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ANNIHILATOR IDEALS IN ALMOST SEMILATTICE

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Abstract. The concept of annihilator ideal is introduced in an Almost Semilattice (ASL) L with 0. It is proved that the set of all annihilator ideals of an ASL L with 0 forms a complete Boolean algebra. The concept of annihilator preserving homomorphism is introduced in an ASL L with 0. A sufficient condition for a homomorphism to be annihilator preserving is derived. Finally, it is proved that the homomorphic image and the inverse image of an annihilator ideal are again annihilator ideals.

1. Introduction

There is only one reasonable way of defining what is to be meant by an ideal in a lattice. Recall that, Dedekind's definition of an ideal in a ring R is that it is a collection J of elements of R which (1) contains all multiples such as ax or ya of any of its elements a, and (2) contains the difference $a-b$, and hence the sum $a+b$, of any two of its elements a and b. By analogy, a collection J of elements of a lattice L is called an ideal if (1) it contains all multiples $a \cap x$ of any of its elements, and (2) it contains the lattice sum $a \cup b$ of any two of its elements a and b. The analogy is that the greatest lower bound , or lattice meet $a \cap b$ corresponds to product in a ring, and the least upper bound, or lattice join $a \cup b$ corresponds to the sum of two elements in a ring.

An Almost Semilattice (ASL) was introduced by authors as an algebra $(L, \circ)$ of type (2) which satisfies all most all the properties of semilattice except possibly the commutative of $\circ$. In this paper, the concept of annihilator ideal in an ASL with 0 is introduced with suitable examples and proved some basic properties of the annihilator ideals. Also, proved that the set $\mathcal{A}(L)$, of all annihilator ideals of an ASL L with 0 is a complete Boolean algebra.

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The concept of annihilator preserving homomorphism is introduced and a sufficient condition for a homomorphism to be annihilator preserving is derived. It is proved that the image and the inverse image under a homomorphism of an annihilator ideal are again annihilator ideals. Finally, it is proved that for any ideal I of L, there exists a homomorphism f from L into an ASL $Hom_L(I)$, of all homomorphisms defined on I such that $Ker(f) = I^{\star}$.

2. Preliminary

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. [**4**] *A semilattice is an algebra $(S, \star)$ where S is nonempty set and $\star$ is a binary operation on S satisfying:*

(1) $x \star (y \star z) = (x \star y) \star z$

(2) $x \star y = y \star x$

(3) $x \star x = x$ *, for all* $x, y, z \in S$.

In other words, a semilattice is an idempotent commutative semigroup. The symbol $\star$ can be replaced by any binary operation symbol, and in fact we use one of the symbols of $\wedge$, $\vee$, $+$ or $.$, depending on the setting. The most natural example of a semilattice is $(\mathcal{P}(X), \cap)$, or more generally any collection of subsets of X closed under intersection. A sub semilattice of a semilattice $(S, \star)$ is a subset of S which is closed under the operation $\star$. A homomorphism between two semilattices $(S, \star)$ and $(T, \star)$ is a map $h : S \to T$ with the property that $h(x \star y) = h(x) \star h(y)$ far all $x, y \in S$. An isomorphism between two semilattices is a homomorphism that is $1 - 1$ and onto. It is worth nothing that, because the operation is determined by the order and vice versa. Also, it can be easily observed that two semilattices are isomorphic if and only if they are isomorphic as ordered sets.

Definition 2.2. [**5**] *An algebra $(L, \circ)$ of type (2) is called an almost semilattice if it satisfies the following axioms:*

(AS_1) $(x \circ y) \circ z = x \circ (y \circ z)$	*(associative law)*
(AS_2) $(x \circ y) \circ z = (y \circ x) \circ z$	*(almost commutative law)*
(AS_3) $x \circ x = x$	*(idempotent law)*

If L has an element 0 and satisfies $0 \circ x = 0$ along with the above properties, then L is called an ASL with 0.

Theorem 2.1. [**5**] *Let L be an ASL. Define $a \leqslant b$ if and only if $a \circ b = a$ for all $a, b \in L$. Then $\leqslant$ is a partial ordering on L.*

Lemma 2.1. [**5**] *Let L be an ASL with 0. Then we have the following properties:*

(a) $a \circ (a \circ b) = a \circ b$

(b) $(a \circ b) \circ b = a \circ b$

(c) $b \circ (a \circ b) = a \circ b$

(d) $a \circ b = b \circ a$ *whenever* $a \leqslant b$

(e) a *is a minimal element of* L *if and only if* $x \circ a = a$ *for all* $x \in L$

(f) $a \circ 0 = 0$ *for all* $a \in L$

(g) $a \circ b = 0$ *if and only if* $b \circ a = 0$

(h) $a \leqslant b$ *implies that* $a \circ x \leqslant b \circ x$ *and* $x \circ a \leqslant x \circ b$.

Definition 2.3. [**6**] *A nonempty subset* I *of an ASL* L *is said to be an ideal if* $x \in I$ *and* $a \in L$*, then* $x \circ a \in I$.

Definition 2.4. [**6**] *A proper ideal* P *of an almost semilattice* L *is said to be prime ideal if for any* $a, b \in L$ *such that* $a \circ b \in P$*, then either* $a \in P$ *or* $b \in P$.

Theorem 2.2. [**6**] *Let* S *be a nonempty subset of an ASL* L*. Then* $(S] = \{(\circ_{i=1}^{n} s_i) \circ x |\ x \in L, s_i \in S$ *where* $1 \leqslant i \leqslant n$ *and* n *is a positive integer*$\}$ *is the smallest ideal of* L *containing* S.

Corollary 2.1. [**6**] *Let* L *be an ASL and* $a \in L$*. Then* $(a] = \{a \circ x |\ x \in L\}$ *is an ideal of* L*, and is called principal ideal generated by* a.

Lemma 2.2. [**6**] *For any* a, b *in an ASL* L*, we have the following:*

(a) $a \in (b]$ *if and only if* $a = b \circ a$.

(b) $b \in (a]$ *if and only if* $(b] \subseteq (a]$

(c) $(a] \subseteq (b]$ *whenever* $a \leqslant b$

(d) $(b \circ a] = (a \circ b] = (a] \cap (b]$.

Corollary 2.2. [**6**] *Let* I *be an ideal of* L*. Then, for any* $a, b \in L, a \circ b \in I$ *if and only if* $b \circ a \in I$.

Definition 2.5. [**5**] *Let* L *be a nonempty set. Define a binary operation* $\circ$ *on* L *by* $x \circ y = y$*, for all* $x, y \in L$*. Then* $(L, \circ)$ *is an ASL and is called discrete ASL.*

Definition 2.6. [**5**] *An element* $m \in L$ *is said to be unimaximal if* $m \circ x = x$ *for all* $x \in L$.

Theorem 2.3. [**5**] *Every unimaximal element in an ASL* L *is a maximal element.*

Theorem 2.4. [**6**] *The set* $\mathfrak{I}(L)$*, of all ideals of an ASL* L *is a distributive lattice with respect to set inclusion, where for any* $I, J \in \mathfrak{I}(L), I \wedge J = I \cap J$ *and* $I \vee J = I \cup J$.

Theorem 2.5. *Let* $(P, \leqslant)$ *be a poset which is bounded above. If every nonempty subset of* P *has glb, then every nonempty subset of* P *has lub and hence* P *is a complete lattice.*

Definition 2.7. *A complemented distributive lattice is called a Boolean Algebra.*

3. Annihilator Ideals

In this section, we introduce the concept of an annihilator ideal in an almost semilattice (ASL) L with 0 and prove some basic properties of the annihilator ideals. Also, we prove that the set $\mathcal{A}(L)$, of all annihilator ideals form a complete Boolean algebra. First we begin with the following definition.

Throughout the remaining of this section, by L we mean an ASL with 0 unless otherwise specified.

DEFINITION 3.1. *For any nonempty subset A of an ASL L with 0, define $A^\star = \{x \in L \mid x \circ a = 0, for\ all\ a \in A\}$. Then $A^\star$ is called the annihilator of A.*

Note that, if $A = \{a\}$, then we denote $A^\star = \{a\}^\star$ by $[a]^\star$. In the following we prove that for any nonempty subset A of L, $A^\star$ is an ideal.

THEOREM 3.1. *For any nonempty subset A of L, $A^\star$ is an ideal of L.*

PROOF. Since $a \circ 0 = 0$ for all $a \in A$, $0 \in A^\star$. Hence $A^\star$ is nonempty. Let $x \in A^\star$ and $t \in L$. Then $x \circ a = 0$ for all $a \in A$. Now, let $b \in A$. Then $(x \circ t) \circ b = (t \circ x) \circ b = t \circ (x \circ b) = t \circ 0 = 0$. Therefore $x \circ t \in A^\star$. Thus $A^\star$ is an ideal of L. □

Recall that, every ideal is an initial segment. It follows that for any subset A of L, $A^\star$ is an initial segment of L. In the following we prove some properties of annihilator ideals.

LEMMA 3.1. *For any subset A of L, $A \cap A^\star = \{0\}$.*

PROOF. Suppose A is a subset of L and suppose $x \in A \cap A^\star$. Then $x \in A$ and $x \circ a = 0$, for all $a \in A$. It follows that $x = x \circ x = 0$. Therefore $A \cap A^\star = \{0\}$. □

THEOREM 3.2. *For any ideals I, J of L, we have the following:*
(1) $I^\star = \bigcap_{a \in I} (a]^\star$
(2) $(I \cap J)^\star = (J \cap I)^\star$
(3) $I \subseteq J \implies J^\star \subseteq I^\star$
(4) $I^\star \cap J^\star \subseteq (I \cap J)^\star$
(5) $(I \cap J)^{\star\star} = I^{\star\star} \cap J^{\star\star}$
(6) $I \subseteq I^{\star\star}$
(7) $I^{\star\star\star} = I^\star$
(8) $I^\star \subseteq J^\star \iff J^{\star\star} \subseteq I^{\star\star}$
(9) $I \cap J = (0] \iff I \subseteq J^\star \iff J \subseteq I^\star$
(10) $(I \cup J)^\star = I^\star \cap J^\star$.

PROOF. (1) Let $t \in I^\star$. Then $t \circ a = 0$ for all $a \in I$. Hence $t \in (a]^\star$ for all $a \in I$. Therefore $t \in \bigcap_{a \in I} (a]^\star$. Thus $I^\star \subseteq \bigcap_{a \in I} (a]^\star$. Clearly, $\bigcap_{a \in I} (a]^\star \subseteq I^\star$. Hence $I^\star = \bigcap_{a \in I} (a]^\star$.

(2) Since $I \cap J = J \cap I$, proof (2) is clear.

(3) Suppose $I \subseteq J$ and $x \in J^\star$. Then $x \circ a = 0$, for all $a \in J$. Hence $x \circ a = 0$ for all $a \in I$. Thus $x \in I^\star$, and hence $J^\star \subseteq I^\star$.

(4) Since $I \cap J \subseteq I, J$, by (3), we get $I^\star, J^\star \subseteq (I \cap J)^\star$. Therefore $I^\star \cap J^\star \subseteq (I \cap J)^\star$.

(5) Let $I, J \in \mathfrak{I}(L)$. Then we have $I \cap J \subseteq I, J$. Hence by (3), we get $I^\star, J^\star \subseteq (I \cap J)^\star$. It follows that $(I \cap J)^{\star\star} \subseteq I^{\star\star}, J^{\star\star}$. Thus $(I \cap J)^{\star\star} \subseteq I^{\star\star} \cap J^{\star\star}$. Conversely, let $x \in I^{\star\star} \cap J^{\star\star}$ and $y \in (I \cap J)^\star$. Then for any $i \in I$ and $j \in J$, we have $i \circ j \in I \cap J$. Hence $(y \circ i) \circ j = y \circ (i \circ j) = 0$. Therefore $y \circ i \in J^\star$. Again, since $x \in J^{\star\star}$ and $y \circ i \in J^\star$, we get $(x \circ y) \circ i = x \circ (y \circ i) = 0$. Hence $x \circ y \in I^\star$. Since $x \in I^{\star\star}$, we get $x \circ y \in I^{\star\star}$. Thus $x \circ y \in I^\star \cap I^{\star\star} = \{0\}$. Hence $x \circ y = 0$. Therefore $x \in (I \cap J)^{\star\star}$. Thus $I^{\star\star} \cap J^{\star\star} \subseteq (I \cap J)^{\star\star}$. Hence $(I \cap J)^{\star\star} = I^{\star\star} \cap J^{\star\star}$.

(6) Suppose $x \in I$ and $y \in I^\star$. Then $y \circ a = 0$ for all $a \in I$. In particular, $y \circ x = 0$. Hence $x \in I^{\star\star}$. Thus $I \subseteq I^{\star\star}$.

(7) Suppose $x \in I^\star$ and $a \in I^{\star\star}$. Then $x \circ a \in I^\star \cap I^{\star\star} = \{0\}$. Hence $x \circ a = 0$. Therefore $x \in I^{\star\star\star}$ and hence $I^\star \subseteq I^{\star\star\star}$. Converse follows by (3) and (6). Hence $I^\star = I^{\star\star\star}$.

(8) Its proof follows by (3) and (7).

(9) Suppose $I \cap J = (0]$. Let $x \in I$ and $a \in J$. Then we get $x \circ a \in I$ and $x \circ a \in J$. Hence $x \circ a \in I \cap J = (0]$. Therefore $x \circ a = 0$. It follows that $x \in J^\star$. Thus $I \subseteq J^\star$. Conversely, suppose $I \subseteq J^\star$. Let $x \in I \cap J$. Then $x \in I$ and $x \in J$. Since $I \subseteq J^\star$, $x \in J^\star$. It follows that $x \circ x = 0$. Therefore $x = 0$. Thus $I \cap J = (0]$. Similarly we can prove that $I \cap J = (0]$ if and only if $J \subseteq I^\star$.

(10) We have $I, J \subseteq I \cup J$. Therefore by (3), we get $(I \cup J)^\star \subseteq I^\star, J^\star$. Hence $(I \cup J)^\star \subseteq I^\star \cap J^\star$. Conversely, let $x \in I^\star \cap J^\star$. Then $x \in I^\star$ and $x \in J^\star$. Hence $x \circ a = 0$, for all $a \in I$ and $x \circ b = 0$, for all $b \in J$. Therefore $x \circ t = 0$, for all $t \in I \cup J$ and hence $x \in (I \cup J)^\star$. Thus $I^\star \cap J^\star \subseteq (I \cup J)^\star$. Therefore $I^\star \cap J^\star = (I \cup J)^\star$. □

Corollary 3.1. *If $\{I_i \mid i \in \Delta\}$ is a family of ideals of L, then*

$$(\bigcap_{i \in \Delta} I_i)^{\star\star} = \bigcap_{i \in \Delta} (I_i)^{\star\star}.$$

Theorem 3.3. *For any $x, y \in L$, we have the following:*

(1) $x \leqslant y \implies [y)^\star \subseteq [x)^\star$

(2) $[x)^\star \subseteq [y)^\star \implies [y)^{\star\star} \subseteq [x)^{\star\star}$

(3) $x \in [x)^{\star\star}$

(4) $(x]^\star = [x)^\star$

(5) $(x] \cap [x)^\star = \{0\}$

(6) $[x \circ y)^\star = [y \circ x)^\star$

(7) $[x)^\star \cap [y)^\star \subseteq [x \circ y)^\star$

(8) $[x \circ y)^{\star\star} = [x)^{\star\star} \cap [y)^{\star\star}$

(9) $[x)^{\star\star\star} = [x)^\star$

(10) $[x]^\star \subseteq [y]^\star$ *if and only if* $[y]^{\star\star} \subseteq [x]^{\star\star}$.

Proof. (1) Suppose $x, y \in L$ such that $x \leqslant y$ and $t \in [y]^\star$. Then $t \circ y = 0$. Since $x \leqslant y$, we get $t \circ x \leqslant t \circ y = 0$. Therefore $t \circ x = 0$ and hence $t \in [x]^\star$. Thus $[y]^\star \subseteq [x]^\star$.

(2) Suppose $[x]^\star \subseteq [y]^\star$. Let $s \in [y]^{\star\star}$. Then $s \circ a = 0$ for all $a \in [y]^\star$. Hence $s \circ a = 0$ for all $a \in [x]^\star$. Therefore $s \in [x]^{\star\star}$. Thus $[y]^{\star\star} \subseteq [x]^{\star\star}$.

(3) Let $t \in [x]^\star$. Then $t \circ x = 0$. Thus $x \in [x]^{\star\star}$.

(4) Let $t \in (x]^\star$. Then $t \circ s = 0$ for all $s \in (x]$. In particular $t \circ x = 0$, since $x \in (x]$. Hence $t \in [x]^\star$. Therefore $(x]^\star \subseteq [x]^\star$. Conversely, suppose $t \in [x]^\star$. Then $t \circ x = 0$. Let $s \in (x]$. Then $x \circ s = s$. Now, $t \circ s = t \circ (x \circ s) = (t \circ x) \circ s = 0 \circ s = 0$. Hence $t \circ s = 0$ for all $s \in (x]$. Therefore $t \in (x]^\star$. Hence $[x]^\star \subseteq (x]^\star$. Thus $[x]^\star = (x]^\star$.

(5) Let $t \in (x] \cap [x]^\star$. Then $t \in (x]$ and $t \in [x]^\star$. Hence $x \circ t = t$ and $t \circ x = 0$. Hence $t = 0$. Thus $(x] \cap [x]^\star = \{0\}$.

(6) We have $x \circ y = 0$ if and only if $y \circ x = 0$. It follows that $[x \circ y]^\star = [y \circ x]^\star$.

(7) Suppose $t \in [x]^\star \cap [y]^\star$. Then $t \in [x]^\star$ and $t \in [y]^\star$. Therefore $t \circ x = 0$ and $t \circ y = 0$ and hence $t \circ x \circ y = 0$. It follows that $t \in [x \circ y]^\star$. Thus $[x]^\star \cap [y]^\star \subseteq [x \circ y]^\star$.

(8) Let $x, y \in L$. Then we have $x \circ y \leqslant y$ and $y \circ x \leqslant x$. Therefore by (1), we get $[y]^\star \subseteq [x \circ y]^\star$ and $[x]^\star \subseteq [y \circ x]^\star = [x \circ y]^\star$. Hence $[x]^\star, [y]^\star \subseteq [x \circ y]^\star$. Therefore by (2), $[x \circ y]^{\star\star} \subseteq [x]^{\star\star}, [y]^{\star\star}$, and hence $[x \circ y]^{\star\star} \subseteq [x]^{\star\star} \cap [y]^{\star\star}$. Conversely, suppose $t \in [x]^{\star\star} \cap [y]^{\star\star}$ and $s \in [x \circ y]^\star$. Then $t \in [x]^{\star\star}$, $t \in [y]^{\star\star}$ and $s \circ (x \circ y) = 0$. It follows that $s \circ x \in [y]^\star$. Since $t \in [y]^{\star\star}$, we get $t \circ (s \circ x) = 0$. It follows that $t \circ s \in [x]^\star$. Now, $t \circ s = (t \circ t) \circ s = t \circ (t \circ s) = 0$, since $t \in [x]^{\star\star}$. Therefore $t \in [x \circ y]^{\star\star}$. Hence $[x]^{\star\star} \cap [y]^{\star\star} \subseteq [x \circ y]^{\star\star}$. Thus $[x \circ y]^{\star\star} = [x]^{\star\star} \cap [y]^{\star\star}$.

(9) Let $t \in [x]^\star$ and $s \in [x]^{\star\star}$. Then $t \circ s = [x]^\star \cap [x]^{\star\star} = \{0\}$. Hence $t \in [x]^{\star\star\star}$. Therefore $[x]^\star \subseteq [x]^{\star\star\star}$. But by (3), we get $[x]^{\star\star\star} \subseteq [x]^\star$. Thus $[x]^\star = [x]^{\star\star\star}$.

(10) Proof (10), follows by conditions (2) and (9). □

Recall that M_o is the least element in the distributive lattice $\mathfrak{I}(L)$ of all ideals of L which contains precisely all minimal elements in L. In the following, we define annihilator of a nonempty set in another form.

Definition 3.2. *For any nonempty subset* S *of* L*, define* $[S]^\star = \{x \in L | x \circ s \in M_o, for\ all\ s \in S\}$.

It can be easily seen that, if x is a minimal element, then $x \circ a$ is also a minimal element for all $a \in L$, and hence we have the following theorem.

Theorem 3.4. *Let* L *be an ASL with a minimal element. Then, for any non empty subset* S *of* L*,* $[S]^\star$ *is an ideal of* L*.*

Proof. Suppose L has a minimal element. Then clearly $[S]^\star$ is nonempty. Let $x \in [S]^\star$ and $t \in L$. Then $x \circ s \in M_o$ for all $s \in S$. Let $s \in S$. Now, consider $(x \circ t) \circ s = (t \circ x) \circ s = t \circ (x \circ s) = x \circ s$, since $x \circ s \in M_o$ which is minimal. Hence $(x \circ t) \circ s \in M_o$. Therefore $x \circ t \in [S]^\star$. Thus $[S]^\star$ is an ideal of L. □

If L is an ASL with 0, then it can be easily observed that $S^\star = [S]^\star$ and $L^\star = M_o$. Now, we have the following corollary whose proof is straight forward.

COROLLARY 3.2. *For any nonempty set S of L, $[S]^\star = [(S]]^\star$ where $(S]$ is an ideal generated by S.*

COROLLARY 3.3. *Let L be an ASL with a minimal element. Then for any nonempty subset S of L, $(S] \cap [S]^\star = M_o$.*

PROOF. We have M_o is the least element in $\mathfrak{I}(L)$. Therefore $M_o \subseteq (S] \cap [S]^\star$. Conversely, let $t \in (S] \cap [S]^\star$. Then $t \in (S]$ and $t \in [S]^\star = [(S]]^\star$. Thus $t \circ s \in M_o$ for all $s \in (S]$. In particular $t = t \circ t \in M_o$, since $t \in (S]$. Therefore $t \in M_o$ and hence $(S] \cap [S]^\star \subseteq M_o$. Thus $M_o = (S] \cap [S]^\star$. □

Now, we define the concept of annihilator ideal in an ASL L with 0.

DEFINITION 3.3. *Let L be an ASL with 0. An ideal I of L is called an annihilator ideal if $I = S^\star$ for some nonempty subset S of L.*

It can be easily seen that if I is annihilator ideal, then $I = I^{\star\star}$. Note that, the set of all annihilator ideals of L is denoted by $\mathcal{A}(L)$. In the following, we give some examples of annihilator ideals.

EXAMPLE 3.1. *Let X be a discrete ASL with 0 and with at least two elements, other than 0. Then $(X^n, \circ, 0^{'})$ is an ASL with zero $0^{'} = (0, 0, ..., 0)$, where $\circ$ defined coordinate-wise. Put, $I = \{(0, ..., a_i, ..., 0) \mid a_i \in X\}$. Then clearly I is an ideal of X^n. Also, clearly $I^\star = \{(a_1, a_2, ..., a_{i-1}, 0, a_{i+1}, ..., a_n) \mid a_i \in X\}$ and $I^{\star\star} = \{(0, ..., a_i, ..., 0) \mid a_i \in X\} = I$. Hence I is an annihilator ideal of L.*

EXAMPLE 3.2. *Let $(R, +, ., 0)$ be a commutative ring with unity. For any $a \in L$, let a^0 be the unique idempotent element in L such that $aR = a^0R$. For any $x, y \in R$, define $x \circ y = x^0 y$. Then clearly $(R, +, ., 0)$ is an ASL with 0. Now, consider $I = (x^0]$ and $J = (1 - x^0]$. Since $x^0 \circ (1 - x^0) = 0$, we get that $(x^0] \subseteq (1 - x^0]^\star$ and $(1 - x^0] \subseteq (x^0]^\star$. Now, $a \in (x^0]^\star$ implies $a \circ x^0 = 0$. So $a^0 x^0 = 0$. Now, $a(1 - x^0) = a - ax^0 = a - 0 = a$. Hence $a \in (1 - x^0]$. Thus $(x^0]^\star \subseteq (1 - x^0] = J$. Similarly we can obtain $J^\star = (1 - x^0]^\star = (x^0] = I$. Hence I and J are the annihilator ideals in L.*

EXAMPLE 3.3. *Let $L = \{0, a, b, c\}$ and defined $\circ$ on L as follows:*

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then clearly $(L, \circ, 0)$ is an ASL with 0. Consider the set $I = \{0, a\} \subseteq L$. Then clearly I is an ideal in L. Now, $I^\star = \{0, c\}$ and also $I^{\star\star} = \{0, a\} = I$. Thus I is an annihilator ideal in L. Similarly, the ideal $J = \{0, c\}$ of L, is another annihilator ideal in L.

Example 3.4. *Let* $L = \{0, a, b, c\}$ *and define* $\circ$ *on* L *as follows:*

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly $(L, \circ, 0)$ *is an ASL with* 0. *Consider the ideal* $I = \{0, c\}$. *Then* $I^{\star\star} = (0]^{\star} = L$. *Therefore* I *is not an annihilator ideal in* L.

In the following, we prove some properties of annihilator ideals.

Theorem 3.5. *For* $I, J \in \mathcal{A}(L)$, *we have* $I \cap J = (I^{\star} \cup J^{\star})^{\star}$.

Proof. Since $I^{\star}, J^{\star} \subseteq I^{\star} \cup J^{\star}$, we get $(I^{\star} \cup J^{\star})^{\star} \subseteq I^{\star\star}, J^{\star\star}$. Hence $(I^{\star} \cup J^{\star})^{\star} \subseteq I, J$. Therefore $(I^{\star} \cup J^{\star})^{\star} \subseteq I \cap J$. Conversely, suppose $x \in I \cap J$ and $y \in I^{\star} \cup J^{\star}$. Then $y \in I^{\star}$ or $y \in J^{\star}$. Since $x \in I \cap J$ and $y \in I^{\star} \cup J^{\star}$, $x \circ y \in I \cap I^{\star}$ or $x \circ y \in J \cap J^{\star}$. It follows that $x \circ y = 0$. Therefore $x \in (I^{\star} \cup J^{\star})^{\star}$ and hence $I \cap J \subseteq (I^{\star} \cup J^{\star})^{\star}$. Thus $I \cap J = (I^{\star} \cup J^{\star})^{\star}$ □

Recall that, for any ideal I in L $I^e = \{(a] \mid a \in I\}$ is an ideal of an ASL $P\mathfrak{I}(L)$ of all principal ideal in L. Now, we prove the following theorem which express the relation between ideals of L and ideals of $P\mathfrak{I}(L)$.

Theorem 3.6. *Let* L *be an ASL with* 0. *Then* I *is an annihilator ideal in* L *if and only if* I^e *is an annihilator ideal in* $P\mathfrak{I}(L)$.

Proof. Suppose I is an annihilator ideal in L. Since I^e is an ideal, by *theorem* 3.4(6) we have $I^e \subseteq I^{e\star\star}$. Let $(a] \in I^{e\star\star}$ and $b \in I^{\star}$. Then for any $c \in I$, $(b] \cap (c] = (b \circ c] = (0]$. Hence $(b] \in I^{e\star}$. Since $(a] \in I^{e\star\star}$, we get $(a] \cap (b] = (0]$. Therefore $(a \circ b] = (0]$. Which implies that $a \circ b = 0$. Hence $a \in I^{\star\star} = I$. It follows that $(a] \in I^e$. Therefore $I^{e\star\star} \subseteq I^e$. Hence $I^e = I^{e\star\star}$. Thus I^e is an annihilator ideal of $P\mathfrak{I}(L)$. Conversely, suppose I^e is an annihilator ideal in $P\mathfrak{I}(L)$. We have always $I \subseteq I^{\star\star}$. Let $a \in I^{\star\star}$ and $(b] \in I^{e\star}$. Now, for any $c \in I$, $(c] \in I^e$. Hence $(b] \cap (c] = (0]$. Therefore $(b \circ c] = (0]$. Which implies that $b \circ c = 0$. Therefore $b \in I^{\star}$. Now, $a \in I^{\star\star}$ and $b \in I^{\star}$ and hence $a \circ b = 0$. Therefore $(a] \cap (b] = (a \circ b] = (0]$. It follows that $(a] \in I^{e\star\star} = I^e$. Thus $a \in I$. Hence $I^{\star\star} \subseteq I$. We get that $I = I^{\star\star}$. Therefore I is an annihilator ideal in L. □

If L is an ASL, then we know that $(\mathfrak{I}(L), \cap, \cup)$ is a distributive lattice. But $\mathcal{A}(L)$ is not a sub-lattice $\mathfrak{I}(L)$. For, in *example* 3.3, consider the ideals $I = \{0, a\}$ and $J = \{0, c\}$. Now, $I^{\star} = \{0, c\} = J$ and $J^{\star} = \{0, a\} = I$. Hence $I^{\star\star} = \{0, a\} = I$ and $J^{\star\star} = \{0, c\} = J$. Thus I and J are both annihilator ideals in L. Now, $I \cup J = \{0, a, c\}$. So $(I \cup J)^{\star} = \{0\}$. Hence $(I \cup J)^{\star\star} = L$. Therefore $I \cup J$ is not an annihilator ideal in L. However, we prove in the following that $\mathcal{A}(L)$ is a complete Boolean algebra on its own.

Theorem 3.7. *Let* L *be an ASL with* 0. *Then the set* $\mathcal{A}(L)$, *of all annihilator ideals of* L *forms a complete Boolean Algebra, on its own.*

PROOF. Let $I, J \in \mathcal{A}(L)$. Define $I \wedge J = I \cap J$ and $I \underline{\vee} J = (I^\star \cap J^\star)^\star$. Since $I, J \in \mathcal{A}(L)$, $I^{\star\star} = I$ and $J^{\star\star} = J$. Hence $(I \cap J)^{\star\star} = I^{\star\star} \cap J^{\star\star} = I \cap J$. Thus $I \cap J \in \mathcal{A}(L)$. Also, $(I \underline{\vee} J)^{\star\star} = ((I^\star \cap J^\star)^\star)^{\star\star} = (I^\star \cap J^\star)^{\star\star\star} = (I^\star \cap J^\star)^\star = I \underline{\vee} J$. Hence $I \underline{\vee} J \in \mathcal{A}(L)$. It can be easily seen that with respect to set inclusion, $(\mathcal{A}(L), \subseteq)$ is a poset. Clearly, $I \cap J$ is the g.l.b of I, J. Now, we have $I, J \subseteq I \cup J$. By *theorem* 3.2(3) and (10), we get $I^\star \cap J^\star = (I \cup J)^\star \subseteq I^\star, J^\star$ and hence $I^{\star\star}, J^{\star\star} \subseteq (I^\star \cap J^\star)^\star$. It follows that $I, J \subseteq I \underline{\vee} J$. Therefore $I \underline{\vee} J$ is an upper bound of I, J. Suppose $H \in \mathcal{A}(L)$ is an upper bound of I, J. Then $I, J \subseteq H$. By *therorem* 3.2(3), $H^\star \subseteq I^\star, J^\star$. Therefore $H^\star \subseteq I^\star \cap J^\star$ and hence $(I^\star \cap J^\star)^\star \subseteq H^{\star\star}$. Thus $I \underline{\vee} J \subseteq H$ and hence $I \underline{\vee} J$ is a l.u.b of I, J. This implies that $(\mathcal{A}(L), \wedge, \underline{\vee})$ is a lattice. Since $(0]^\star = L$ and $L^\star = (0]$, It follows that $(0], L \in \mathcal{A}(L)$. Clearly $(0]$ and L are the least and greatest elements of $\mathcal{A}(L)$. Therefore $(\mathcal{A}(L), \wedge, \underline{\vee})$ is a bounded lattice. Let $I \in \mathcal{A}(L)$. Then clearly $I^\star \in \mathcal{A}(L)$ since $I^\star = I^{\star\star\star}$. Also, $I \cap I^\star = (0]$ and $I \underline{\vee} I^\star = (I^\star \cap I^{\star\star})^\star = (I^\star \cap I)^\star = (0]^\star = L$. Thus $I^\star$ is a complement of I. Therefore $(\mathcal{A}(L), \wedge, \underline{\vee}, ^\star, (0], L)$ is a complemented lattice. Let $I, J, K \in \mathcal{A}(L)$. We shall prove that $I \underline{\vee} (J \wedge K) = (I \underline{\vee} J) \wedge (I \underline{\vee} K)$. Clearly $I \underline{\vee} (J \wedge K) \subseteq (I \underline{\vee} J) \wedge (I \underline{\vee} K)$. Now, we prove that $(I \underline{\vee} J) \wedge (I \underline{\vee} K) \subseteq I \underline{\vee} (J \wedge K)$. We first prove that $(I \underline{\vee} J) \wedge K \subseteq I \underline{\vee} (J \wedge K)$. We have $I \cap K \cap [I^\star \cap (J \cap K)^\star] = (0]$, it follows by *theorem* 3.2(9) $K \cap I^\star \cap (J \cap k)^\star \subseteq I^\star$. Similarly we can prove that $K \cap I^\star \cap (J \cap K)^\star \subseteq J^\star$. Hence $K \cap I^\star \cap (J \cap K)^\star \subseteq I^\star \cap J^\star$. Therefore $[K \cap I^\star \cap (J \cap K)^\star] \cap (I^\star \cap J^\star)^\star = (0]$. Hence $I^\star \cap (J \cap K)^\star \cap [K \cap (I^\star \cap J^\star)^\star] = (0]$. Thus $K \cap (I^\star \cap J^\star)^\star \subseteq [I^\star \cap (J \cap K)^\star]^\star$. Hence, we get $(I \underline{\vee} J) \wedge K \subseteq I \underline{\vee} (J \wedge K)$. Now, $(I \underline{\vee} J) \cap (I \underline{\vee} K) \subseteq I \underline{\vee} [J \cap (I \underline{\vee} K)] = I \underline{\vee} [(I \underline{\vee} K) \cap J] \subseteq I \underline{\vee} [I \underline{\vee} (K \cap J)]$. Thus $(\mathcal{A}(L), \wedge, \underline{\vee}, ^\star, (0], L)$ is a Boolean Algebra. By *corollary* 3.1, $\{\bigcap_{i \in \Delta} A_i\}^{\star\star} = \bigcap_{i \in \Delta} {A_i}^{\star\star} = \bigcap_{i \in \Delta} A_i$, since each $A_i \in \mathcal{A}(L)$. It follows that $(\mathcal{A}(L), \wedge, \underline{\vee}, ^\star, (0], L)$ is a complete Boolean Algebra. □

4. Annihilator Preserving Homomorphisms

In this section, we introduce the concept of annihilator preserving homomorphisms and derive a sufficient condition for a homomorphism to be annihilator preserving. We prove that the image and inverse image of annihilator ideal are again annihilator ideals. Finally, we prove that for any ideal I of L there exists a homomorphism f from L into $Hom_L(I)$ such that $Ker(f) = I^\star$.

DEFINITION 4.1. *Let L and $L^{'}$ be two ASLs with zero elements 0 and $0^{'}$ respectively. Then a mapping $f : L \to L^{'}$ is called a homomorphism if it satisfies the following:*

(1) $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in L$

(2) $f(0) = 0^{'}$.

The kernel of the homomorphism $f : L \longrightarrow L^{'}$ (both L and $L^{'}$ are $ASLs$ with 0 and $0^{'}$ respectively) is defined by $Ker(f) = \{x \in L \mid f(x) = 0^{'}\}$, and it can be easily observed that $Ker(f)$ is an ideal of L.

LEMMA 4.1. *Let L and $L^{'}$ be two ASLs with zero elements 0 and $0^{'}$ respectively and $f : L \to L^{'}$ a homomorphism. Then we have the following:*

(1) If f is onto, then for any ideal I of L, $f(I)$ is an ideal of $L^{'}$.

(2) For any ideal J of $L^{'}$, $f^{-1}(J)$ is an ideal of L containing $Ker(f)$.

PROOF. (1) Suppose f is onto and I is an ideal of L. Then clearly $f(I) = \{f(x) \mid x \in I\}$ is nonempty. Let $f(x) \in f(I)$ and $b \in L^{'}$. Since f is onto, there exists $a \in L$ such that $b = f(a)$. Now, $f(x) \circ b = f(x) \circ f(a) = f(x \circ a) \in f(I)$ since $x \circ a \in I$. Thus $f(I)$ is an ideal of $L^{'}$.

(2) Suppose J is an ideal of $L^{'}$. We have $f^{-1}(J) = \{x \in L \mid f(x) \in J\}$. Since J is an ideal of $L^{'}$, $0^{'} = f(0) \in J$. Hence $f^{-1}(J)$ is nonempty. Let $x \in f^{-1}(J)$ and $a \in L$. Then $f(x) \in J$ and $f(a) \in f(L) \subseteq L^{'}$. Therefore $f(x \circ a) = f(x) \circ f(a) \in J$. Hence $x \circ a \in f^{-1}(J)$. Thus $f^{-1}(J)$ is an ideal of L. Let $x \in Ker(f)$. Then $f(x) = 0 \in J$. Hence $x \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is an ideal of L containing $Ker(f)$. □

THEOREM 4.1. *Let $L, L^{'}$ be an ASLs with 0 and $f : L \to L^{'}$ be a homomorphism. Then for any nonempty subset A of L, we have $f(A^{\star}) \subseteq (f(A))^{\star}$.*

PROOF. Let $a \in f(A^{\star})$ and $y \in f(A)$. Then there exists $b \in A^{\star}$ and $x \in A$ such that $a = f(b)$ and $y = f(x)$. Now, $a \circ y = f(b) \circ f(x) = f(b \circ x) = f(0) = 0^{'}$. Therefore $a \circ y = 0^{'}$. Hence $a \in (f(A))^{\star}$. Thus $f(A^{\star}) \subseteq (f(A))^{\star}$. □

But, converse of the above theorem need not be true. For, consider the following example.

EXAMPLE 4.1. *Let $L = \{0, a, b, c\}$ be a discrete ASL. Define a mapping $f : L \to L$ by $f(x) = 0$ for all $x \in L$. Then clearly f is a homomorphism. Put $A = \{a, b\}$. Then clearly $A^{\star} = \{0\}$ and hence $f(A^{\star}) = \{0\}$. Now, we have $f(A) = \{0\}$ and hence $(f(A))^{\star} = (\{0\})^{\star} = L$. Thus $(f(A))^{\star} \nsubseteq f(A^{\star})$.*

In view of the above observation, we define the concept of annihilator preserving homomorphism.

DEFINITION 4.2. *Let $L, L^{'}$ be an ASLs with zero elements 0 and $0^{'}$ respectively and let $f : L \to L^{'}$ be a homomorphism. Then f is called annihilator preserving if $f(A^{\star}) = (f(A))^{\star}$, for any $\{0\} \subset A \subset L$.*

EXAMPLE 4.2. *Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ASLs. Write $L = A \times B = \{(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. Then $(L, \circ, \bar{0})$ is an ASL with 0 under point-wise operations, where the zero elements in L is $\bar{0} = (0, 0)$. Let $L^{'} = \{0, d, e, f\}$ be another ASL in which the operation $\circ$ is defined as follows:*

$\circ$	0	d	e	f
0	0	0	0	0
d	0	d	0	d
e	0	0	e	e
f	0	d	e	f

Now, define the mapping $f : L \to L^{'}$ *by* $f((0,0)) = 0$, $f((a,0)) = d$, $f((0,b_1)) = f((0,b_2)) = e$, $f((a,b_1)) = f((a,b_2)) = f$. *Then clearly* f *is a homomorphism from* L *onto* $L^{'}$. *It can also be verified that* f *is annihilator preserving.*

Definition 4.3. *An element* a *in an ASL* L *with* 0 *is said to be dense element if* $[a]^{\star} = \{0\}$.

It can be easily observed that every unimaximal element is dense. But, dense element need not be unimaximal. For, consider the following example.

Example 4.3. *Let* $A = \{0, a\}$ *and* $B = \{a, b_1, b_2\}$ *are two discrete ASLs. Let* $L = A \times B = \{(0,0), (0,b_1), (0,b_2), (a,0), (a,b_1), (a,b_2)\}$. *Define a binary operation* $\circ$ *on* L *under point-wise:*

$\circ$	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(0, b_1)	(0, 0)	(0, b_1)	(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)
(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)
(a, 0)	(0, 0)	(0, 0)	(0, 0)	(a, 0)	(a, 0)	(a, 0)
(a, b_1)	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)
(a, b_2)	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)

Now, let $L^{'} = \{(0,0), (0,b_1), (0,b_2), (a,b_1), (a,b_2)\}$. *Then* $L^{'}$ *is a sub ASL of* $(L, \circ, 0^{'})$. *In* $L^{'}$, $(a,b_1), (a,b_2)$ *are only unimaximal elements. Now,* $((0,b_1)]^{\star} = \{(0,0)\}$. *So that* $(0,b_1)$ *is a dense element, but not a unimaximal element in* $L^{'}$. *Because* $(0,b_1) \circ (a,b_1) = (0,b_1) \neq (a,b_1)$. *Similarly* $(0,b_2)$ *is also a dense element but not a unimaximal element.*

Definition 4.4. *An ASL* L *with* 0 *is said to be dense if* $[a]^{\star} = \{0\}$ *for all* $a(\neq 0) \in L$.

It can be easily seen that every discrete ASL is a dense ASL. More generally, we have the following theorem whose proof is straight forward.

Theorem 4.2. *Let* L *and* $L^{'}$ *be two dense ASLs. Then every homomorphism from* L *into* $L^{'}$ *is annihilator preserving.*

It can be easily seen that f is one-one implies that $Ker(f) = \{0\}$. But, converse need not be true. For, consider the following example.

Example 4.4. *Let* $L = \{0, a, b\}$ *and* $L^{'} = \{0^{'}, c\}$ *be two discrete ASLs. Define a mapping* $f : L \to L^{'}$ *by* $f(0) = 0^{'}$ *and* $f(a) = f(b) = c$. *Then clearly* f *is a homomorphism from* L *into* $L^{'}$. *Clearly* f *is onto. Also* $Kerf = \{0\}$. *But* f *is not one-one.*

In the following, we give sufficient condition for a homomorphism to become annihilator preserving.

Theorem 4.3. *Let* L *and* $L^{'}$ *be two ASLs with zero elements* 0 *and* $0^{'}$ *respectively and let* $f : L \to L^{'}$ *be a homomorphism. If* $Ker(f) = \{0\}$ *and* f *is onto, then both* f *and* f^{-1} *are annihilator preserving.*

Proof. Let A be a subset of L such that $(0] \subset A \subset L$. Then by *theorem* 4.1, $f(A^\star) \subseteq (f(A))^\star$. Now, let $x \in (f(A))^\star$. Since f is onto, there exists $y \in L$ such that $f(y) = x \in (f(A))^\star$. Hence $f(y) \circ f(a) = 0^{'}$ for all $a \in A$. This implies that $f(y \circ a) = 0^{'}$ and hence $y \circ a \in Ker(f) = \{0\}$. It follows that $y \circ a = 0$ for all $a \in A$. Therefore $y \in A^\star$. Hence $x = f(y) \in f(A^\star)$. Therefore $(f(A))^\star \subseteq f(A^\star)$. Thus $f(A^\star) = (f(A))^\star$.

Again, let $(0] \subset A \subset L^{'}$. It is enough to prove that $f^{-1}(A^\star) = (f^{-1}(A))^\star$. Let $x \in (f^{-1}(A))^\star$. Then $x \circ a = 0$ for all $a \in f^{-1}(A)$. Hence $x \circ a = 0$ for all $f(a) \in A$. It follows that $f(x) \circ f(a) = f(x \circ a) = f(0) = 0^{'}$ for all $f(a) \in A$. Therefore $f(x) \in A^\star$ and hence $x \in f^{-1}(A^\star)$. Thus $(f^{-1}(A))^\star \subseteq f^{-1}(A^\star)$. Conversely, suppose $x \in f^{-1}(A^\star)$ and $a \in f^{-1}(A)$. Then $f(x) \in A^\star$ and $f(a) \in A$. Hence $f(x \circ a) = f(x) \circ f(a) = 0^{'}$. Thus $x \circ a \in Ker(f) = \{0\}$. Therefore $x \circ a = 0$, for all $a \in f^{-1}(A)$. Hence $x \in (f^{-1}(A))^\star$. Therefore $f^{-1}(A^\star) \subseteq (f^{-1}(A))^\star$. Thus $f^{-1}(A^\star) = (f^{-1}(A))^\star$. □

Now, we prove some properties of annihilator preserving homomorphisms.

Theorem 4.4. *Let L and $L^{'}$ be two ASLs with zero elements 0 and $0^{'}$ respectively. Let $f : L \to L^{'}$ be annihilator preserving homomorphism such that $Ker(f) = \{0\}$. Then $A^\star = B^\star$ if and only if $(f(A))^\star = (f(B))^\star$ for any nonempty subsets A and B of L.*

Proof. Suppose $A^\star = B^\star$. Then clearly $f(A^\star) = f(B^\star)$. Since f is annihilator preserving, $(f(A))^\star = (f(B))^\star$. Conversely, assume that $(f(A))^\star = (f(B))^\star$. Let $t \in A^\star$. Then $t \circ a = 0$ for all $a \in A$. Hence $f(t \circ a) = f(0) = 0^{'}$. Therefore $f(t) \circ f(a) = 0^{'}$ for all $a \in A$. It follows that $f(t) \in (f(A))^\star$ and hence $f(t) \in (f(B))^\star$. Therefore $f(t) \circ f(b) = 0^{'}$ for all $b \in B$. Hence $f(t \circ b) = 0^{'}$. Therefore $t \circ b \in Ker(f) = \{0\}$ for all $b \in B$. We get $t \circ b = 0$ for all $b \in B$. Therefore $t \in B^\star$. Hence $A^\star \subseteq B^\star$. Similarly we can prove that $B^\star \subseteq A^\star$. Therefore $A^\star = B^\star$. □

Theorem 4.5. *Let L and $L^{'}$ be two ASLs with zero elements 0 and $0^{'}$ respectively and let $f : L \to L^{'}$ a homomorphism. Then we have the following:*

(a) If f is annihilator preserving and onto, then $f(I)$ is annihilator ideal of $L^{'}$ for every annihilator ideal I of L.

(b) If f^{-1} preserves annihilators, then $f^{-1}(J)$ is an annihilator ideal of L for every annihilator ideal J of $L^{'}$.

Proof. (a) Suppose f is annihilator preserving homomorphism which is onto and suppose I is an annihilator ideal of L. Then by *lemma* 4.1(1), $f(I)$ is an ideal of $L^{'}$. Since f is annihilator preserving, $(f(I))^{\star\star} = ((f(I))^\star)^\star = (f(I^\star))^\star = f(I^{\star\star}) = f(I)$. Therefore $f(I)$ is an annihilator ideal of $L^{'}$.

(b) Suppose f^{-1} preserves annihilators. Let J be an annihilator ideal of $L^{'}$. Then by *lemma* 4.1(2), $f^{-1}(J)$ is an ideal of L. Since f^{-1} preserves annihilators, we get $(f^{-1}(J))^{\star\star} = ((f^{-1}(J))^\star)^\star = (f^{-1}(J^\star))^\star = f^{-1}(J^{\star\star}) = f^{-1}(J)$. Therefore $f^{-1}(J)$ is an annihilator ideal of L. □

COROLLARY 4.1. *Let L and $L^{'}$ be two ASLs with zero elements 0 and $0^{'}$ respectively and let $f : L \to L^{'}$ be homomorphism such that f^{-1} preserves annihilators. Then $Ker(f)$ is an annihilator ideal of L.*

PROOF. Since $Ker(f) = f^{-1}((0^{'}])$ and $(0^{'}]$ is annihilator ideal of $L^{'}$,by above theorem, $Ker(f)$ is an annihilator ideal. □

Recall that if I is an ideal of an ASL L with 0, then I is a sub ASL with 0. Finally, we prove that if I is an ideal of L, then there exists a homomorphism whose kernel is the annihilator of I. First we need the following lemma.

LEMMA 4.2. *Let L be an ASL with 0 and I be an ideal of L. Then the set $Hom_L(I)$, of all endomorphisms on I is an ASL under the operation $\underline{\circ}$ defined on $Hom_L(I)$ by $(f \,\underline{\circ}\, g)(x) = f(x) \circ g(x)$ for all $x \in I$.*

PROOF. Clearly $Hom_L(I)$ is a nonempty set since the identity map on I belonging to $Hom_L(I)$. Also, clearly $Hom_L(I)$ is an ASL under the binary operation $\underline{\circ}$. Now, define $f_o : I \to I$ by $f_o(x) = 0$ for all $x \in I$. Then clearly $f_o \in Hom_L(I)$. Also, for any $f \in Hom_L(I)$ and $x \in I$, consider, $(f_o \,\underline{\circ}\, f)(x) = f_o(x) \circ f(x) = 0 \circ f(x) = 0 = f_o(x)$. Therefore $f_o \,\underline{\circ}\, f = f_o$. Hence f_o is the zero element in $Hom_L(I)$. Thus $Hom_L(I)$ is an ASL with zero element f_o. □

THEOREM 4.6. *Let L be an ASL with 0. Then for any ideal I of L there exists a homomorphism f from L to $Hom_L(I)$ such that $Ker(f) = I^{\star}$.*

PROOF. Let I be an ideal of L. Now, fix $r \in L$ and define $\theta_r : I \to I$ by $\theta_r(x) = x \circ r$ for all $x \in I$. We shall prove that $\theta_r \in Hom_L(I)$. Since I is an ideal of L, we get $\theta_r(x) = x \circ r \in I$. Therefore θ_r is well-defined. Let $x, y \in I$. Then $\theta_r(x \circ y) = (x \circ y) \circ r = (x \circ y) \circ (r \circ r) = x \circ (y \circ (r \circ r)) = x \circ ((y \circ r) \circ r) = x \circ ((r \circ y) \circ r) = (x \circ (r \circ y)) \circ r = ((x \circ r) \circ y) \circ r = (x \circ r) \circ (y \circ r) = \theta_r(x) \circ \theta_r(y)$. Also, $\theta_r(0) = 0 \circ r = 0$. Thus θ_r is a homomorphism. Hence $\theta_r \in Hom_L(I)$. Now, define $f : L \to Hom_L(I)$ by $f(r) = \theta_r$ for all $r \in L$. Then clearly f is well-defined. Now, let $r, s \in L$. Then $f(r \circ s) = \theta_{r \circ s}$. Now, for any $x \in I$, $\theta_{r \circ s}(x) = x \circ (r \circ s) = (x \circ x) \circ (r \circ s) = x \circ (x \circ (r \circ s)) = x \circ ((x \circ r) \circ s) = x \circ ((r \circ x) \circ s) = x \circ (r \circ (x \circ s)) = (x \circ r) \circ (x \circ s) = \theta_r(x) \circ \theta_s(x) = (\theta_r \circ \theta_s)(x)$. Therefore $\theta_{r \circ s} = \theta_r \circ \theta_s$. Thus $f(r \circ s) = f(r) \circ f(s)$. Also, $f(0) = \theta_0$. Now, $\theta_0(x) = x \circ 0 = 0 = f_o(x)$ for all $x \in I$. Therefore $\theta_0 = f_o$. Thus f is a homomorphism. Hence $Ker(f)$ is an ideal of L. We now prove that $Ker f = I^{\star}$. Consider,

$r \in Ker(f) \iff f(r) = \theta_0$, which is the zero element of $Hom_L(I)$.

$\iff \theta_r = \theta_0$

$\iff \theta_r(x) = \theta_0(x)$ for all $x \in I$

$\iff x \circ r = \theta_0(x) = 0$ for all $x \in I$

$\iff r \in I^{\star}$.

Therefore $Ker(f) = I^{\star}$. □

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