Extension of Fourier-Stieltjes Transform

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Abstract— A linear phase shift introduced in time domain signals results in a frequency domain is a Modulation theorem of integral transform and the Parseval's theorem is the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform. The properties of the Fourier transform provided valuable insight into how signal operations in the time-domain are described in the frequency-domain.

In the present work, Fourier-Stieltjes Transform is extended in distributional generalized sense. Adjoint operators for the distributional Fourier-Stieltjes Transform is obtained. Also,Modulation and Parseval's Theorem for the Fourier-Stieltjes Transform are proved.

Keywords— Fourier Transform, Steiltjes Transform, Inversion Formula of Fourier-Stieltjes Transform, Fourier-Steiltjes Transform, Generalized function.

INTRODUCTION

Mathematically, the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform is a Parseval's Identity(Parseval's Theorem). This theorem originates from series by Marc-Antoine Parseval's in 1799, which was later applied to the Fourier series. Although the term "Parseval's theorem" is often used to describe the unitarily of any Fourier transform, especially in Physics and Engineering, the most general form of this property is more properly called the Plancherel theorem.

Modulation is an important property for Transform that $\mathcal{F}[\cos(2\pi k_0 x) f(x)](k)$ can be expressed in terms of $\mathcal{F}[f(x)] = F(k)$ as follows [5]-

$$\mathcal{F}[\cos(2\pi k_0 x) f(x)](k) = \frac{1}{2}[F(k-k_0) + F(k+k_0)].$$

In the same manner we have, $\mathcal{F}[\sin(2\pi k_0 x) f(x)](k) = \frac{1}{2i}[F(k-k_0) - F(k+k_0).$

We live in the time-domain. The integral transform converts a signal or system representation to the frequency-domain, which provides another way to visualize a signal or system convenient for analysis and design. The properties of the transform provided valuable insight into how signal operations in the time-domain are described in the frequency-domain.

Fourier transform is a technique employed to solve ODE's, PDE's, Initial value problems, boundary value problems, Integral equations, Signal processing, in computers, image processing and so on[7].

Similarly, The Stieltjes transform is also used in many areas such as continued fraction, probability and signal processing. It was first introduced by T.S. Stieltjes in connection to the moment problems for semi-infinite interval [6]. In the present work of this paper, we also defined the adjoint operators of Fourier- Stieltjes Transform.

We had already define the Conventional Fourier-Stieltjes Transform of a complex valued smooth function $f(t,x)$ is defined by convergent integral [3] as -

$$FS\{f(t,x)\} = F(s,p) = \int_0^\infty \int_0^\infty f(t,x)\ e^{-ist}\ (x+y)^{-p}\ dt\ dx\ .$$

In the same manner, we had defined the Distributional Fourier- Stieltjes Transform in [3] as-

$$FS\{f(t,x)\} = F(s,p) = \langle f(t,x), e^{-ist}(x+y)^{-p} \rangle$$

Where, FS_α^* is dual space consist of continuous linear function on FS_α and $f(t,x) \in FS_\alpha$, for some $s > 0$ and $k > Re\, p$.

From our previous work, Inversion formula for the Distributional Fourier-Stieltjes Transform is defined as follows-

$$f(t,x) = \lim_{r,q\to\infty} \frac{1}{4\pi^2} \int_{-r}^{r} \int_0^q F(s,p)\ e^{ist}(x+y)^{(p+1)}\ dp\ ds$$

Outline of this paper-

In section 1, we developed the Modulation theorem for Fourier-Stieltjes Transform. Section 2, gives the Parseval's theorem for the Fourier-Stieltjes Transform. Section 3, defines the adjoint operators for Distributional Fourier-Stieltjes Transform. The notation and terminology is given by A.H. Zemanian[2].

1. Modulation Property

i] Prove that-

$FS\{f(t,x)\cos(at+bx)\}(s,p)$

$$=\frac{1}{2}\{FS[e^{ibx}f(t,x)](s-a,p)+FS[e^{-ibx}\,f(t,x)](s+a,p)\}$$

Proof:- We have

$FS\{f(t,x)\cos(at+bx)\}(s,p)$

$$=\int_0^\infty\int_0^\infty f(t,x)\cos(at+bx)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx$$

$$=\int_0^\infty\int_0^\infty f(t,x)\left[\frac{e^{i(at+bx)}+e^{-i(at+bx)}}{2}\right]e^{-ist}\,(x+y)^{-p}\,dt\,dx$$

$$=\frac{1}{2}\int_0^\infty\int_0^\infty[f(t,x)e^{i(at+bx)}e^{-ist}(x+y)^{-p}+f(t,x)\,e^{-i(at+bx)}e^{-ist}\,(x+y)^{-p}]\,dt\,dx$$

$$=\frac{1}{2}\int_0^\infty\int_0^\infty f(t,x)e^{i(at+bx)}e^{-ist}(x+y)^{-p}\,\,dt\,dx$$

$$+\frac{1}{2}\int_0^\infty\int_0^\infty f(t,x)\,e^{-i(at+bx)}e^{-ist}\,(x+y)^{-p}\,\,dt\,dx$$

$$=\frac{1}{2}\{\int_0^\infty\int_0^\infty e^{ibx}f(t,x)e^{-i(s-a)t}(x+y)^{-p}\,\,dt\,dx$$

$$+\int_0^\infty\int_0^\infty e^{-ibx}f(t,x)\,e^{-i(s+a)t}\,(x+y)^{-p}\,\,dt\,dx\}$$

$FS\{f(t,x)\cos(at+bx)\}(s,p)$

$$=\frac{1}{2}\{FS[e^{ibx}f(t,x)](s-a\,,p)+FS[e^{-ibx}f(t,x)](s+a\,,p)\}$$

ii] Prove that-

$$FS\{f(t,x)\sin(at+bx)\}(s,p)=\frac{1}{2i}\{FS[e^{ibx}f(t,x)](s-a,p)-FS[e^{-ibx}\,f(t,x)](s+a,p)\}$$

Proof: - We have

$$FS\{f(t,x)\sin(at+bx)\}(s,p)=\int_0^\infty\int_0^\infty f(t,x)\sin(at+bx)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx$$

$$=\int_0^\infty\int_0^\infty f(t,x)\left[\frac{e^{i(at+bx)}-e^{-i(at+bx)}}{2i}\right]e^{-ist}\,(x+y)^{-p}\,dt\,dx$$

$$=\frac{1}{2i}\int_0^\infty\int_0^\infty[f(t,x)e^{i(at+bx)}e^{-ist}(x+y)^{-p}-f(t,x)\,e^{-i(at+bx)}e^{-ist}\,(x+y)^{-p}]\,dt\,dx$$

$$=\frac{1}{2i}\int_0^\infty\int_0^\infty f(t,x)e^{i(at+bx)}e^{-ist}(x+y)^{-p}\,\,dt\,dx$$

$$-\frac{1}{2i}\int_0^\infty\int_0^\infty f(t,x)\,e^{-i(at+bx)}e^{-ist}\,(x+y)^{-p}\,\,dt\,dx$$

$$=\frac{1}{2i}\{\int_0^\infty\int_0^\infty e^{ibx}f(t,x)e^{-i(s-a)t}(x+y)^{-p}\,\,dt\,dx$$

$$-\int_0^\infty \int_0^\infty e^{-ibx} f(t,x)\, e^{-i(s+a)t}\, (x+y)^{-p}\, dt\, dx\}$$

$FS\{f(t,x)\sin(at+bx)\}(s,p)$

$$= \frac{1}{2i}\{FS[e^{ibx} f(t,x)](s-a\,,p) - FS[e^{-ibx} f(t,x)](s+a\,,p)\}$$

2. Parseval's Identity for the distributional Fourier- Steiltjes Transform:

Theorem: If $FS\{f(t,x)\}(s,p) = F(s,p)$ and $FS\{g(t,x)\}(s,p) = G(s,p)$ then

i] $\int_0^\infty \int_0^\infty f(t,x)\ \overline{g(t,x)}\, dt\, dx$

$$= \frac{1}{2\pi^2}\int_0^\infty \int_0^\infty FS\{(x+y)f(t,x)\}(s,p)\ \overline{G(s,p)}\, dp\, ds$$

ii] $\int_0^\infty \int_0^\infty |f(t,x)|^2\ dt\, dx$

$$= \frac{1}{2\pi^2}\int_0^\infty \int_0^\infty |F(s,p)|^2\ dp\, ds$$

Proof:-We have

$FS\{f(t,x)\}(s,p) = F(s,p) = \int_0^\infty \int_0^\infty f(t,x) e^{-ist}\, (x+y)^{-p}\, dt\, dx$

By using Inversion formula for Fourier-Stieltjes Transform, We have-

$g(t,x) = \frac{1}{4\pi^2}\int_{-\infty}^\infty \int_0^\infty G(s,p)\, e^{ist}(x+y)^{p+1}\, dp\, ds$

$g(t,x) = \frac{1}{2\pi^2}\int_0^\infty \int_0^\infty G(s,p)\, e^{ist}(x+y)^p(x+y)dpds$

Now it's conjugate is-

$\overline{g(t,x)} = \frac{1}{2\pi^2}\int_0^\infty \int_0^\infty \overline{G(s,p)}\, e^{-ist}(x+y)^p(x+y)dpds$

Consider,

$\int_0^\infty \int_0^\infty f(t,x)\ \overline{g(t,x)}\, dt\, dx$

$$= \int_0^\infty \int_0^\infty f(t,x)\, dt\, dx \left\{\frac{1}{2\pi^2}\int_0^\infty \int_0^\infty \overline{G(s,p)}\, e^{-ist}(x+y)^p(x+y)dpds\right\}$$

$$= \frac{1}{2\pi^2}\left[\int_0^\infty \int_0^\infty f(t,x)\, dt\, dx\{\int_0^\infty \int_0^\infty \overline{G(s,p)}\, e^{-ist}(x+y)^p(x+y)dpds\}\right]$$

$$= \frac{1}{2\pi^2}\left[\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \overline{G(s,p)}\ f(t,x)\, e^{-ist}(x+y)^p(x+y)dpds\, dt\, dx\right]$$

$$= \frac{1}{2\pi^2}\left[\int_0^\infty \int_0^\infty \overline{G(s,p)}\, dpds\left\{\int_0^\infty \int_0^\infty f(t,x)\, e^{-ist}(x+y)^p \frac{(x+y)^{-p}}{(x+y)^{-p}}(x+y)dt\, dx\right\}\right]$$

$$= \frac{1}{2\pi^2}\left[\int_0^\infty \int_0^\infty \overline{G(s,p)}\, dpds\{\int_0^\infty \int_0^\infty (x+y)^{2p+1}\, f(t,x)\, e^{-ist}(x+y)^{-p}\, dt\, dx\}\right]$$

$\int_0^\infty \int_0^\infty f(t,x)\ \overline{g(t,x)}\, dt\, dx$

$$= \frac{1}{2\pi^2}\int_0^\infty \int_0^\infty FS\{(x+y)^{2p+1} f(t,x)\}(s,p)\, \overline{G(s,p)}\, dpds$$

$$= \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty (x+y)^{2p+1} \, F(s,p) \, \overline{G(s,p)} \, dpds$$

Putting-

$f(t,x) = g(t,x)$,

$F(s,p) = G(s,p)$,

$\overline{F(s,p)} = \overline{G(s,p)}$

By using the above result we have-

$$\int_0^\infty \int_0^\infty f(t,x) \, \overline{f(t,x)} \, dt \, dx$$

$$= \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty F(s,p) \, \overline{F(s,p)} \, ds \, dp$$

$$\int_0^\infty \int_0^\infty |f(t,x)|^2 \, dt \, dx$$

$$= \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty |F(s,p)|^2 \, ds \, dp$$

Hence Proved.

3. Adjoint Operators on Distributional Fourier-Stieltjes Transform

3.1 Proposition

The Adjoint shifting operator is a continuous function from FS_α^* to FS_α^* . The adjoint operator is $f(t,x) \to f(t-\tau,x)$ leads to the operation transform formula-

$$FS\{f(t-\tau,x) = e^{-is\tau} \, FS\{f(t,x)\}$$

Proof:- Consider

$$FS\{f(t-\tau,x) = \langle f(t-\tau,x), e^{-ist}(x+y)^{-p} \rangle$$

$$= \langle f(t-\tau,x), e^{-is(t+\tau)}(x+y)^{-p} \rangle$$

$$= \langle f(t,x), e^{-is\tau} \, e^{-ist}(x+y)^{-p} \rangle$$

$$= e^{-is\tau} \langle f(t,x), e^{-ist}(x+y)^{-p} \rangle$$

$$FS\{f(t-\tau,x) = e^{-is\tau} FS\{f(t,x)$$

3.2 Proposition

The adjoint differential operator $f \to D_t f$ is continuous linear mapping from the dual space FS_α^* into itself. Corresponding transform formula is

$$FS\{D_t f(t,x)\} = (is) FS\{f(t,x)\}$$

Proof:- Consider,

$$FS\{D_t f(t,x)\} = \langle D_t f(t,x), e^{-ist}(x+y)^{-p}\rangle$$

$$= \langle f(t,x), -\,D_t e^{-ist}(x+y)^{-p}\rangle$$

$$= \langle f(t,x), -(-is)e^{-ist}(x+y)^{-p}\rangle$$

$$= is\langle f(t,x), e^{-ist}(x+y)^{-p}\rangle$$

$$FS\{D_t f(t,x)\} = (is)FS\{f(t,x)\}$$

3.3 Proposition

The adjoint differential operator $f \to D_x f$ is continuous linear mapping from the dual space FS_α^* into itself. Corresponding transform formula is

$$FS\{D_x f(t,x)\} = (p)FS\{f(t,x)\}$$

Proof:- Consider,

$$FS\{D_t f(t,x)\} = \langle D_x f(t,x), e^{-ist}(x+y)^{-p}\rangle$$

$$= \langle f(t,x), -\,D_x e^{-ist}(x+y)^{-p}\rangle$$
$$= \langle f(t,x), -(-p)e^{-ist}(x+y)^{-p-1}\rangle$$
$$= p\,\langle f(t,x), e^{-ist}(x+y)^{-(p+1)}\rangle$$

$$FS\{D_t f(t,x)\} = (p)F(s,p+1)$$

3.4 Proposition

The adjoint operator $f \to \theta f$ is a continuous linear mapping of FS_α^* into itself. The adjoint operator is $f(t,x) \to e^{-i\tau t}(x+y)^{-\alpha} f(t,x)$. Corresponding operator transform formula is

$$FS\{e^{-i\tau t}(x+y)^{-\alpha} f(t,x)\} = F(s+\tau, p+1)$$

Proof: Consider,

$$FS\{e^{-i\tau t}(x+y)^{-\alpha} f(t,x)\} = \langle e^{-i\tau t}(x+y)^{-\alpha} f(t,x), e^{-ist}(x+y)^{-p}\rangle$$

$$= \langle f(t,x), e^{-i\tau t}(x+y)^{-\alpha} e^{-ist}(x+y)^{-p}\rangle$$

$$= \langle f(t,x), e^{-it(s+\tau)}(x+y)^{-(p+\alpha)}\rangle$$

$$= F(s+\tau, p+\alpha)$$

$$FS\{e^{-i\tau t}(x+y)^{-\alpha} f(t,x)\} = F(s+\tau, p+1)$$

3.5 Proposition

Noting above proposition the another adjoint operator is $f(t,x) \rightarrow (-it)^{k_1}[-\log(x+y)]^{k_2} f(t,x)$. Corresponding operator transform formula is $FS\{(-it)^{k_1}[-\log(x+y)]^{k_2} f(t,x)\} = D_s^{k_1} D_p^{k_2}\, F(s,p)$

Proof: Consider,

$$FS\{(-it)^{k_1}[-\log(x+y)]^{k_2} f(t,x)\} = \langle (-it)^{k_1}[-\log(x+y)]^{k_2} f(t,x), e^{-ist}(x+y)^{-p}\rangle$$

$$= \langle f(t,x), (-it)^{k_1}[-\log(x+y)]^{k_2} e^{-ist}(x+y)^{-p}\rangle$$

$$= \langle f(t,x), D_s^{k_1} D_p^{k_2} e^{-ist}(x+y)^{-p}\rangle$$

$$= D_s^{k_1} D_p^{k_2} \langle f(t,x), e^{-ist}(x+y)^{-p}\rangle$$

$$FS\{(-it)^{k_1}[-\log(x+y)]^{k_2} f(t,x)\} = D_s^{k_1} D_p^{k_2}\, F(s,p)$$

CONCLUSION

In the present work, Fourier-Stieltjes Transform is extended in distributional generalized sense. Adjoint operators for the distributional Fourier-Stieltjes Transform is obtained. Also,Modulation and Parseval's Theorem for the Fourier-Stieltjes Transform are proved.

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