseteq [int(A)]^\star \cap [int(B)]^\star \subseteq A \cap B$. This implies that $A \cap B$ is mildly $\mathcal{I}_g$-ω-closed by Theorem 3.1. □

REMARK 4.3. *In an ideal topological space $(X, \tau, \mathcal{I})$, the union of two mildly $\mathcal{I}_g$-ω-closed subsets need not be mildly $\mathcal{I}_g$-ω-closed.*

EXAMPLE 4.2. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and ideal $\mathcal{I} = \{\phi\}$, for $A = (\mathbb{R} - \mathbb{Q})_+ =$ the set of positive irrationals and $B = (\mathbb{R} - \mathbb{Q})_- =$ the set of negative irrationals, $int(A) = \phi$ and $int(B) = \phi$ respectively. So $(int(A))^\star = \phi^\star = \phi \subseteq A$ and thus A is mildly $\mathcal{I}_g$-ω-closed by Theorem 3.1. Similarly B is also mildly $\mathcal{I}_g$-ω-closed. But $int(A \cup B) = int(\mathbb{R} - \mathbb{Q}) = \mathbb{R} - \mathbb{Q}$. So $[int(A \cup B)]^\star = (\mathbb{R} - \mathbb{Q})^\star = cl(\mathbb{R} - \mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{R} - \mathbb{Q} = A \cup B$. Hence $A \cup B$ is not mildly $\mathcal{I}_g$-ω-closed.*

THEOREM 4.3. *In an ideal topological space $(X, \tau, \mathcal{I})$, a subset G is mildly $\mathcal{I}_g$-ω-open if and only if $G \subseteq int^{\star}(cl(G))$.*

PROOF. G is mildly $\mathcal{I}_g$-ω-open $\Leftrightarrow X\backslash G$ is mildly $\mathcal{I}_g$-ω-closed $\Leftrightarrow X\backslash G$ is $pre^{\star}_{\mathcal{I}}$-closed by (5) of Theorem 3.7 $\Leftrightarrow G$ is $pre^{\star}_{\mathcal{I}}$-open $\Leftrightarrow G \subseteq int^{\star}(cl(G))$. □

THEOREM 4.4. *In an ideal topological space $(X, \tau, \mathcal{I})$, if the subset G is mildly $\mathcal{I}_g$-ω-closed, then $cl^{\star}(int(G))\backslash G$ is mildly $\mathcal{I}_g$-ω-open in $(X, \tau, \mathcal{I})$.*

PROOF. Since G is mildly $\mathcal{I}_g$-ω-closed, $cl^{\star}(int(G))\backslash G = \phi$ by (3) of Theorem 3.7. Thus $cl^{\star}(int(G))\backslash G$ is mildly $\mathcal{I}_g$-ω-open in $(X, \tau, \mathcal{I})$. □

THEOREM 4.5. *In an ideal topological space $(X, \tau, \mathcal{I})$, if G is mildly $\mathcal{I}_g$-ω-open, then $int^{\star}(cl(G)) \cup (X - G) = X$.*

PROOF. Since G is mildly $\mathcal{I}_g$-ω-open, $G \subseteq int^{\star}(cl(G))$ by Theorem 4.3. So $(X-G)\cup G \subseteq (X-G)\cup int^{\star}(cl(G))$ which implies $X = (X-G)\cup int^{\star}(cl(G))$. □

THEOREM 4.6. *In an ideal topological space $(X, \tau, \mathcal{I})$, if G is mildly $\mathcal{I}_g$-ω-open and H is a subset such that $int^{\star}(cl(G)) \subseteq H \subseteq G$, then H is mildly $\mathcal{I}_g$-ω-open.*

PROOF. Since G is mildly $\mathcal{I}_g$-ω-open, $G \subseteq int^{\star}(cl(G))$ by Theorem 4.3. By assumption $int^{\star}(cl(G)) \subseteq H \subseteq G$. This implies $G \subseteq int^{\star}(cl(G)) \subseteq H \subseteq G$. Thus $G = H$ and so H is mildly $\mathcal{I}_g$-ω-open. □

COROLLARY 4.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, if G is a mildly $\mathcal{I}_g$-ω-open set and a closed set, then $int^{\star}(G)$ is mildly $\mathcal{I}_g$-ω-open.*

PROOF. Let G be a mildly $\mathcal{I}_g$-ω-open set and a closed set in $(X, \tau, \mathcal{I})$. Then $int^{\star}(cl(G)) = int^{\star}(G) \subseteq int^{\star}(G) \subseteq G$. Thus, by Theorem 4.6, $int^{\star}(G)$ is mildly $\mathcal{I}_g$-ω-open in $(X, \tau, \mathcal{I})$. □

DEFINITION 4.1. *A subset H of an ideal topological space $(X, \tau, \mathcal{I})$ is called a $G_{\mathcal{I}}$-set if $H = M \cup N$ where M is $g\omega$-closed and N is $pre^{\star}_{\mathcal{I}}$-open.*

PROPOSITION 4.1. *Every $pre^{\star}_{\mathcal{I}}$-open (resp. $g\omega$-closed) set is a $G_{\mathcal{I}}$-set.*

REMARK 4.4. *The separate converses of Proposition 4.1 are not true in general as shown in the following Example.*

EXAMPLE 4.3.
(1) *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1 and $G = \mathbb{R} - \mathbb{Q}$. Since G is closed, it is $g\omega$-closed and hence a $G_{\mathcal{I}}$-set. But $int^{\star}(cl(G)) = int^{\star}(G) = \mathbb{R}\backslash cl^{\star}(\mathbb{Q}) = \mathbb{R}\backslash cl(\mathbb{Q}) = \mathbb{R}\backslash\mathbb{R} = \phi \not\supseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is not $pre^{\star}_{\mathcal{I}}$-open.*
(2) *In Example 4.1, for $G = \mathbb{R} - \mathbb{Q}$, $int^{\star}(cl(G)) = int^{\star}(\mathbb{R}) = \mathbb{R} \supseteq G$. Thus G is $pre^{\star}_{\mathcal{I}}$-open and hence a $G_{\mathcal{I}}$-set. But G is open and $G \subseteq G$, whereas $cl_{\omega}(G) = \mathbb{R} \nsubseteq G$. Hence G is not $g\omega$-closed.*

REMARK 4.5. *The following example shows that the concepts of $pre^{\star}_{\mathcal{I}}$-openness and $g\omega$-closedness are independent of each other.*

EXAMPLE 4.4. *In Example 4.3(1), $G = \mathbb{R} - \mathbb{Q}$ is $g\omega$-closed but not $pre^{\star}_{\mathcal{I}}$-open. In Example 4.3(2), $G = \mathbb{R} - \mathbb{Q}$ is $pre^{\star}_{\mathcal{I}}$-open but not $g\omega$-closed.*

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O. Nethaji: Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai District, Tamil Nadu, India.

E-mail address: jionetha@yahoo.com

I. Rajasekaran: Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai District, Tamil Nadu, India.

E-mail address: rajasekarani@yahoo.com

O. Ravi: Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai District, Tamil Nadu, India.

E-mail address: siingam@yahoo.com.