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Conceptualizing Generalization

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Abstract

The purpose of this paper is to encourage mathematics educators to refine their notions of generalizing and generalization. The paper describes generalization and how definitions of generalization have evolved from conceptions of transfer. In particular, the paper demonstrates how definitions of transfer have shaped definitions of generalizing, and specifically how constructivist learning perspectives have influenced definitions of mathematical generalization. Furthermore, the activity of generalizing is explained in the context of algebra as a way to support early algebra learning. Lastly, the final section offers tasks as a resource for mathematics educators who aim to promote generalizing in their classrooms.

Keywords: *generalization, generalizing-activity, early algebra*
ZDM Subject Classification: ***D30, D40***

1. Introduction

Generalization is the activity of lifting and communicating reasoning to a level where the focus is no longer on a particular instance, but rather on patterns and relationships of those particular instances [41]. Generalizing is the most authentic practice of the mathematics classroom. The process of generalizing a set of particular instances, and justifying and formalizing the generalization is fundamental to mathematics. Yet, as a review of the literature will reveal, conceptions of generalization vary.

Generalizing is often described as the core of algebra [18, 43, 54]. To this end, generalization has garnered increased attention; particularly as standards and initiatives suggest that generalizing have a stronger role in elementary grade mathematics (e.g., [57, 58, 59, 60]). As the field moves forward in striving to maximize students' engagement with generalizing, mathematics educators need to refine their notions of generalizing and generalization.

Furthermore, some researchers argue that generalizing is natural and occurs without an instructor or curriculum promoting it. Children have a natural inclination to notice and discuss regularities and patterns in the number system, and that is the foundation for constructing, testing, and justifying generalizations [69]. Many traditional elementary mathematics concepts are platforms for generalizing, such as the Commutative Property of Addition—the order in which two terms are added does not affect the sum. When learning these concepts, students begin to explore generality [69]. Educators must be able to identify these opportunities and be aware of ways they can make use of students' "natural powers" to generalize [55]. Thus, mathematics educators need to familiarize themselves with the various ways generalization is defined, so that they can identify the phenomena of generalizing and support students' as they engage in generalizing activity.

Over the last twenty years definitions of generalization have broadened. In particular, researchers have shifted their conceptions of generalization from an individual cognitive action (e.g., [13, 21, 24, 31, 56]) to a situated activity, distributed across multiple tools, resources, and people [15, 16, 23, 25, 26, 27, 30, 40, 52, 70]. This paper describes generalization and how definitions of generalization have evolved from conceptions of transfer. In particular, the next section of this paper demonstrates how definitions of the act of transfer have shaped definitions of the activity of generalizing. The following two sections describe specifically how constructivist learning perspectives have influenced definitions of mathematical generalization. Furthermore, in the fourth section, the activity of generalizing is explained in the context of algebra as a way to support early algebra learning. Lastly, the fifth section offers example tasks as one practical way for mathematics educators to promote generalizing in their classrooms.

2. The Act of Transfer Versus the Activity of Generalizing

The activity of generalizing is critical to learning beyond and outside of mathematics as well as within mathematics. In fact, education is "aimed at helping students develop robust understandings that will generalize to decision making and problem solving in other situations, both inside and outside the classroom" [51, p. 431]. Likewise, transfer is a topic that often sits at the core of debates on learning; therefore, similar arguments have been made regarding transfer [16]. Hence, the ways in which mathematics education researchers conceptualize generalization stems from the ways in which researchers conceptualize transfer, and transfer and generalization are closely related.

Transfer refers to the act of applying knowledge that was learned in one situation to a different situation [50]. Although transfer is often viewed as central to learning, the term transfer has garnered negative attention in some research communities, particularly in mathematics education research. The negative connotation may stem from an extensive body of research that conceptualizes transfer as an isolated, individual cognitive action.

Furthermore, research on transfer in mathematics education has found that transferring knowledge can be counterproductive when learning mathematics. In some cases, transferring knowledge leads to automatic behavior, which lacks mathematical insight and often results in incorrect problem solving. Abramovich [1] discusses this phenomenon in a chapter on the Einstellung effect, which occurs when problem solvers demonstrate a tendency to apply previously acquired strategies, regardless of the appropriateness of the strategy, over mathematical insight. In his chapter, Abramovich compares productive and reproductive thinking, equating reproductive thinking to transfer. Using a task similar to those in section 5 of this paper, Abramovich reflects on a prior study (viz., [4]) in which the researchers observed a pre-service teacher engage in reproductive thinking by reasoning proportionally about a relationship that was not directly proportional.

In the late 1990s, various reviews and rejoinders between research communities regarding transfer from traditional cognitive learning theories and situated, sociocultural learning theories took place. Anderson, Reder, and Simon [5, 6] and Greeno [33] are two of the well-known exchanges in which authors discussed their stance on transfer. Responses regarding this particular exchange (e.g., [17]) refer to Anderson et al. [5, 6] as the first wave of cognitive learning theories and Greeno [33] as the second wave of cognitive learning theories.

From the first wave perspective, transfer is demonstrating the acquisition of decontextualized knowledge by applying knowledge in a situation other than the situation in which it was acquired [50, 51]. Thus, learning is not context-specific; rather learning depends on acquiring knowledge that corresponds to an external reality. At a theoretical level, this perspective presumes knowledge is ontologically independent of perception.

In traditional transfer studies, research on lateral transfer is prevalent. Lateral transfer studies have investigated isomorphic transfer by assessing if students apply knowledge to the same problem at different points in time (e.g., [36]), homomorphic transfer by assessing if students apply knowledge to similar problems, which are problems with the same structure and underlying question (e.g., [65]), and general lateral transfer between similar but slightly modified problems (e.g., [7, 64]). Furthermore, some researchers distinguish between near and far transfer. Near transfer occurs when knowledge is transferred between similar situations, whereas far transfer occurs when knowledge is transferred between different contexts (e.g., [7]).

Traditional transfer research has yielded a few major findings. The most prevalent finding of these studies is that practicing certain problems results in little or no transfer to similar or related problems (e.g., [37, 38, 64]). Namely, these studies show that students rarely, if ever, transfer prior knowledge. Yet, nearly all theories of learning suppose that learning involves prior knowledge. Furthermore, everyone can remember a time in which they generalized knowledge or witnessed someone else generalize knowledge. Hence, researchers have coined this finding the paradox of traditional transfer research [51]. This paradox may have contributed to the field moving away from studying transfer in the traditional sense, which had developed a negative connotation, and towards a reconceived view of transfer.

First wave transfer research was part of a stagnant period, which ended in the mid-1980s when theorists expressed dissatisfaction with traditional perspectives and the limited definition of transfer. As Lobato [51] said, researchers began questioning "the assumptions about knowing, knowers, learning, and context" undermining the traditional conceptions of transfer. This shift began an era of constructivist research along with the second wave of cognitive research (e.g., [33]), which both contributed to a reconceived understanding of transfer, more appropriately referred to as generalization in mathematics education.

3. Constructivist Conceptualizations of Generalization

Constructivist perspectives have shaped definitions of transfer, and thus have contributed to researchers' conceptualizations of generalization. As previously stated, at a theoretical level, the traditional definitions of transfer presume knowledge is ontologically independent of perception. Conversely, constructivist learning theorists account for individuals' perceptions and view individuals' cognitive processes as intentional and generative [40]. Thus, constructivists reject the traditional definition of transfer because it disregards perception. Embedded in a constructivist definition of generalization is the assumption that mathematical understanding is dependent on an individuals' perception (cf. [6]).

Building on Piaget's theory of learning, constructivist research on generalization (e.g., [21, 31, 56] defines the construct in terms of abstraction. By doing so, this perspective conceptualizes generalization as a process of ideas becoming more abstract. That is, generalization is the process of moving away from the concrete situation, or the process of abstracting what is similar and salient in the structure of objects, relationships, or operations [45]. The relationship between generalization and transfer is evident because, at a surface level, this definition aligns with a traditional definition of transfer. Bassok and Holyoak's [7] three-part study demonstrates this point.

Bassok and Holyoak [7] is representative of research conducted from the traditional perspective. In this study, the authors conducted three experiments to examine students' abilities to generalize skills obtained in practicing arithmetic-progression problems in algebra to constant-acceleration problems in physics. The researchers evaluate students' lateral transfer by measuring if knowledge can be accessed in a situation other than the one in which it was directly taught. In other words, the researchers evaluate students' ability to generalize knowledge from algebra to physics. While this example shows the relationship between transfer and generalization, a deeper understanding of generalization from the constructivist perspective will further reveal the differences between traditional transfer and generalization.

Mitchelmore and White [56] believe generalization is the process in which one identifies the essence of an idea through an interconnected web of knowledge about that idea and then creates a model that serves as the basis for the idea and related ideas. Whereas, Font and Contreras [31] explain that the product of the process of generalizing is generalization, which is no different from an individual's objectification, idealization, or abstraction. On the other hand, also from the constructivist perspective, Harel and Tall [35] define generalization with a hint of a social context, as "the process of applying a given argument to a broader context," (p. 38) while emphasizing that the process depends on the individual's current knowledge. In general, researchers from the constructivist perspective describe generalization as a process by which an individual constructs and applies a general idea. At a theoretical level, this perspective assumes knowledge is individual and cognitive.

While the traditional perspective might also describe transfer as "individual and cognitive," compared to a constructivist definition of generalization, the traditional perspective puts less emphasis on the individual. Unlike Bassok and Holyoak [7] and other traditional research, constructivist research accounts for the categories and representations of individuals' perceptions that they construct to make sense of the world. A case in point is Dubinsky's [24] description of generalization in terms of his Action-Process-Object-Schema (APOS) cycle. Based on Piaget's cognitive theory, the final stage of APOS is reflective abstraction. In this stage, the objects constructed by the learner exist only in the learner's mind. Hence, according to Dubinsky generalization is the application of an *individual's* schema to a broader context.

Both the traditional and constructivist definitions may use "individual" to exclude the social aspect of transfer. However, the constructivist perspective uses "individual" as a way to emphasize individual intention and perception, whereas the traditional approach uses "individual" to avoid the social. Constructivist definitions of generalization are compatible with situated and sociocultural theories. Although examples of theoretically dichotomous research in generalization exist (see [24] vs. [30]) most researchers do not view the constructivist and situated/sociocultural perspectives, or second wave perspectives, as mutually exclusive, and hence locate themselves somewhere on a continuum (e.g., [22, 35]).

4. Generalizing as a Situated Activity

Situated and socially oriented definitions of transfer emerged in concert with the second wave of cognitive research. By studying learning in non-traditional settings, researchers addressed assumptions about learning as an individual and cognitive act, by developing broad theories, such as sociocultural theory [71, 72], distributed cognition [39, 61], situated cognition [34, 49], and embodied cognition [32]. Since transfer is critical to learning, as researchers began broadening their perspectives, they simultaneously redefined transfer as a dynamic, situated construct, more appropriately referred to as generalization. That is, one reason for broadened definitions of generalization is that definitions of generalization are shaped by theories of learning. Generalization is fundamental to learning [16], so a reconceptualization of learning necessitates a reconceptualization of generalization.

According to Carraher and Schliemann [16], generalization is the mental construct that results from adapting, adjusting, and reorganizing a wealth of previous learning experiences. Whereas Ellis [29] defines generalization as the product or outcome of generalizing, demonstrated through activity and talk, and generalizing as the activity, tied to a specific socio-mathematical context, through which people construct generalizations. On the other hand, Jurow [40] defines generalization as the social process of identifying and claiming that some rule can be applied to multiple situations or objects, shaped by the people and the representations or tools involved.

Despite various nuances in the way that researchers define generalization and use theory, a salient point of agreement among these examples is that generalization is dynamic and shaped by social and contextual factors. Since second wave perspectives are flexible perspectives, some authors even develop their own "brands" of a theory by specifying how their definition may differ from others. For instance, Dörfler [22] describes his "brand" as a semiotic constructivist understanding of generalization because he bases his definition on the assumption that meaning is constructed from social sign systems. However, the fundamental idea that generalization is shaped by contextual factors remains.

5. The Activity of Generalizing in Practice

Contributions

Studies on generalization as a social and situated activity produce findings that are practical and applicable to the classroom context. In a rejoinder comparing the situated and traditional cognitive perspectives, Anderson et al. [6], who advocate for the traditional perspective, argue that the prevailing theory of learning will be the theory that improves education. Although the authors were implying that the traditional approach would prevail, their statement holds true. In current generalization research, the situated and sociocultural perspectives dominate because they produce practical results by documenting the ways that context (e.g., instruction, activities, curricula, discourse, etc.) influences learning.

Research on generalization has three main contributions. First, findings have informed instruction [9, 27, 28, 30, 40, 52, 53, 73]. For instance, research has identified focusing phenomena in an instructional or problem solving environment that foster generalization (e.g., [52]); instructional mechanisms that support students' engagement with generalizing and justifying as interrelated practices [27, 53]; and contextual framing [30] and linking and conjecturing as ways to support generalizing [40].

Second, findings have impacted the ways in which the field characterizes generalization (e.g., [25, 26, 29, 68]. For instance, research has categorized and described the actions that foster developing or refining generalizing actions and generalizations [25, 26, 29] and identified forms of generalization [68].

Third, findings have contributed to the field's understanding of factors shaping generalizing [16, 62, 66, 68]. For instance, research has explained the role of iconicity and contraction processes in the mathematical experience of generalizing [63], the role of gestures, speech, semiotic signs [62], and concrete instruments or environments in generalizing [16]. Additionally, several studies have investigated the relationship between generalizing and justifying (e.g., [8, 25, 26, 27, 29, 47, 48].

Most notable are the practical findings, that suggest how to or what may foster generalization in the classroom. The research listed above defines generalization in a variety of ways. However, one similarity among these studies is that generalization is conceptualized as a situated activity, distributed across multiple tools, resources, and people. Research that defines generalization in this way employs a situated perspective. As a result, this research is conducted in the context of an authentic classroom; in turn, the research findings can be applied in the context of a classroom.

Studies from the situated perspective that focus on generalization inform mathematics learning in general because understanding the context of generalizing helps researchers make sense of how students learn mathematics. Jurow's [40] analysis of students' studying guppy population growth focuses on the communication and shared understandings of students' generalizations. By examining learning as participation in a community of practice, Jurow focuses on the context of generalizing. Jurow's emphasis on context results in an interesting and unique contribution to the field. Since generalizing is a fundamental practice in mathematics, understanding how students generalize "help(s) us understand how students enter into the specialized disciplinary discourse of mathematics" [40, p. 280]. That is, understanding the context of generalizing informs researchers about how students learn mathematics in general.

Another example of a study from the situated perspective on generalization that informs mathematics learning in general is Ellis [26]. Contrasted with Jurow [40], this study focuses more on the process of generalization than the context of generalization. Specifically, Ellis [26] aims to understand how students develop increasingly sophisticated generalizations, and by doing so, she gains insight in how students develop this kind of mathematical expertise. Moreover, Ellis argues that by understanding what students generalize about, researchers will gain understanding about the aspects of the phenomena that were most important to the student, further contributing to understanding mathematics learning in general.

As researchers continue to conceptualize and explore generalization as a social and situated activity, research will become generatively practical by contributing to the growing understanding of students' generalizing *in situ* and how to study students' generalizing *in situ*. Finally, research on generalization is especially productive because it contributes to knowing how students learn the authentic thinking practices of mathematics by moving "away from the predominant preoccupation with numerical calculations", and placing the "focal emphasis on typical and important ways of mathematical thinking" [22, p. 159].

Generalizing and Algebraic Thinking

At this point, the paper has defined generalization in the context of research. The relationship between transfer research, and constructivist, situated learning theories, and generalization has been explained, and generalization and the importance of generalization have been described. What is generalizing in the context of a classroom? The remainder of this paper addresses this question. In the following section, generalization is presented as a way to engage students in algebraic reasoning.

Recently, early algebra has emerged as a developing subfield in mathematics education research. In response to high failure rates in algebra, initiatives (e.g., [57, 58, 59, 60]) and conferences (e.g., the US Department of Education *Algebra Initiative Colloquium*, 1993; the *Nature and Role of Algebra in the K-14 Curriculum Conference*, 1998; and the Mathematics Learning Committee of the National Research Council, 2001) have re-conceptualized algebra, suggesting that algebraic thinking have a stronger role in elementary grade mathematics, specifically, that algebra should be treated as a longitudinal K–12 strand of thinking. This curricular shift in elementary mathematics is referred to as early algebra.

Early algebra conceptualizes algebra as a mental activity. More commonly, algebra is condensed to one or two classes; it exists as its own entity, with no explicit interweaving with other mathematics topics. This curricular structure implies that, upon completion of the class, algebra is complete. In contrast, early algebra initiatives reframe algebra as a longitudinal strand of thinking that extends throughout grade levels and topics, and generalizing is a fundamental aspect of this longitudinal strand of thinking.

According to Kaput [41], algebraic thinking is the process of generalizing mathematical ideas from a set of particular instances, justifying those generalizations through discourse, and then expressing them in age-appropriate formal ways. Essentially, generalizing is the core of algebra [18, 43, 54].

Generalized arithmetic is building on arithmetic by recognizing and articulating mathematical structure and relationships and using insights to generalize [11, 14, 20, 42]. Historically, elementary mathematics focuses mainly on arithmetic and computational fluency. Therefore, generalized arithmetic is often outlined as one pathway into early algebra [42].

As previously noted, some researchers even argue that generalizing is a natural way of thinking for students, and suggest that the inclination to notice and discuss regularities and patterns in the number system is the foundation for constructing, testing, and justifying generalizations [69]. The field offers substantial research that describes the algebraic way of thinking in elementary grades. Research verifies the feasibility of early algebra by demonstrating students' capabilities (e.g., [55]) and by showing how proper tool use and instruction can promote young children to reason algebraically. For example, Abramovich [3] demonstrated how a graphics software enabled second-grade students to engage in informal algebraic reasoning, which not only raised the cognitive demand of the mathematics curricula, but also supported students in naturally developing informal problem-solving skills.

Students naturally engage in the activity of generalizing. Moreover, algebra is a significant portion of the K-12 mathematics curriculum in most countries. Therefore, generalizing is likely occurring in mathematics classrooms, ranging from elementary to secondary grades. It is critical for educators to be aware of students' natural abilities to generalize and the opportunities to encourage generalization. In order to do this, educators must have a clear definition of generalization and be able to identify the activities of and opportunities for generalizing.

Generalizing is a practical way to support algebraic thinking at any grade level. By understanding definitions of generalization, educators may be able to identify and encourage it in their classroom. Although "algebrafied" instructional materials are uncommon [10, 11], mathematics educators can incorporate opportunities to generalize and promote generalization in their classroom using any mathematics curriculum by modifying existing tasks or supplementing their curriculum with tasks that promote generalizing.

As indicated in a review of literature on generalization, the area of research is extensive. Moreover, studies often publish the mathematical tasks, which were successful in promoting generalization in their research experience. The final section is a review of mathematical tasks, developed and refined in research that may promote generalization in the context of a mathematics classroom. The purpose of this review is to provide mathematics educators with tasks that they could implement and examples to communicate some of the characteristics and aspects of tasks that have proven successful in promoting students' generalizing elsewhere.

Generalization Activities/Tasks

The Stacking Cubes task (see Fig. 1) was adapted from a study that examined the ways in which instructional materials can support students' engagement in functional thinking [11]. The task was used with elementary students. The researchers found that one way to introduce generalization into curriculum is to vary task parameters. For instance, if a functional thinking task, similar to the Stacking Cubes task (see Fig. 1), did not ask students to describe the surface area for fifty cubes, students may not think beyond the scope of the present number of cubes. Similarly Lannin et al. [48] found that by asking students to solve for a value that is large enough that they are unable to model the problem using that value encourages them to go beyond arithmetic reasoning by identifying and generalizing a pattern or relationship.

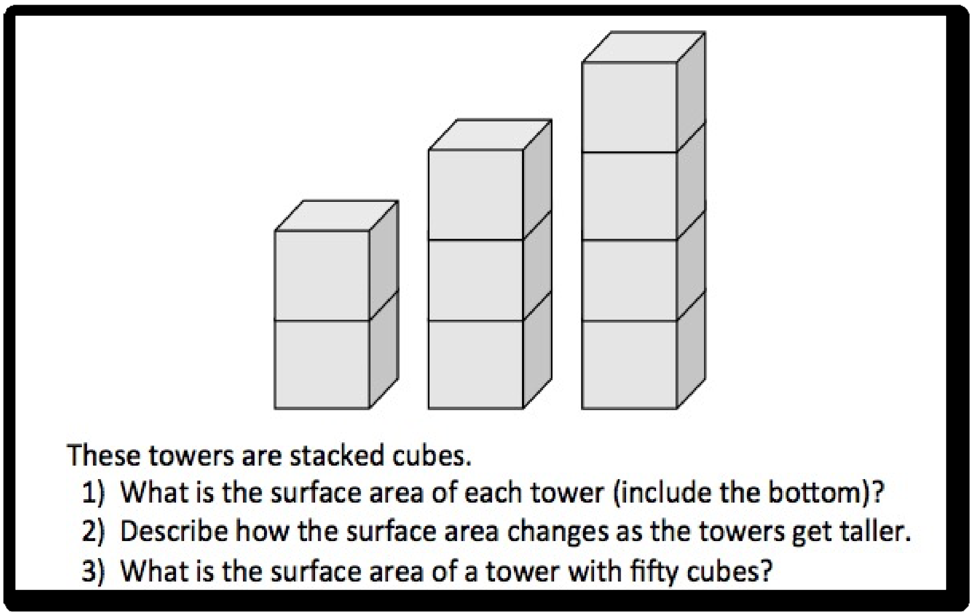

Fig. 1. Stacking Cubes (adapted from [57, p. 160]).

The Straw Task (see fig. 2) is well known; the task or variations of the task can be found in many places. One variation of the task appears in Abramovich and Brouwer [4], a previously mentioned study in which the researchers observed the Einstellung effect (see [1] for further elaboration of the Einstellung effect).

The Straw Task was also used to examine the sophistication of students' justification in grade six [47]. All of the tasks used in the study involved iconic representations, so that students were encouraged to find connections between their calculations and the representation. Then, the researchers could make inferences about students' justification strategies by examining how they related their calculations to the representation. Furthermore, the tasks involved geometric relationships, so that students could easily draw connections between their calculations and the problem context. This strategy was again employed and successful in a later study (viz., [48]).

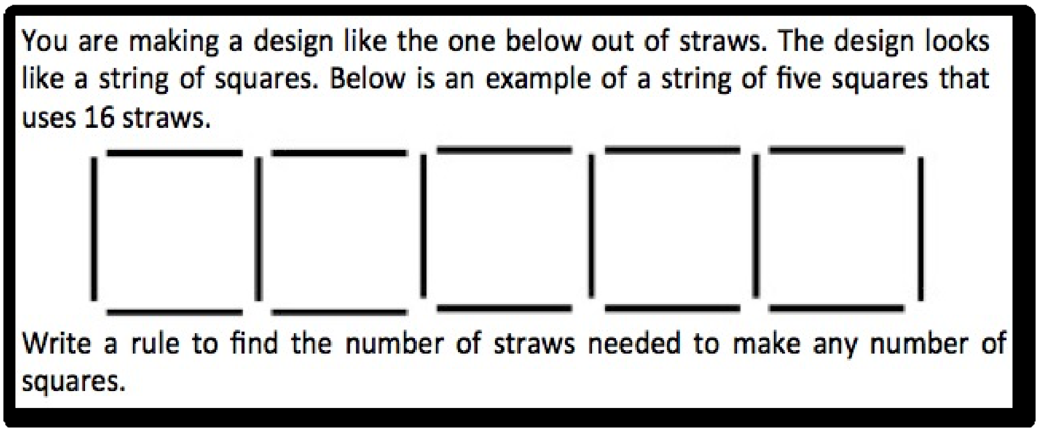

Fig. 2. The Straw Task (adapted from [47]).

Blanton et al. [12] study the development and sophistication of students' thinking about generalized functional relationships. The study employed a teaching experiment conducted with 6-year old children. Interestingly, the study finds that students, as early as 6 years old, engage in sophisticated thinking about functional relationships. Dogs and Dog's Noses (see Fig. 3) is a modified version of a task used in this study. The task was given prior to the teaching experiment. That is, students had no experience with functional reasoning when they engaged with this task.

In the article, the researchers list many tasks used throughout the teaching experiment, all intended to foster generalizing about functional relationships. One slightly more advanced task, that appears later in the teaching experiment, is a simple modification of Dogs and Dog's Noses (see Fig. 3). The modified task asks students to consider the relationship between the number of dogs and the total number of legs on the dogs. In other words, the task involves counting dogs and their legs, instead of their noses. Despite the variety of tasks shared by Blanton et al. [12], the tasks were designed for young children, and thus would not be intellectually stimulating for a more mathematically advanced audience.

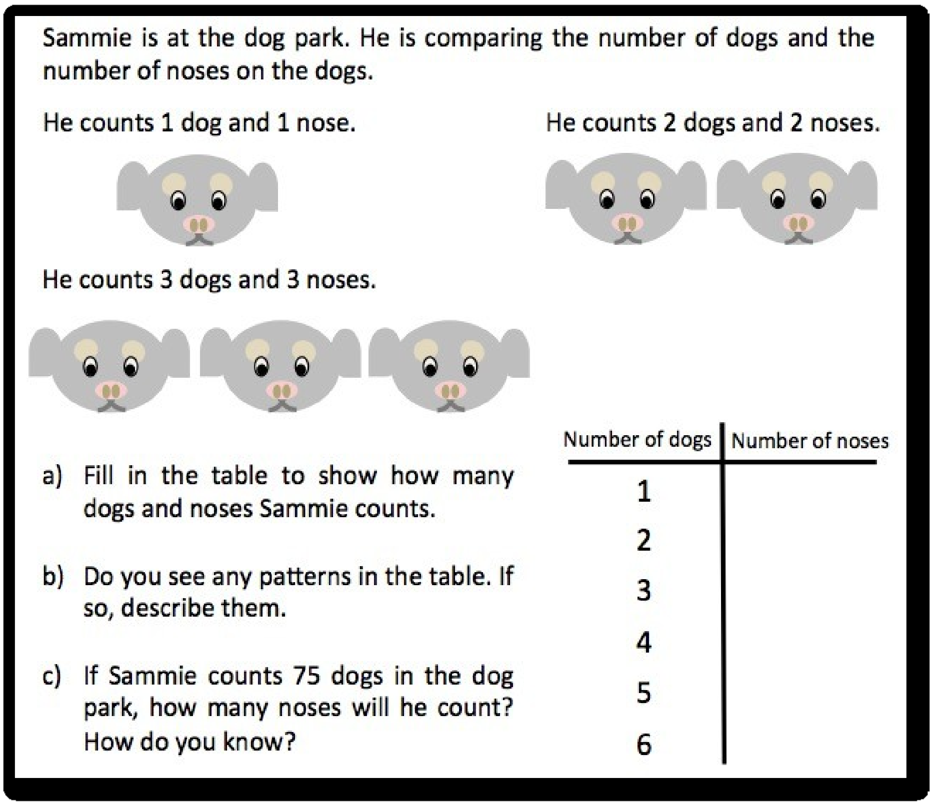

Fig. 3. Dogs and Dog's Noses (adapted from [12]).

Examples of tasks that are appropriate for mathematically advanced students can be found in recent issues of this journal. Abramovich [2] discusses how technology can shape the way pre-service teachers learn to develop problem-posing skills and construct and refine tasks. This author discusses a framework for characterizing tasks that has supported pre-service teachers in learning the "craft of task design" (as cited in [2]). One rich example that illustrates his argument and could be implemented elsewhere to support generalization was a task given in a teacher education course called "Creative problem solving." Initially, he shared the task in Fig. 4 with the students – pre-service teachers. After analyzing the task according to the previously mentioned framework, the pre-service teachers revised the task so that the context was more interesting and exoteric, meaning it was likely to be accessible to students (see Fig. 5).

Natural numbers are put in groups as follows:
(1), (2, 3), (4, 5, 6), (7, 8, 9, 10), (11, 12, 13, 14, 15), ...
Find the sum of numbers in the 10th group.
Find the sum of numbers in the nth group.

Fig. 4. Grouping number groups [2, p. 125].

A group of people is in a room together for some kind of meeting. Each person is expected to become part of a group. The first group will have one person, the second group will have two people, the third group will have three people, and so on. The person in the group one is assigned the number one. The persons in the group two are assigned the numbers two and three. The persons in the group three are assigned the numbers four, five, six, and so on with the remaining groups. Each person in each group is given a piece of candy according to their number. For example, person one gets one piece of candy, person two gets two pieces of candy, person three gets three pieces of candy, and so on. The person handing out the candy wants to put each group's candy in a zip lock bag prior to handing it out and needs to know the total pieces of candy each group will get. Help this person to solve the problem.

Fig. 5. Bags of candy [2, pp. 125-126].

However, after revising the task, the pre-service teachers decided that the task would benefit from further modification. They explained that the task required students to find the sum of consecutive natural numbers from a generalized number without providing any scaffolds to help them do this. In response, the class explored how using a spreadsheet might initially support students in the problem-solving process.

Thus far, all the tasks in this paper have encouraged students to test particular instances, compare and relate those instances, and then search for a pattern or relationship among the particular instances. The remaining tasks vary slightly in that they promote justification by inviting students to agree or disagree with a conjecture. This scenario initiates a dialogue about the validity of the conjecture, thereby promoting students to justify—to provide evidence or backing for their argument.

Researchers have found that justification is critical to generalization (e.g., [26, 29, 47]. Thus, asking students "why or why not?" to promote justification, in turn supports generalizing. Moreover, constructing a scenario, such as the scenarios used in Figs. 6 through 9, initiates argumentation and justification because it creates a situation in which the student can agree or disagree. Conversely, if a textbook states a conjecture directly students are unlikely to interpret the conjecture as negotiable.

Cooper et al. [19] used two of these tasks (see figs. 6 and 7) in semi-structured interviews with middle school students. The goal of the research was to identify the strategies that students used to evaluate and justify their conjectures. The last two tasks were used in a study with students in grades 6-8. The study aimed to understand students' ability to justify and prove [44]. Although these tasks were used to promote proof, they are valuable in encouraging students to generalize as well. As previously noted, students are naturally capable of noticing, expressing, and representing characteristics and properties of numbers, which these tasks support [55]. Additionally, these tasks foster generalized arithmetic because they prompt students to notice and discuss mathematical relationships, and then generalize about those ideas.

Tellie says that if you pick any whole number and add this number to the number before it and the number after it, then your answer will always equal 3 times the number you started out with.

Do you agree with Tellie?

Why or why not?

Fig. 6. Tellie's "Times Three" Conjecture (adapted from [19])

Tellie says that if you pick any even number and add this number to half of itself, then your answer will always be divisible by 3.
Do you agree with Tellie?
Why or why not?

Fig. 7. Tellie's "Divisible by Three" Conjecture (adapted from [19])

Una says that if you add any two consecutive numbers, then your answer will always be odd.
Do you agree with Una?
Why or why not?

Fig. 8. Una's "Sum of Neighboring Numbers" Conjecture (adapted from [44])

Una says that if you add any two even numbers, then your answer will always be even.
Do you agree with Una?
Why or why not?
Next, Una is thinking about what happens if you add any number of even numbers. What do you think about adding any number of even numbers?

Fig. 9. Una's "Adding Evens" Conjecture (adapted from [44])

6. Conclusion

The ways in which mathematics educators understand and define generalization continues to evolve. In many ways, "generalization" resembles what Lacan [46] calls a sliding signifier. Depending on one's epistemology and beliefs about learning, generalization is conceptualized differently. Furthermore, generalizing appears in countless forms, at various levels of sophistication, and in relation to many different concepts. The aim of this paper is to provide an overview of those definitions, and offer some historical context in hope to explain major shifts in definitions of generalization. With this information, readers can glean their own understanding of the concept, and relate it to their classroom experience.

Since this paper is not an empirical study, the author's contribution is limited in that the ideas presented here are a synthesis of previously shared and published ideas. The goal of this synthesis is to provide examples that will help refine readers' definitions of generalization, and offer teachers and researchers resources for encouraging generalization in the classroom. Moving forward, I hope teachers and researchers will build on these ideas with an aim to contribute to the growing understanding of generalizing in the classroom.

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