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AN EMPIRICAL STUDY IN GROWTH OF DISCRETE AND CONTINUOUSLY DEFINED FUNCTIONS VIA THE COMBINED ACTION OF A SOCRATIC INTERVIEW AND A COMPUTER-GENERATED TOOL

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Abstract. Starting with an intuitive understanding of the idea of one-to-one correspondence via counting collections, we develop the notion of functional relationship in discrete and continuous settings. This concept image build up enables us to pursue the study of growth and rate of growth in the context of sequences and simple functions given by analytic expressions as well as the study under which conditions growth and its rate coincide, triggering the appearance of the number *e* as an added bonus. Under the constructivist perspective which focuses in individual thinking and the continuing act of creating learning opportunities, we shall use a carefully planned semi-structured Socratic interview in conjunction with a computer-generated tool. The dialog tests the solidity of those ideas and arguments, which students bring and that are based in prior educative experiences, when challenged to dig out and verbalize their own beliefs and questioning techniques try to ease the difficulties most students have with concept translations to and from verbal representations (some more considerable than others) which are the key to understanding and communicating effectively. The tool, apart from easing the burden of calculations, promotes higher levels of engagement encouraging students' participation, provides them with numerical data and visual realizations which, paradoxically as it may sound, facilitate abstraction by linking the processes of discovery, understanding and conceptualization.

Key words: growth, rate of growth, function, number *e*

1. Introduction

In [3], the authors conducted a study where functions appeared as models of certain relationships that were observed, trying to interest students in explaining changes verbally and transferring to their graphical (Cartesian) representations leading to the idea of rate of change and local rate of change, i.e. the derivative. In [2], the authors focused on the transfer from verbal to symbolic representation to justify the delicate equality 0.999…=1. Both studies were structured as empirical interviews presenting data in a semi-fictitious Socratic narrative considering typical misconceptions and barriers students might encounter and how these might be overcome as well as emphasizing the role and need for a software tool. Since the same presentation is followed here, we refer to our considerations there, where justification for choosing this methodology was provided.

The cognitive structure associated with a mathematical concept which includes all mental images, visual representations, experiences and impressions as well as associated properties and processes is called concept image [8]. Learning, understanding, applying and developing mathematical concepts involves the construction of this kind of structure in the mind [5,6]. By means of a semi-structured clinical interview, we shall provide the means for the construction of a suitable concept image of functional relationship which incorporates visual, numerical and algebraic connotations enabling us to proceed to the study of growth and rate of growth in sequences and sequence-inspired functions. For this purpose and with the help of a mathematical assistant like MATLAB, we shall design a tool, technological in nature, covering all those aspects. For a study of growth in the context of numerical series, we refer the interested reader to [4].

Our experiment was conducted with high school students which had been exposed in the classroom to a notion of function (as a relationship between changing variables symbolized by $y = f(x)$, a single rule over a whole domain) and to its associated bi-dimensional graph in the Cartesian plane, but not to any limiting process aiming at the definition and manipulation of continuity, differentiation or integration. Around twenty interviews were carried out. It was not our purpose to measure the extent of students' previous knowledge, but to guide them in a journey of discovery of significant particularities of growth, via the progressive construction of a concept image of the notion of function suitable to our purposes and adjusted to their prior understanding and compatible with their prior understanding of it. We tested at every stage of the experiment the foundations of their beliefs and their ways of reasoning, very much in the Socratic spirit and with special care on whether former and new information were successfully integrated. Our questioning line is inspired by the platonic dialogue 'Meno' which deals with the method of hypothesis, and the distinction between knowledge and true belief, a very suitable reading for all those aiming at a PhD in Mathematical Education.

It took considerable time to design the interview and some failed attempts were needed in order to reach a satisfactory one. After that, twenty interviews with High School students of Valencia and Seville were carried out, each successful one consuming roughly one and a half hour time. Students were selected on the basis of their willingness to participate and every volunteer was accepted. They agreed to the audio recording of the interviews and also to the use of their corresponding anonymous transcriptions in our analysis. No substantial differences in their previous mathematical knowledge were noticed, because all were endowed with the usual consistent (but sadly declining) one provided by the Baccalaureate in Spain.

2. The tool

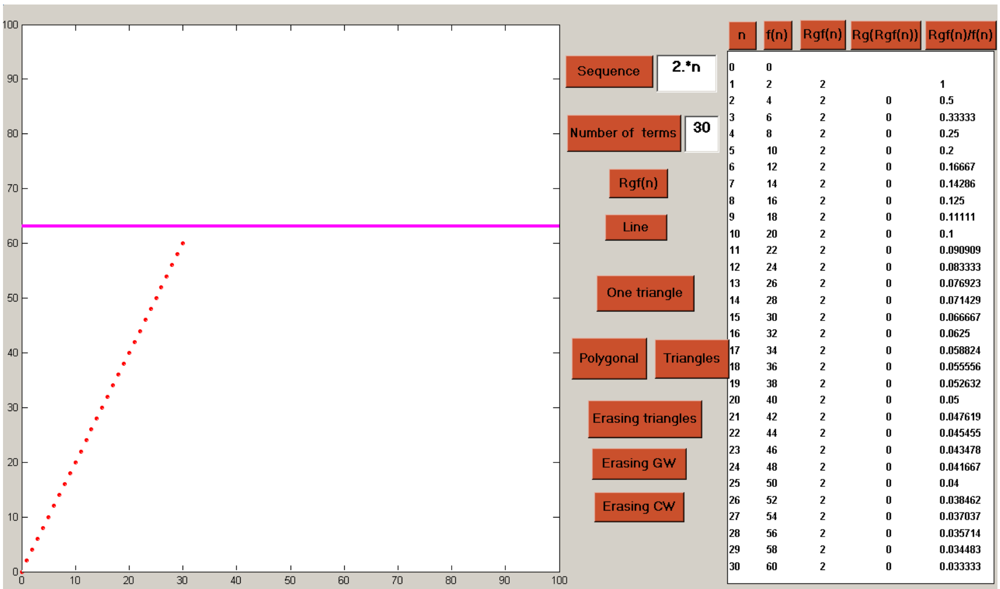


Figure 1: First interactive screen.

With the help of MATLAB 6.1 we have designed a tool exhibiting an introductory screen which is the door to three interactive screens, each of them consisting of a graphical (GW) and a computational (CW) window to provide the interviewee with means of analyzing visually growth of sequences (the two first screens) and functions establishing connections between visual and computational information.

No previous knowledge of the mathematical assistant used in the design is needed: all of them are endowed with an editable text box to enter the expression of the sequence/function to be studied as well as the rest of numerical data needed for a desired graphical representation. All commands are activated using the mouse on suitable pushbuttons. Zooming capability is available at all time.

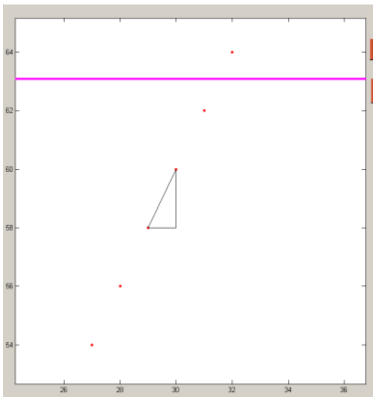


Figure 2: GW1 screen.

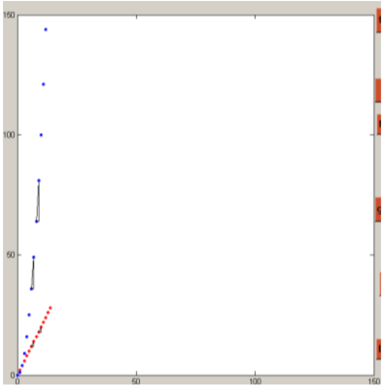


Figure 3: GW2 screen.

In our first screen (Figure 1), the window GW1 allows the bi-dimensional visualization of a numerical sequence as pairs (position, term) where the sequence can be selected as well as the number of points to be seen by using suitable editable text boxes. Other capabilities are available: it is possible to use the mouse to fix horizontal lines to see whether the sequence is bounded or not by them and consecutive or isolated right triangles may be drawn to visualize rate of growth (Figure 2). CW1 shows several numerical columns: the first indicates position *n*; the second, the terms *f(n)* of the sequence; the third, the rate of growth *Rg(f(n))*, defined as *f(n+1)-f(n)*, between consecutive terms; the fourth, the rate of growth *Rg(Rg(f(n))* of our former rate of growth and the fifth puts *Rg(f(n))* in proportion with *f(n).*

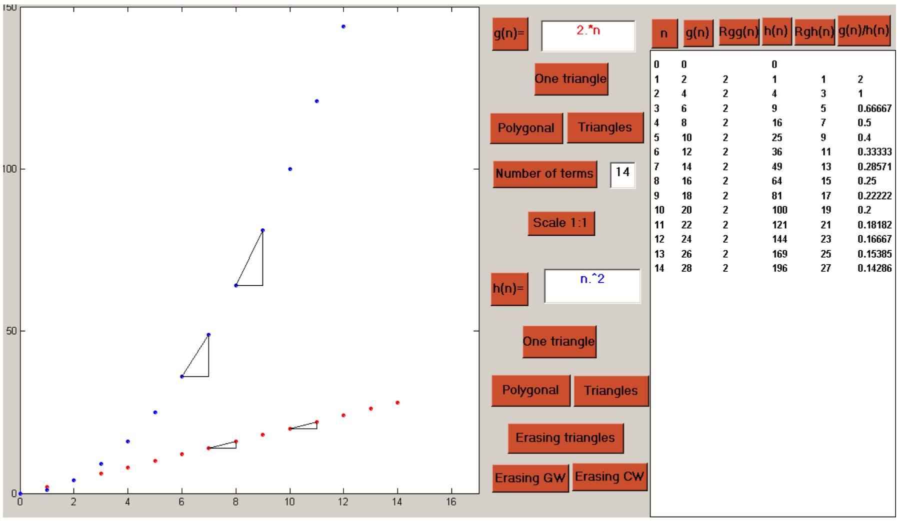


Figure 4: Second interactive screen.

Our second screen is devoted to compare the behavior of two sequences (Figure 4). GW2 allows the simultaneous plotting of both sequences *f(n)* and *g(n)*, whose expressions are entered at suitable editable text boxes and CW2 is endowed with four columns: the first for n, the second and third take care of *f(n)* and *g(n)* and the fourth is reserved for their respective quotients *f(n)/g(n)*. GW2 allows drawing right triangles between two consecutive points of the sequences as a mean of comparing respective growths. It is also possible to draw a polygonal between points for a given sequence. Scale 1:1 can be recalled by using the corresponding pushbutton (Figure 3).

The third screen deals with functions (Figure 5). GW3 allows the comparison of the graph of the function with the (discrete) plotting of the rates of growth corresponding to a previously chosen partition in the domain (Figure 5) and a similar polygonal can be adjoined (Figure 6). The scale of the plot can be adjusted at will. CW3 shows four columns: the first for the variable *x*; the second for the values *f(x)*: the third for the rate of growth *Rgf(x)* and the fourth for the quotients *Rgf(x)/f(x)*. Several plots of functions and their respective discrete plots of rates of growth can be seen in GW3, although numerical data in CW3 correspond only to the first plotted function (Figure 6). The tool is available on request.

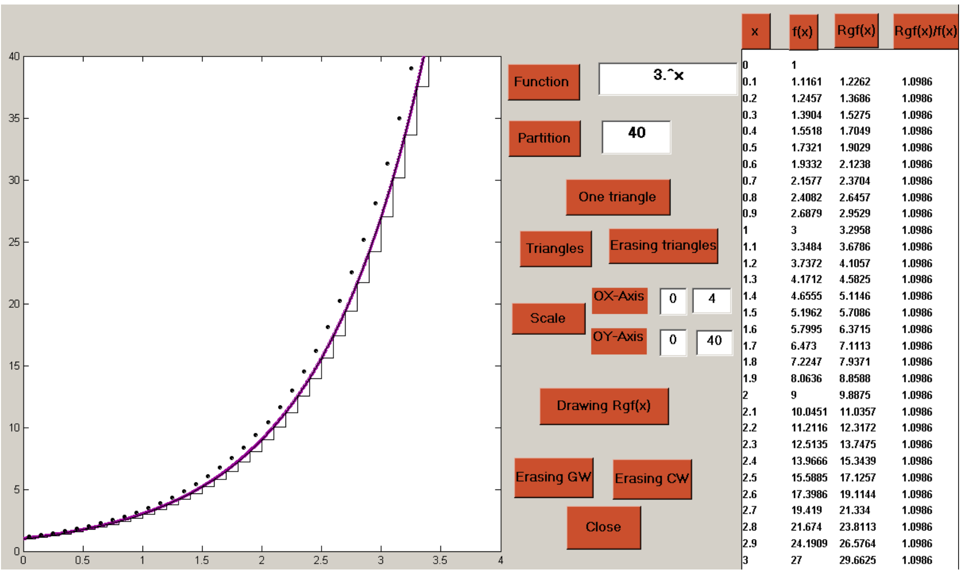


Figure 5: Third interactive screen.

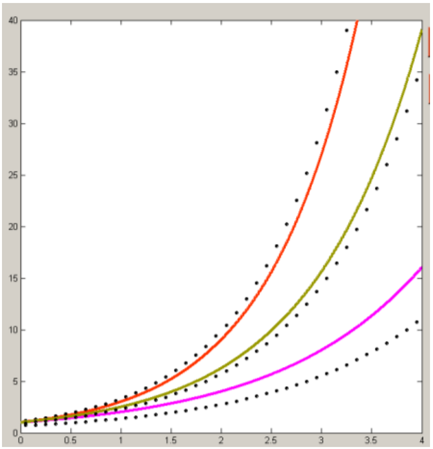


Figure 6: GW3 screen.

3. The interview and student's responses

Hoping that the experience might offer some guidance to teachers wanting to introduce the topic to their students, we have unified our transcriptions and edited heavily the final text in order to present a narrative in mature language avoiding clumsiness, false starts, hesitations and fragmented and incoherent arguments hopefully preserving what we were able to experience in terms of clarity of thought and adaptability of our most successful students, if not in their exact actual words. Overcoming difficulties contribute to conceptual learning but in order to keep this article within reasonable bounds, dysfunctions do not show in the following transcription of our interview, since the responses belong to the more gifted students.

3.1 Natural numbers and correspondences.

Thinking of natural numbers is, well, natural for human beings. Thinking of growth in term of numbers is also a usual activity. Besides, we all are aware (consciously or not) of the idea of one-to-one correspondence: we all have been waiting to receive some service with a number assigned to us and no one of us expect to find another person possessing the same number as ours as well as not having to go through all ordinals until our turn comes.

Should we all stand in a line and should we all be respectful of others' rights, we all understand that an assigned number is not necessary, that is, the correspondence does not need to be explicit using numbers. Not such a familiarity is present if we want to deal with non finite collections of objects and this is the purpose of our introductory remarks: we are going to deal with different representations (discrete and continuously defined functions) of growth phenomena. Starting with the simplest example of growth (natural numbers) we deal with what 'how many', 'very large' and 'infinite' means which will connect us with the notion of potentially infinite collections which, in turn, brings us to the idea of one-to-one correspondence as a first manifestation of the notion of function.

We begin the experiment by reminding our interviewee that mathematics is axiomatic, which means that it does not inherently reflect what **"is true"** or **"is not true"** but rather it describes the consequences of certain simple assumptions. For example, we assume that there is some abstract quantity called a "one" and that you can add it to numbers to get other numbers, which, to give them names, we might as well call "two" and "three" and so on. We define the **natural numbers** as the collection of numbers that are finite sums where all terms are 1. For instance, $1 + 1 + 1 + 1 + 1 = 5$ is a natural number, where for practical reasons we have gone from numbers to symbols (numerals). We engage in questions relative to why symbolism might be useful enabling us to utilize short term memory to better effect or whether it should depict what is trying to represent or not. Concerning numbers, we remark that we do often use the *word* "number" to mean "numeral" when it isn't important to make that distinction. We go a step further by assigning a *letter*, capital N, to denote the collection of all natural numbers as we seem to get used to using particular letters so that we remember what they stand for more quickly. In any case, we emphasize that whenever we use a letter or a symbol in mathematics we should make clear what kind of mathematical entity it represents. Given a collection of objects, can he describe what *counting it* means?

St.: I can take out objects one by one ; each time I take out an object, I say a natural number out loud, in sequential order "one, two, three, four, ..." and eventually I will run out of objects. The last name indicates the number of objects present and I have counted the collection.

We ask him if he agrees in interpreting what he just said as establishing a *one-to-one correspondence* between the collection of objects and a collection of numerals, let us say, {1,2,3,4,5,6,7} for a collection of seven objects. He hesitates, uses paper and pencil drawing the familiar arrows and complies. Could he dispose of the arrows by putting the objects of our collection in a **list** understood as the first, the second, and so on? Listing means putting them sequentially from left to right and I do not need to mention {1,2,3,4,5,6,7} again, he answers. In other words we emphasize, counting a collection is the possibility of listing all objects in the collection.

Pr.: When I say that a collection is finite, what do I mean?
St.: I guess that a collection is finite when it is *possible* to list all the elements, that is, it has a specific number of elements which I can count.
Pr.: Some collections are *infinite* (not finite); these collections have more than n elements for any natural n.
St.: (puzzled) I do not understand. Do you mean that there is no way of listing all of them?
Pr.: Hmm ..., can you count collections as large as you want?
St.: Yes, again say 1, 2, 3, ... until we run out of things to count.
Pr.: Do you never run out of numbers?
St.: No, never! There is nothing that can ever stop numbers from getting large.
Pr.: There is no end to N.
St.: In fact, it's safe to assume that, since you could count forever, there are an infinite number of numbers in that collection.
Pr.: Careful with the expression "infinite number"! The word "infinity" doesn't represent an actual number.
St.: Is not a number which is bigger than all the others?

We point out that "infinite" just means "without end" (we ask him to check on the Latin root of the word) and it is a way of describing something that never comes to an end: there are infinitely many numbers, because there is no last number and that's really all it means. Infinity is not "a very large number"; in fact, infinity really has nothing to do with numbers, we insist.

Pr.: This is something very special about numbers, which is worth thinking about: numbers are just ideas, not anything you can see or touch, so there is nothing to stop them from going on forever.
St.: (doubtful) But computers put a limit on the largest number they can use.

Pr.: That is only because computers need more memory to store higher numbers, and there is a limit to how much memory a computer has. But we do not have a memory limit, so there is no largest number.
St.: All right. In this sense, {1, 2, 3, 4, 5, 6....} and so on has infinitely many numbers in it.
Pr.: But here's the thing: how do you know what number comes after 19 or after 12,327?
St.: You add one to it. Given any number, there is always one which follows in the ordering.
Pr.: And how do you know which one it is?
St.: Well, if I take 12,327, I know the next one in the list is 12,328
Pr.: Is it because you have memorized somewhere in your head "12,327 + 1 = 12,328"?
St.: (laughing) No, because there is a procedure for adding one to natural numbers: I look at the digit in the one's place and If the digit is between 0 and 8, return the rest of the number with the ones' place incremented.
Pr.: Right, and if the digit is 9?
St.: Add one to the next digit and return this number with a zero in the one's place.
Pr.: This leaves us with no largest number. Observe that the thing about N is that there is a first one '1' and, moreover, if you specify any number, you can construct the following one.

We agree that N is then potentially infinite, from the Latin "*potentia*" meaning possibility as opposed to finite. Although our physical world does not contain infinite collections, our mind can conceive infinite collections, at least, potentially infinite collections. Our imaginations are bigger than the whole universe! And, sure, N is such a collection. Now we ask about the uses of N as an entity.

Pr.: And, for us, N is instrumental in counting arbitrary large finite collections. Can it be used to count itself, that is, the number of objects of N?
St.: (perplexed) No, I can start saying out loud numbers and never end.
Pr.: Can you explain the problem in terms of one-to-one correspondence?
St.: I do not see how a limited part … that is, a finite part of N can be chosen to establish such a correspondence with N as a whole.
Pr.: But isn't it possible to establish a correspondence from N into itself?
St.: Well, yes, of course, every number is assigned to itself. The correspondence can be established but there is no numeral to account for all of them and even if one was invented, it would not make much sense.

We are not going to pursue any further this line of questioning regarding counting infinite collections and the possibility of assigning meaning to such a symbol, but we shall concentrate in the idea of correspondence between infinite collections of numbers, one of them fixed, namely N.

Pr.: Is it possible to establish a one-to-one correspondence between N and, let us say, the even natural numbers or the squares of the natural numbers or the successive powers of 2?
St.: If I were to count them I will make a list which shows me who is the first, second and so on.
Pr.: Well, you don't have to count them but, agreed, to show that there is a correspondence, the only thing that matters is write the even numbers in *a list***,** making sure that you list them all.
St.: It is the same as giving the elements of the collection a waiting number in an infinite line... Well, in this case, the answer is yes: just write them all in a list, 2,4,6,8,10,…, 1,4,9,16,25,... and 2,4,8,16,32,…
Pr.: And no one is missing in all those lists?
St.: Well, a lot of them are missing but insofar as they follow a simple rule which is easy to describe, all missing numbers can be calculated one after another.
Pr.: Right, that is the meaning of the three dots: the pattern established in the first five is followed indefinitely and spotting the pattern enables predictions to be made. And this pattern for our first list is?
St.: Since you spoke of even numbers, the pattern is established: numbers next to each other are linked by the same number since they ascend by equal steps. Well, the next one is 12: you add 2 to a number in the list to obtain the next.
Pr.: Consecutive numbers grow by addition. Is there another way of describing the pattern? Could you describe it by using 'position in the list' as the determining feature? And if affirmative, what changes in your description of the pattern relative to the arithmetic operation used?
St.: Hmm… , since we are dealing with the sixth number in the list, I can also obtain 12 as 6 multiplied by 2. Addition is replaced by multiplication.

3.2 Looking for a symbol.

We agree in calling **sequence** any infinite list of numbers which we call **terms** of the sequence. Is there the first term but not the last one? He agrees.

Pr.: A list like the even numbers might be seen as well as a tabular setting.
St.: (using paper and pencil and writing even numbers). Do you mean a vertical setting instead of horizontally … a kind of table? There is no problem (adding a first column with natural numbers) I can see it as a one or a two-column table.
Pr.: Can you label the first column?
St.: (puzzled) Label as writing something above such as 'natural numbers'?
Pr.: For the sake of economy, can you use just a letter let us say 'n', for instance?
St.: n as taking different values, n as a variable?
Pr.: (looking at the paper) Yes. The second column shows the terms of the sequence. What label for it?
St.: Hmm... , as said before we are using multiplication by two, that is $2n$.
Pr.: Right! Is it any difference between a conventional table and a sequence seen this way?
St.: Usually a table ends; a list doesn´t.
Pr.: Right. But there is something else. If you consider any two-column finite table, it is not even clear if there is any relation between both columns …
St.: In what sense?
Pr.: It does not show if there is some hierarchy between columns: whether the entries of the first determine the entries of the second or the other way round.
St.: You have to guess what this relationship is. The relationship has to exist because otherwise, why should you write two columns together? In our sequence, the relationship is in the labels: we move position and we move along the sequence: that is the order. Position determines term.
Pr.: Right. Even if we admit this kind of dependency, should we have not labeled the columns, there is no direct specification how to compute the entries of the second column in terms of the entries of the first.
St.: I see. A list or a sequence is not just writing down numbers as a table. Precisely, because it is not finite, there has to be some pattern in order to specify each and every one of them.

Three aspects came to the fore in a sequence: unending first column, dependency between columns (the second depends on the first and hence is non finite as well) and pattern to be able to know all of them. Now we want to isolate the pattern in such a way as generating a formal notation which takes care of all three aspects.

On the surface this goal seems fairly innocuous; why not make some abbreviations so that things can be said more economically? But, we have to be careful when introducing formal notations which are known to generate obstacles since most students see this as disconnected information [7]; we have to check that the student is able to transfer information between verbal to symbolic representations and we shall not proceed further until it is certified that, being in possession of a symbolic representation of the pattern, writing the sequence as a list or table is irrelevant. When speaking of a generic sequence, tentatively we propose a symbol such as $n \rightarrow x_n$ to denote it, where the sub-index indicates a position in the sequence (list), x_n a term of the sequence, n runs through all natural numbers and the arrow establishes the linking between position and term. No resistance to accept it is observed.

Pr.: Going back to the sequence of even numbers, what is an appropriate symbol for it?
St.: Let us say, that for a natural n the correspondence sends it to its double 2n, hence $n \rightarrow 2n$ does the trick.
Pr.: Can you provide symbols for the other two lists?
St.: For the first, position is squared hence $n \rightarrow n^2$ describes what is going on. In the other list every term is doubled to obtain the next.
Pr.: The list grows by multiplication. Thus 2^n goes ...
St.: To 2^{n+1}, hence $n \rightarrow 2^n$ should be the symbol.
Pr.: What have in common all sequences considered above?
St.: Each term is smaller than the next one. They grow.

3.3 Looking for visualization.

A sequence has appeared as a rule-based point-wise process. A way of looking at its overall global behavior is providing a graphical representation of it which is more conceptually manageable that our former lists/tables and which take into account all three aspects ('domain', 'dependency', 'pattern') mentioned above. Usually students derive information from tabular representations more easily than from non-linear graphical ones, because graphical understanding needs training *("a picture is worth a thousand words, provided you use a thousand words to explain the picture"*, according to Harold M. Stark, a distinguished number theorist) and to avoid training we need to find a familiar setting. We start by questioning him on how to visualize N. The straight answer comes forward: yes, a straight line, but we need to consider also the number 0; setting a unit length we mark its end as 1 and we repeat the operation to place all the others. Before proceeding further on, we want to be sure that colloquial language triggered by the representation is right on the mark; we ask: If you're comparing two numbers, can you say that the one farther to the left on the number line is "less" and the one farther to the right is "greater"? "Less" and "greater" are concerned with position: where the number lies on the line. We continue: "Small" and "big" on the other hand are concerned not with position, but with … size, he answers, 1 is the smallest number. The closer something is to 1, the smaller it is. We take a detour: is there life in the line between integers? we ask. Fractions and decimal expansions as synonymous are mentioned. We let that pass, no point to talk about irrationality here. Do they qualify as numbers too? Yes, both are instrumental for measuring and comparing purposes, he replies. It is always good to know what we are talking about: after some verbal exchanges, we agree on what is meant by fractions and expansions and we rest in our inquiry: complex numbers are outside his present reach and we do not want to question what a number should or might be. Returning to a first pictorial representation of the sequence of all even numbers, it comes easily to him just by marking them upon the representation of N.

Pr.: And if a bi-dimensional representation of the sequence is what is demanded?
St.: (puzzled) I don't know how to produce one.
Pr.: What bi-dimensional representations of mathematical ideas do you know?
St.: Graphs of functions come to mind. But, is a sequence a function?
Pr.: You'll decide that. What do you remember as the most basic aspect of the idea of function?
St.: A formula connecting two variables?
Pr.: Do you understand a formula as a way of finding one variable with the help of another … a relationship between variable magnitudes …?
St.: Yes, either way.
Pr.: But, would you say that what is really important is what changes and how?
St.: Well, yes again. 'What' are magnitudes; 'how' is the relationship between them, the formula.
Pr.: Which might be expressed as $x \to y$?
St.: Hmm …, usually it is expressed as $y = f(x)$, x and y being the variables. I guess I could use $x \to y$ as well if f is known to me. It doesn`t matter: all that is important is that x, what goes in must determine y, what comes out.
Pr.: What does f stand for?
St.: The relationship: a formula, isn't it?
Pr.: Having to choose between those symbols, which one do you prefer?
St.: Both indicate what changes: x and y. The one with the arrow is … pictorial … it indicates the action of relating numbers although the arrow is not specific about how this relation works, I mean, it does not indicate how they change; the formula is missing. Anyway, the arrow is not a mathematical symbol, right? I think it is better to use $y = f(x)$; moreover, there is an equality and you can operate with it, if needed.
Pr.: What is variable in what the symbol $n \to 2n$ wants to describe?
St.: As n changes from position, the even numbers appear one after the other. I see position works as 'x' and even number as 'y'.
Pr.: Right. Notice that x jumps in steps of magnitude one. And 'f' stands for …?
St.: Mm…, $n \to 2n$ can be written as well as $f(n) = 2n$ and hence it is a function and it should have a graph in the plane.
Pr.: If I write $f(m) = 2m$ where m takes the values 1,2,3,… , are we talking about the same sequence?
St.: Yes, the letter used has no bearing.

So far we have witnessed a successful integration of sequences as functions as he understands them. The need for a graphical representation of a sequence has been responsible for triggering in memory an already planted idea of function and its associated Cartesian graph. Symbols such as $x \to y$ and $y = f(x)$ have been recognized as having the same meaning, namely that input determines in some way output, allowing us to continue undisturbed with our treatment of sequences opening the door to the use of a geometrical model of functional relationship and fortunately avoiding the (usual) perception that, when dealing with functions, changes of a variable are exclusively changes in time.

The fact that the second notation is preferred to the first is indicative of his fixation in understanding functions as analytic formulae (otherwise, an oxymoron for him) as a short description of a computational algorithm which, albeit a limited view, does no harm to our purposes because we are not going to consider split domains.

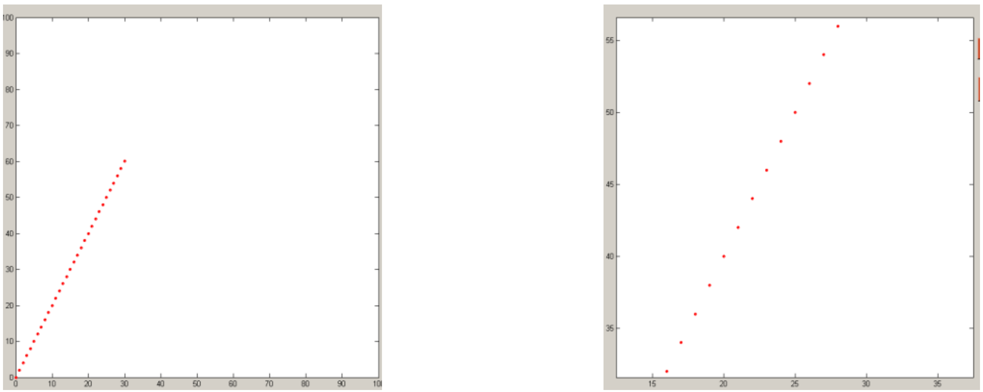
Concerning graphical representations, several facts should be kept in mind in our role as interviewers: (i) using graphs to represent functional relationships is hardly spontaneous; (ii) lack of prior understanding of Cartesian graphs prevents him from interpreting the information that is contained in the graphs; (iii) transfer from graphical to symbolic representation is more difficult than vice versa; (iv) transfer from graphic or symbolic to verbal representation (different from simply reporting some of the features of the relationship) is rarely seen; (v) a graph is a static representation which hides all dynamism of functions and (vi) discontinuous graphs may be interpreted as corresponding to several functions rather than one.

Since he has gone already through the usual drill in the classroom, we don't press those points further, but we keep vigilant and hope for the progressive adjustment of previous knowledge with the flow of ideas this experience brings.

We start by asking for the kind of graph which should appear for $f(n)$, although we expect resistance in accepting a function represented by a discrete set of points. Since students rarely meet graphs other than those of straight lines, using paper and pencil, he offers several points connected by straight lines. We ask him to elaborate on his picture and he seems unsatisfied when we ask him to remove any notion of continuity leaving bare points, even having understood our motivation to do so concerning where the function is defined. It is clear that such a graph does not conform to his previous ideas on what a graph of a function should look alike (idealizations of lines or trajectories of moving points).

3.4 Visualization of growth and rate of growth.

We begin to use our tool by activating the first screen to see an equally spaced cloud of points on the screen keeping the same scale in both axes; by zooming out several times we can appreciate points positions on the plane (Figures 7). We ask him to identify abscissas and ordinates of those points and relate them. He gains confidence: abscissas are positions and he visualizes the process meant by the function by taking abscissas and moving vertically to the point in question and then horizontally to the ordinate axis to check that the (even) number obtained is a term of the sequence, hence performing a (partial) integration of both conceptions of function as an object (cloud) and as a process ($n \rightarrow 2n$).



Figures 7: Visualizing $f(n)= 2n$.

What does the growth imply in such a graph? We zoom out considerably and ask the interviewee to draw a horizontal line at any desired height: can he add points to the graph enough to overreach the line? (see Figure 8). He does so without trouble and we inquire on interpretation of that construction. He suggests that this sequence can be made to exceed any given number, which numerically doesn't come as a surprise to him.

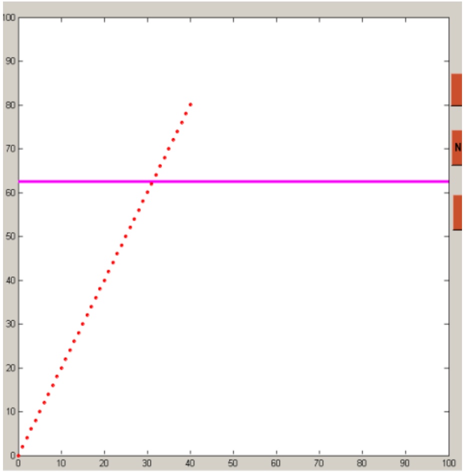


Figure 8: The points overreach the line.

We hand him back some sense of continuity in the graph: in order to help him to appreciate growth in the sequence, we press 'triangles' to draw triangles connecting points hoping to influence him in appreciating growth visually in two steps to highlight the relationship between changing magnitudes: first horizontally, indicating change of position and, second vertically, measuring the difference of altitudes. He recognizes that bases remain unchanged wherever he looks but also altitudes. Activating CW1 to put numbers to our figures (see Figure 9), three columns appear: the first shows *n*, the second *f(n)* and we ask him to interpret the third column.

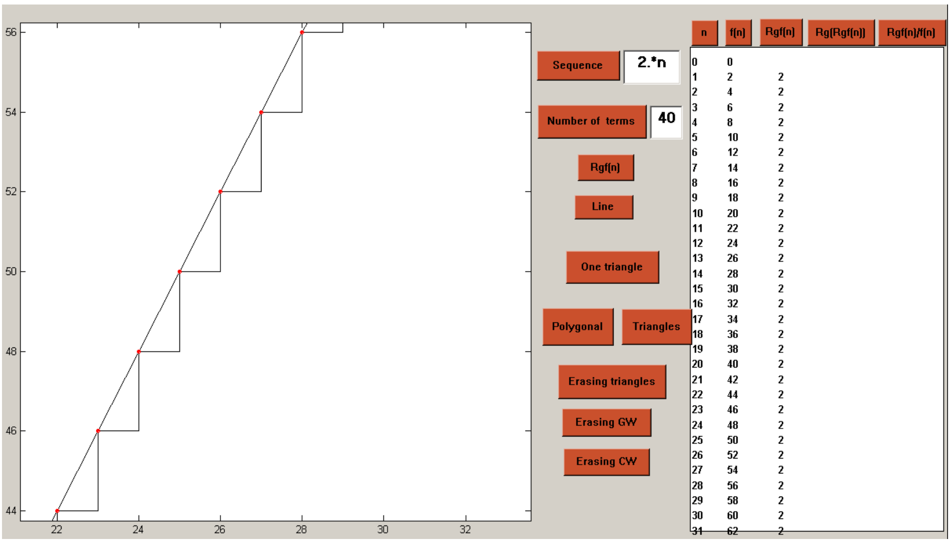


Figure 9: Information about *f(n)*=2*n.*

St.: The third column shows the amount of growth experimented at every step.
Pr.: (Taking paper and pencil and showing operations to him) Changing *n* from 5 to 6 gives us a growth of 12 – 10 = 2 for *f(n).* Rewriting it as **f(6) – f(5)** = 2×6 - 2×5 = 2× **(6 - 5)** = 2, concentrate in **f(6) – f(5)** = 2× **(6 - 5).** Can you see the connection between position and sequence as a quotient?
St.: Well, arithmetically it is evident is that 2 appear as the quotient of *f*(6)–*f*(5*)* by 6-5; as *n* changes in steps of one, *f(n)* grows in steps of two. Since *n* always changes from one to one, step-by-step growth is measured by the numerator.
Pr.: You mentioned that relationship between magnitudes was the idea behind a function. Observe that, when the first two columns are considered not to be significant by themselves, but only in relation to one another, we get the third column. Each entry provides a relative magnitude, a kind of comparative index, a way to measure the change of *f(n)* not by itself but with respect to *n*.

St.: Yes. In order to compare the growths experimented by both position and sequence, you put them in proportion as a quotient; it is what I called step-by-step growth.
Pr.: Right. Let us say then that the third column shows the **rates of growth** of *f(n)* respect to *n*. (Activating GW1) Now to the visual setting. Does this quotient have any interpretation in GW1?
St.: The quotient stands for the inclination of the hypotenuse of the triangle.
Pr.: Is this inclination the same everywhere you look at?
St: (drawing several triangles in GW1) Bases are all clearly the same, as they should be, and it looks as if all hypotenuses have the same inclination, since all heights look alike. (Taking paper and pencil) A change of 11 to 12 gives the same difference: 2×12 - 2×11 = 2× (12 - 11) = 2×1 = 2. (Activates CW1) You can see that in the third column, all entries are equal, namely 2.

We summarize: Even numbers is a sequence that increases by addition of a fixed quantity and hence the difference between successive terms is a constant. By plotting the sequence, we can appreciate that all points lie on a line, which is why we say this kind of growth is "linear". Linear growth is slow and steady, although this sequence can be made to exceed any given number. What happens to 3, 6, 9, 12, 15, …? Much of the same, he replies, the entries of the third column, that is, the rates of growth are all the same number (3 in this case); steady growth, then.

Pr.: 3 is the rate of growth at every step of the second sequence. By the way, it is larger than 2, the rate of growth at every step of the even numbers. What does this fact say about how quick both sequences grow?
St.: It is not clear. It seems to me that rate of change per step is something attached to the very sequence we are dealing with. I mean that, if in a certain sequence, the rates of growth change from step to step, let us say getting bigger all the time, we may speak of a quick growing sequence. In regard to your question, since both have steady growth, the speed of growth should be the same.
Pr.: Thus, it does not mean that 3, 6, 9, 12, 15, … grows faster than 2,4,6,8,10…?
St.: Faster understood as having a larger speed?
Pr.: If you want to compare two different sequences in terms of speed, is it reasonable to put them in proportion term by term?
St.: If I do that, I get 3/2, 6/4, 9/6, 12/8,15/10,… which is always the same number, hence it seems that speed is the same in both sequences. Moreover that number is the quotient of both constant rates of growth.
Pr.: Is it reasonable to say that only when the ratio of their respective sizes becomes larger and larger, we can say that one sequence grows faster than the other?
St.: Yes.

Let us start by seeing that not all sequences have a steady rate of growth and then we shall compare it with the sequence of even numbers term by term. We activate GW1 to plot *g(n)* = n^2. He recognizes that we talking about a growing sequence again.

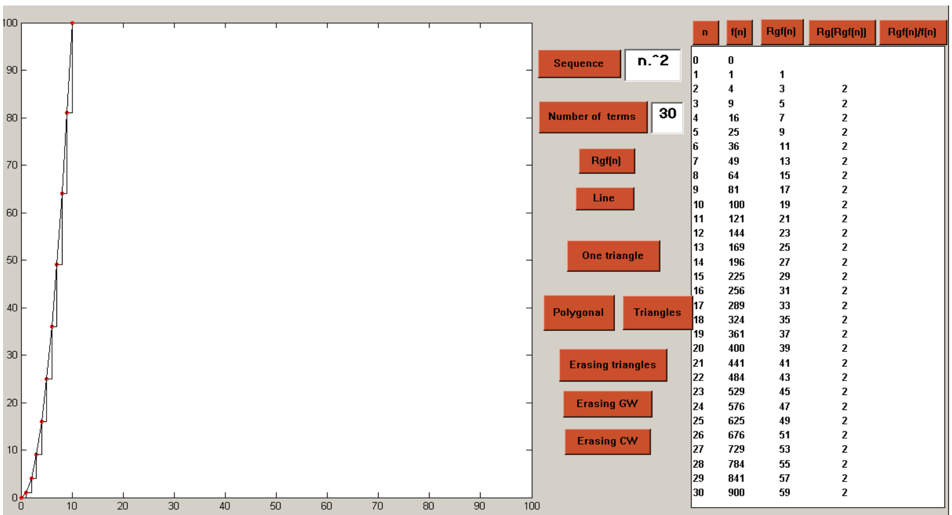


Figure 10: Information about *g(n)*=n^2.

Pr.: But does it grow in the same way as *f(n)?* Please, insert triangles in the plotted sequence (Figure 10).
St.: Their bases keep constant lengths as before but altitudes are increasing more steeply the farther *n* gets from 1.
Pr.: Thus, the rate of growth is not constant …
St.: The growth itself is growing.
Pr.: Hmm…, does it make sense to have a fourth column which measures how the rate of growth grows, that is a column indicating the rate of growth of the rate of growth?
St.: Sure! (activating CW1 to appreciate it numerically in Figure 10). Looking at the fourth column and comparing with the third it is clear that the rate of growth grows linearly, although *g(n)* does not.

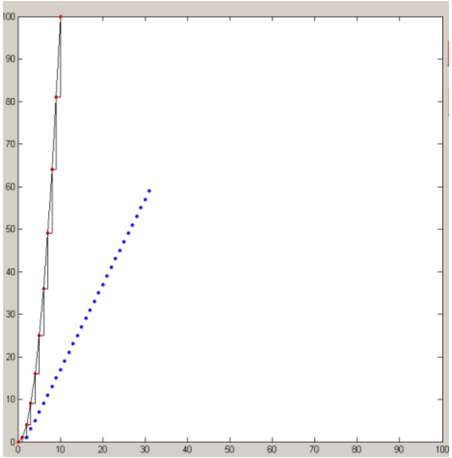


Figure 11: *g(n)* and Rg(*g(n)*).

Pr.: (returning to GW1 plotting *g(n)* and its rates of growth as in Figure 11) All those points seem to lie on a parabola, whereas the points corresponding to the entries of the third column lie in a line. What can you say about its speed of growth in relation to *f(n)*=2*n*?
St.: Plotting both sequences in GW2 gives an idea on what is going on (Figure 12). Taking term by term quotients 1/2, 4/4, 9/6, 16/8, 25/10, … we see the ratio of their respective sizes increasing without stop, hence *g(n)* grows faster than *f(n)*.

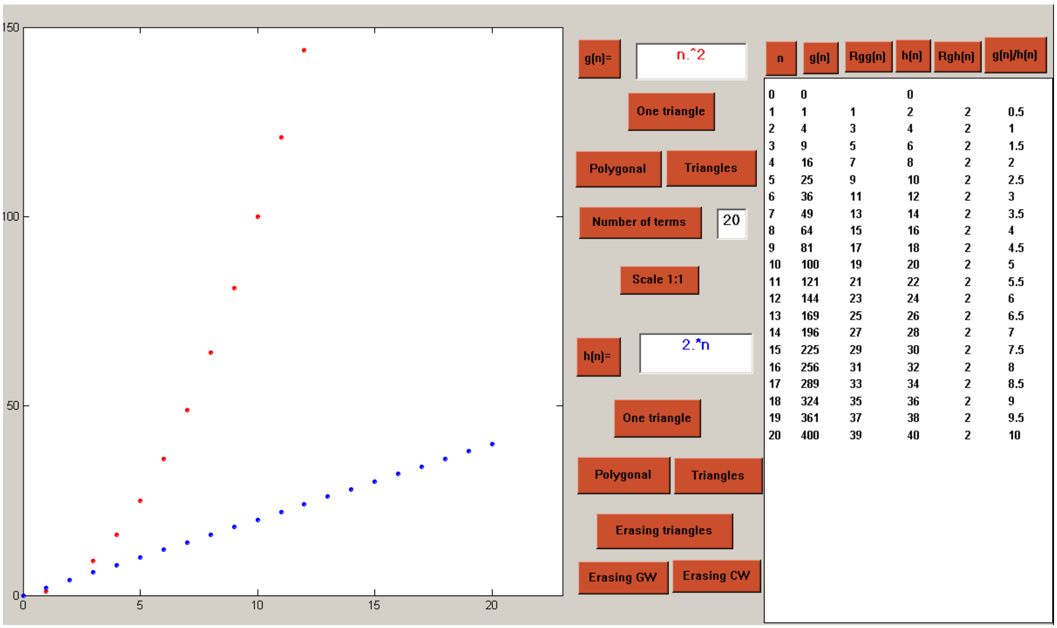


Figure 12: Plotting $g(n)=n^2$ and *f(n)*=2*n.´*

Now we plot $g(n)=n^2$ and $h(n) = 2^n$ simultaneously (Figure 13). What is the difference?

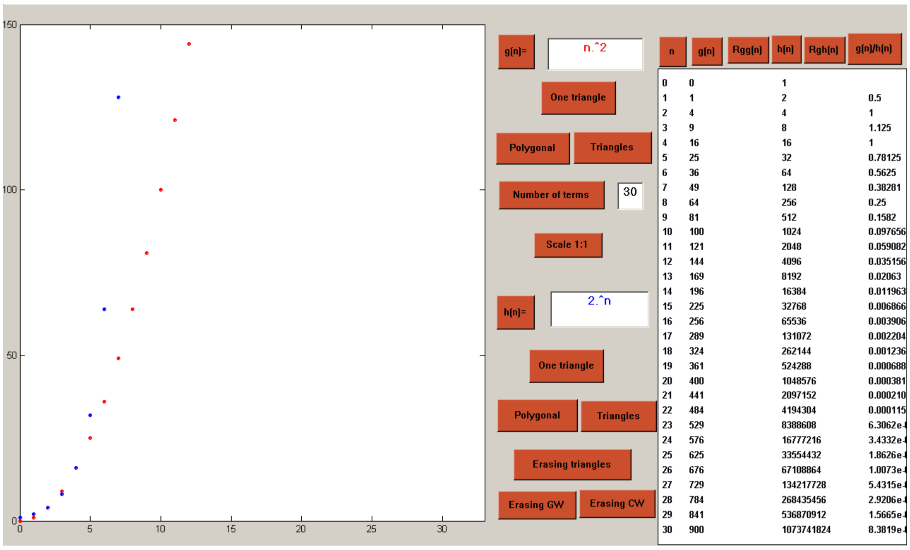


Figure 13: Plotting $g(n)=n^2$ and $h(n)=2^n$.

St.: Near n=1, this looks a lot like *g(n)*, but it grows much more quickly than *g(n)*.
Pr.: Let us explore how much more quickly. First, forget about *g(n)* and study *h(n)* alone.
St.: (activating CW1) Second, third and fourth columns look alike.
Pr.: Do you arrive to a column of constants even if we allow further columns present?
St.: Not a chance! None of the growths ever stops growing, hence it grows fast and it keeps growing faster and faster.
Pr.: Well, that is its behavior. But in comparison with *g(n)*?
St.: 2/1, 4/4, 8/9, 16/16, 32/25, 64/36,…. The ratios keep growing, hence *h(n)* grows faster than *g(n)*.

Now we activate a fifth column in CW1 (Figure 14) which stands for the quotients growth/term. What does this column show?

St.: The growth of the rate of growth is proportional to the sequence itself, well to something less than the original sequence. (Introduces *g(n)* to see its fifth column (Figure 15)). In this case, there is no such proportionality.

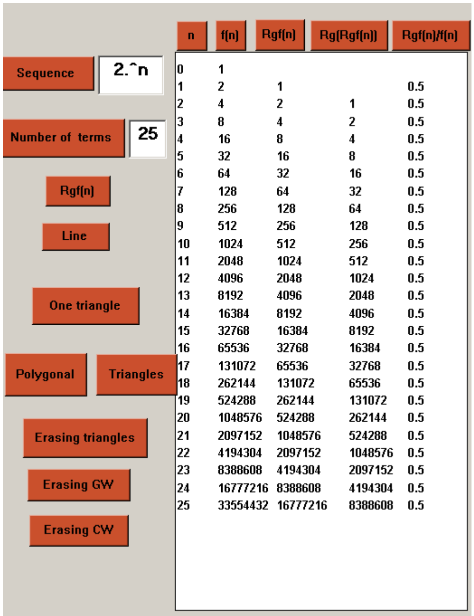


Figure 14: Information about $h(n)=2^n$.

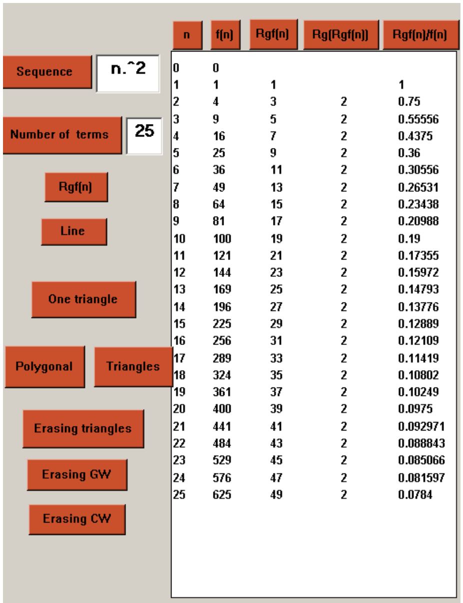


Figure 15: Fifth column of $g(n)=n^2$.

We suspect that students are at ease representing scenarios which encompass exponential growth relationships and we test our guess: we ask if growing sequences such as $n \rightarrow 2^n$ can stand as a model of growth phenomena connected to reality. At first, he seems unable to produce an example and when we suggest either how gossip works (you tell two friends, and they tell two friends, and so on, and so on, and so on), the Indian chess legend or a number of micro-organisms in a culture broth which will grow until an essential nutrient is exhausted, he connects easily. Concerning our last proposal, he asks for a clearer picture of the phenomenon: any further assumptions should be considered? Birth rate is constant and never limited by food or disease and therefore birth rate alone and nothing else controls how population grows. He will ask to quantify about birth rate: we suppose that typically the first organism splits into two daughter organisms every 24 hours, who then each split to form four, who split to form eight, and so on. We caution him that this model of relationship is a distorted image of reality (check our assumptions!) because we want students to be able to distinguish between relationships discovered by experience and the mathematical model of these.

Pr.: Can you establish a connection between the described growth process with a symbol such as $n \to 2^n$ or $h(n) = 2^n$?
St.: Since population doubles every 24 hours, that is, a day the letter n stands for days and *h(n)* is the population present after *n* days which is 2^n.
Pr.: Said another way, doubling for one time period is 100% growth. We can rewrite our formula as $h(n) = (1+100\%)^n$; we separate 2 into the original value 1 plus 100%. Moreover, our formula assumes growth happens in discrete steps: our microorganisms are waiting and they double at the very last moment. Is it realistic?
St.: Depends on what you really want: the formula describes the daily result of growth.
Pr.: But not the growth process itself?
St.: Since it takes one day to produce a doubling, the formula accurately describes the population after each day; I mean the number of completely formed individuals. But the process of growing itself might refer also to the growing of each individual …
Pr.: The model does not cover the instant-by-instant enlarging of the organism which after one day will culminate in two new individuals. What should we have in mind if we wanted to construct a model whose formula accounted for the growth phenomenon described above?
St.: First, we should know the amount of growth per unit of time (hours, minutes or seconds); second, the '*n*' in the formula should run through the time unit chosen: *n* should move second to second or take all possible instants between 0 and 1 hour, between 1 and 2 hours and so on.
Pr.: Something else?
St.: Moreover, we should be able to assign a meaning to, let us say, 1.6 organisms present.

3.5 From discrete to continuous.

Now we suppose we are dealing with growth phenomena where (i) non-integer numbers have a meaning related to growth and (ii) there exists an unspecified capability of measuring growth at numbers between integers. Let us suppose that we know for certain that the information provided by the sequence *h(n)* is reliable. How can we fill the gaps and achieve a global picture of the phenomenon?

St.: Do you ask me to guess at what happens between two values you already have calculated?
Pr.: Right. Can you elaborate on your choice of word 'guess'?
St.: Since the rule used is unspecified, I am only making a guess.
Pr.: That is called interpolation ("inter" is Latin for between).Usually, when you interpolate you are making an assumption that the behavior you're looking at is predictable or that the amount of error in your guess can't be enough to prevent you from using it.
St.: Yes. I might interpolate values to complete the graph and have some confidence that I wouldn't be more than a small quantity off, for instance, by drawing segments joining all data and hence I got complete information. That is what I did before, but you asked me to remove all segments!
Pr.: Right, but remember we were dealing only with discrete information and the joining paths had no relevance there. Now you've discovered 'linear interpolation' as a tool for description and prediction. So interpolation relies on some assumptions about the behavior of whatever you're investigating that makes it possible for you to guess values for which you only have information about what happened "before" and "after". What if growth doesn't behave linearly?
St.: Since we are talking about growth, the path in between should be going from less to more.
Pr.: Yes, but are linear paths the only ones going from less to more, as you said?

St.: (taking paper and pencil) Well, you are right; it could be something like that (draws a concave path), but how do I know? … (Pause) … Then, the rule I am missing should take the form of a function defined everywhere, not only in the natural numbers and hence its graph would be … without gaps.

Once again we have seen that most students tend to provide linear responses. Returning to our sequence *h(n),* we ask him to suppose that a growth phenomenon behaves as its continuous analog and we inquire how this analog should look like. He agrees to change the letter '*n*' for '*x*' as customary and proposes $h(x) = 2^x$. We listen carefully to his explanations concerning the analogies and differences between *h(n)* and *h(x)*, making sure that not all functions are perceived as sequences (in order to avoid the belief that interpolation methods give exact values of functions in intermediate points, although his 'interpolation sense' points out to the presence of a deeper level of his notion of function). He points out that we may consider what happens between natural numbers providing a better textured representation of the growth represented by the function; we may even consider what happens between 0 and 1. Why not for negative *x*? Yes, why not, he replies. Provided with paper, pencil and calculator and aiming to evaluate 2^x for integer *x*, he remembers how exponents work; for negative *x* they start very small, so small that they're practically indistinguishable from 0 and then, for $x = 0$, $2^0 = 1$ and, for $x = 4$ and $x = 5$, 2^4 and 2^5 are too big. He states that it looks like once they start growing, they grow faster and faster. Now we activate our tool to plot *h(x)* scales being 1:1 (Figure 16).

Pr.: Have a look to the negative *x*-axis and how *h(x)* behaves there.
St.: It confirms what we said before concerning how small *h(x)* is.
Pr.: It you look it in reverse, growth is a continual process of cutting in half. This means that this process of growth must spend a very long time at very small values.
St.: In the visual sense, the point is that one very small ordinate is going to look a lot like another, even if the two abscissas in question are lying far apart in the negative *x*-axis.

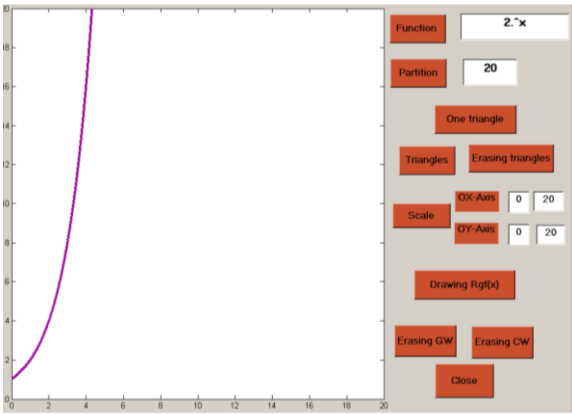


Figure 16: $h(x)=2^x$ (scale 1:1).

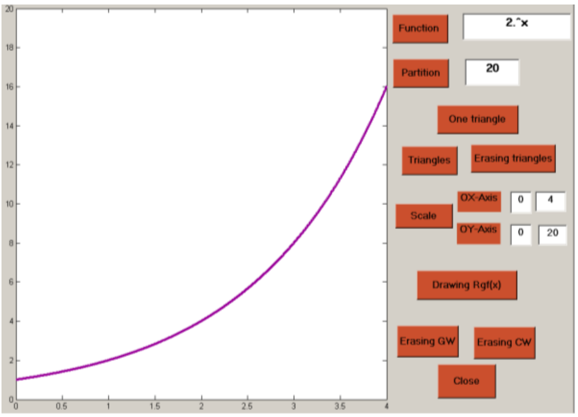
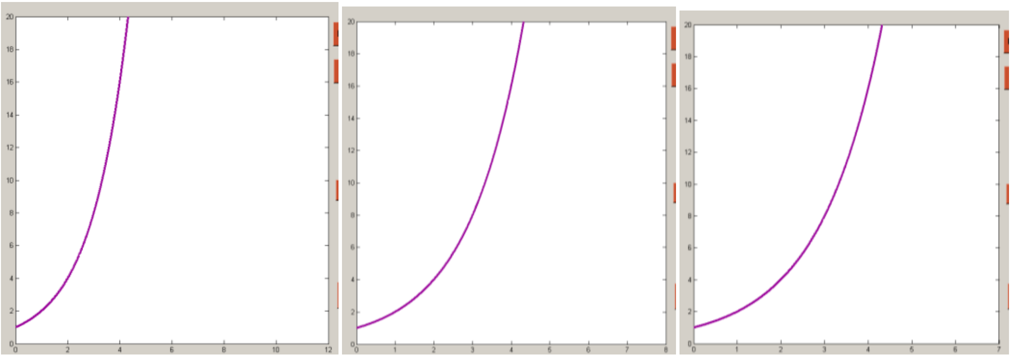


Figure 17: $h(x)=2^x$ (scale 1:5).

It doesn't take too large an *x* to appreciate that the graph shoots right up through the top of the screen. We settle for [0,4] as the domain of *h(x).*

Pr.: What shall we do to see a representative portion of the graph?
St.: If axis scales do not need to match ...
Pr.: The scale on the *x*-axis should be …
St.: Much wider than the scale on the *y*-axis; the scale on the *y*-axis is compressed, compared with that of the *x*-axis.
Pr.: (scaling several times (Figures 18) and keeping finally a scale appropriate to the domain [0,4] (Figure 17)) This technique is useful when graphing functions which grow quickly. Observe that the curve goes steeper the further up you go.
St.: The larger the values of the function, the faster it seems to grow.
Pr.: If *x* stands for time, how do you figure watching something "growing in time" ruled by *h(x)* starting with a very small negative *x*?
St.: This something doesn't grow much for a very long time and all of a sudden it explodes.
Pr.: When does this explosion happen?
St.: I do not know how to express it: let us say, when *x* and *h(x)* are of comparable 'size'?

Prior to that, it looks like not much is happening.



Figures 18: $h(x)=2^x$ (various scales).

Now we inquire about how to translate the concept of rate of growth to this continuous setting. In sequences, growth was measured by steps of 1; here, we can take steps of variable length to appreciate the growth more closely;

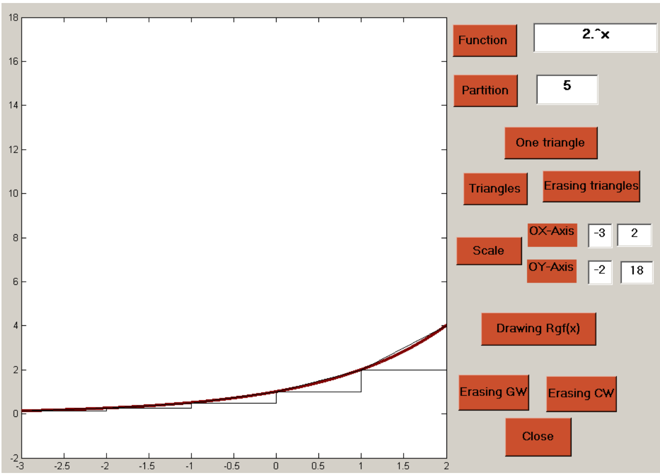


Figure 19: Growth in the negative *x*-axis.

Pr.: But even if we keep 1 as steps in the negative *x*-axis, what happens to the rate of growth there?
St.: (activating triangles in the screen, see Figure 19) Rate of growth remains small wherever I take the triangle: its altitude is very small: the difference between two small numbers is necessarily small.

To benefit from a closer look to growth, he arguments that in order to define rate of change we need to agree on the size of a step, that is, specify two points in abscissas. We ask him to use paper and pencil to precise his ideas: if *x* goes from *a* to *b*, *h* grows from *h(a)* to *h(b)* and, to calculate it, we need to consider the quotient of growths and calculate *(h(b)-h(a))/(b-a)* to which we refer as the **(average) rate of growth** of *h* in this interval [*a*,*b*]. Drawing with the tool any triangle on the graph of the function in GW3 (Figure 20), we ask about the visual meaning of this rate of growth. 'Again, the inclination of the hypotenuse' he replies.

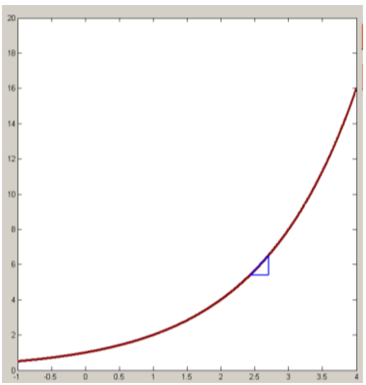


Figure 20: Triangle on the graph.

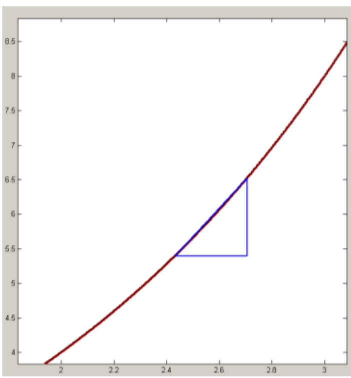


Figure 21: Visual meaning of the rate of growth.

Pr.: What is the advantage of being able to select steps of variable length?
St.: We may calculate rates of growth for steps inside the former jumps between integers and hence have more information on how the curve grows in selected tiny intervals.
Pr.: Almost local information, you mean?
St.: If I select points *a*, *b* very close to each other, drawing the triangle and zooming (Figure 21), the hypotenuse is very close to the portion of the graph, hence its inclination is representative of the steepness of the graph, the more representative the closer *a* and *b* lie.
Pr.: How close can you put both?
St.: As close as I want.
Pr.: But not *a*=*b*? Can you speak of rate of change at the abscissa *a*?
St.: Both abscissas being the same, the segment collapses to a single point and it does not make sense to talk of the inclination of a point.
Pr.: Arithmetically speaking?
St.: (takes paper and pencil) Arithmetically, it does not makes any sense since being the points the same, the quotient is 0/0 which makes impossible to assign a concrete number to this quotient because division undoes what multiplication does: any number will do as 0/0.
Pr.: You are in the situation where you can compute the rate of growth for each tiny interval. Is it right?
St.: Yes. The smaller I make the tiny intervals used in the computation, the closer I will be able to know how growth works close to a point. How close is close?

3.6 When growth and rate of growth coincide.

Let us check if we can get an accurate picture of rate of growth for our function *h(x)* in the interval [0,4] by subdividing it in tiny subintervals of length 0.1, calculating their rates of growth and plotting them in the midpoints of the intervals over which they have been computed together with the function (Figure 22). We ask him to comment on the information showed by the tool: In GW3, it appears that the average rates of growth for *h(x)* are always lagging a little behind the values of the function itself; in CW3 there is a constant proportionality of *h(x)* and its rate of growth.

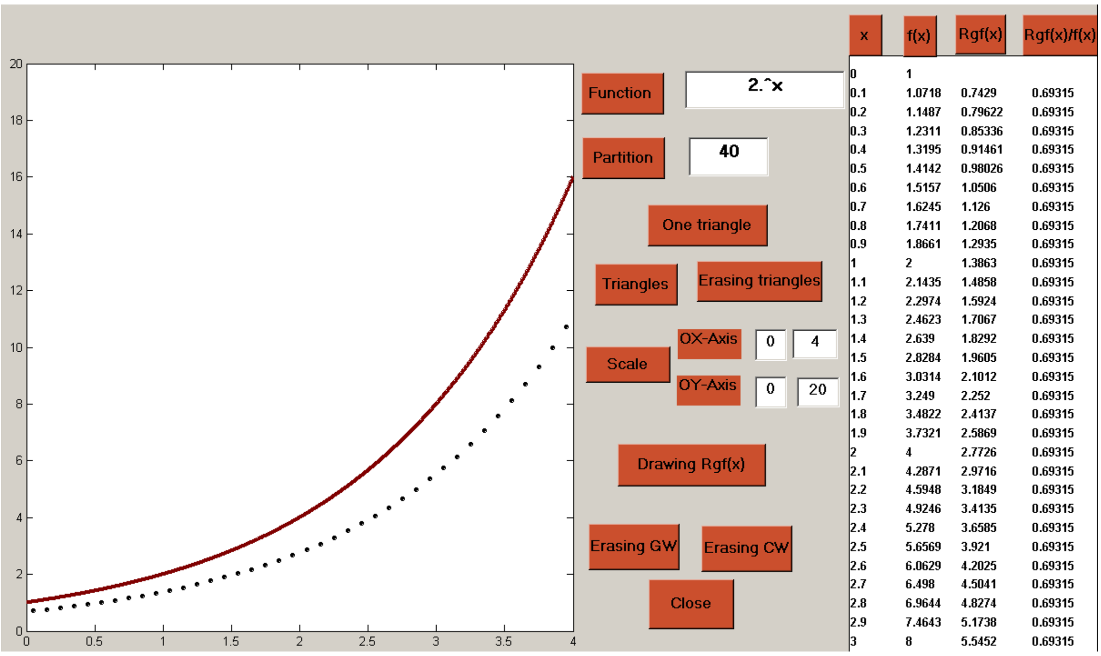


Figure 22: Plotting $h(x)=2^x$ and its rates of growth.

Before we proceed further, we need him to develop a gut feeling on what this proportionality means resorting to an analogy:

Pr.: We're going to imagine climbing a hill. First think about what it would be like if it had the same steepness all the way up.
St. The same rate of growth, then. The hill should look like a straight line.
Pr.: Every pace you give makes you gain altitude.

St.: The same altitude everywhere along the climb.
Pr.: Right. Think about what it would be like if the steepness of the hill kept increasing steadily as you went up.
St.: Steepness as another word for rate of growth? Steady increase of rate of growth, then.
Pr.: What would this hill look like?
St.: A kind of parabola. Every pace gains me more altitude than the preceding one.
Pr.: Now imagine the steepness of the mountain at any point is proportional to the altitude at that point.
St.: .: Hmm …, the shape of *h(x)* comes to mind. Depending on where I start to climb, either each pace gains me a very small altitude and this happens for a long time or, if I am high up, then it's steep; but if it's steep, my next pace gains a lot of altitude.
Pr.: But it's now going to be even steeper; and …
St.: My next pace will gain altitude even faster than before, but that means it'll now be even steeper; and… I understand, very comfortable at the beginning and very stressing afterwards.

Now we return to our tool and proceed with $j(x) = 3^x$; it appears that those related points for *j(x)* are always a little ahead and CW3 shows proportionality again (Figure 23). While neither curve's values coincide exactly with its growth rates, the match is a bit closer for *j(x)*. A reasonable question then seems to be: Is there a number, somewhere between two and three (and perhaps a bit closer to three), that is the base of a function for which the growth rates are *exactly the same* as the values at any point?

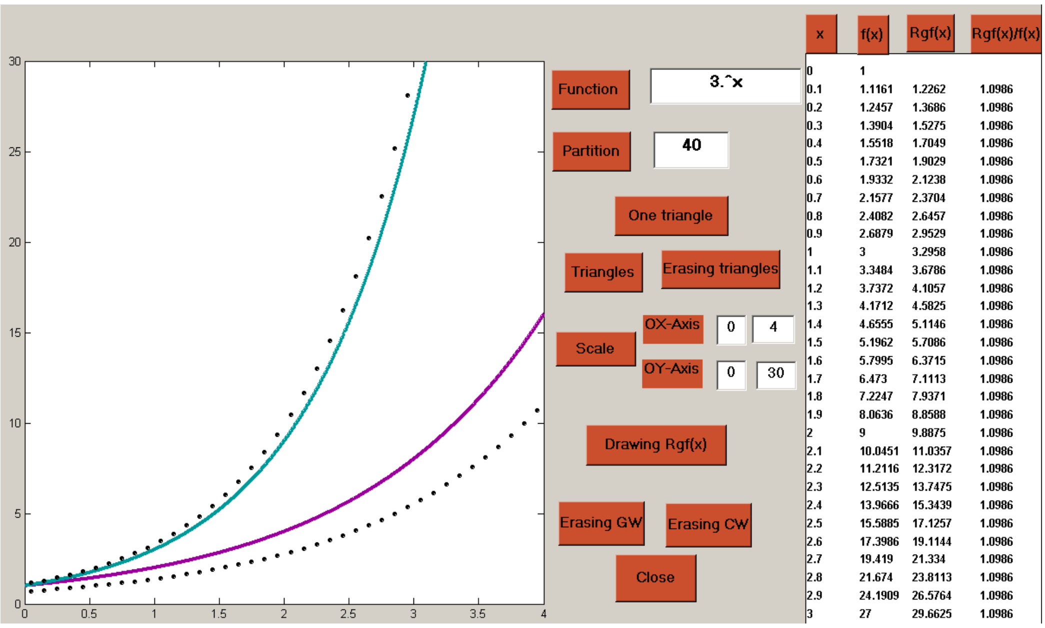


Figure 23: Plotting 2^x and 3^x and their rates of growth.

Pr.: The fourth column in CW3 shows the factor of proportionality between rate of growth and function. Can you imagine what effect would this fact have on GW3 if this factor should be one?
St.: The points corresponding to rates of growth should appear on the graph of the function.
Pr.: We have considered so far the functions $h(x) = 2^x$ and $j(x) = 3^x$. Do you have any suggestions on how to proceed to construct a function where points should appear on its graph?
St.: We should try an expression such as a^x for a suitable a between 2 and 3. Let us start with a=2.5 and go from there (he activates GW3 to produce Figure 24). The fourth column indicates 0.916 as proportionality factor … close to 1 but not quite … the points are not lying on the graph.
Pr.: Should we try a smaller or a larger value for a?
St.: Clearly a larger value for a; let us take a=2.7 (Figure 25).
Pr.: Is 2.7^x the function we are looking for?
St.: Not quite, the fourth column shows 0.99325. Zooming I can see that the points are still below the curve (Figure 26). I should try a larger value for a such as …

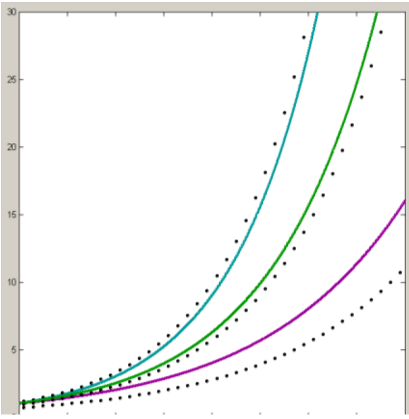


Figure 24:*2.5*^{x} second from above.

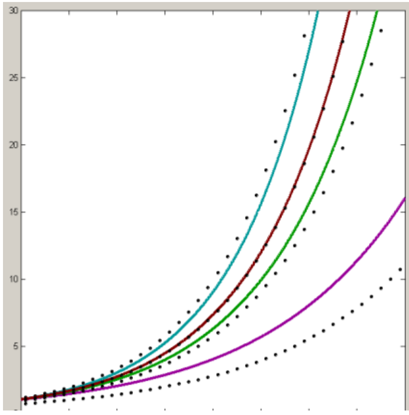


Figure 25:*2.7*^{x} second from above.

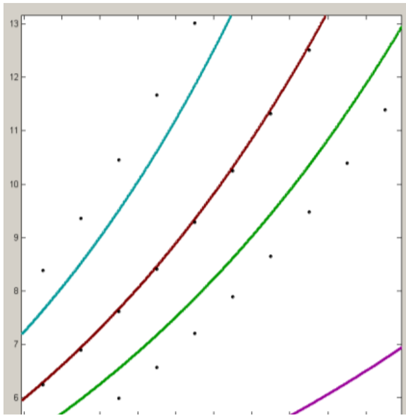


Figure 26: Zooming the graphs.

Pr.: (manipulating the tool) Try 2.71^x (Figure 27).
St.: The fourth column shows 0.99695; we are getting closer, but still not what we are looking for (he zooms again to see Figure 28). We may try 2.72^x (produces Figure 29); now the factor is 1.0006 and the points are lying above the curve (Figure 30). Well, the value for *a* should be something in between 2.71 and 2.72.

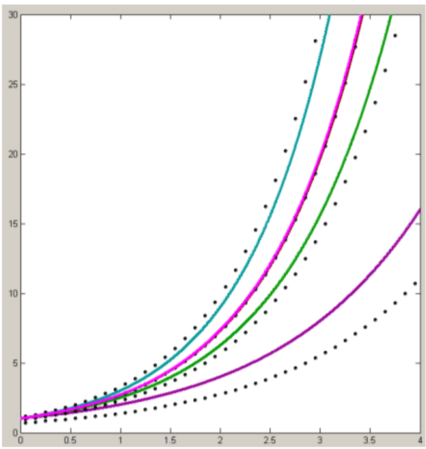


Figure 27: 2.71^x second from above

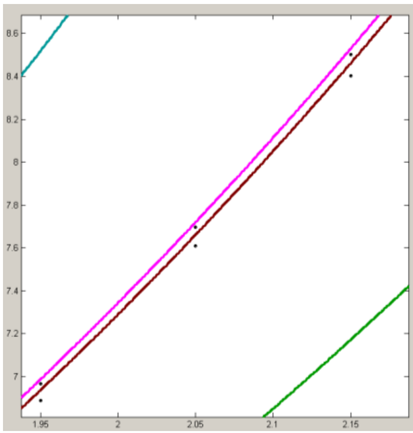


Figure 28: Zooming the graphs.

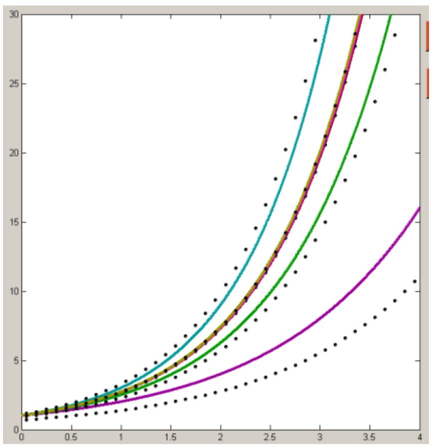
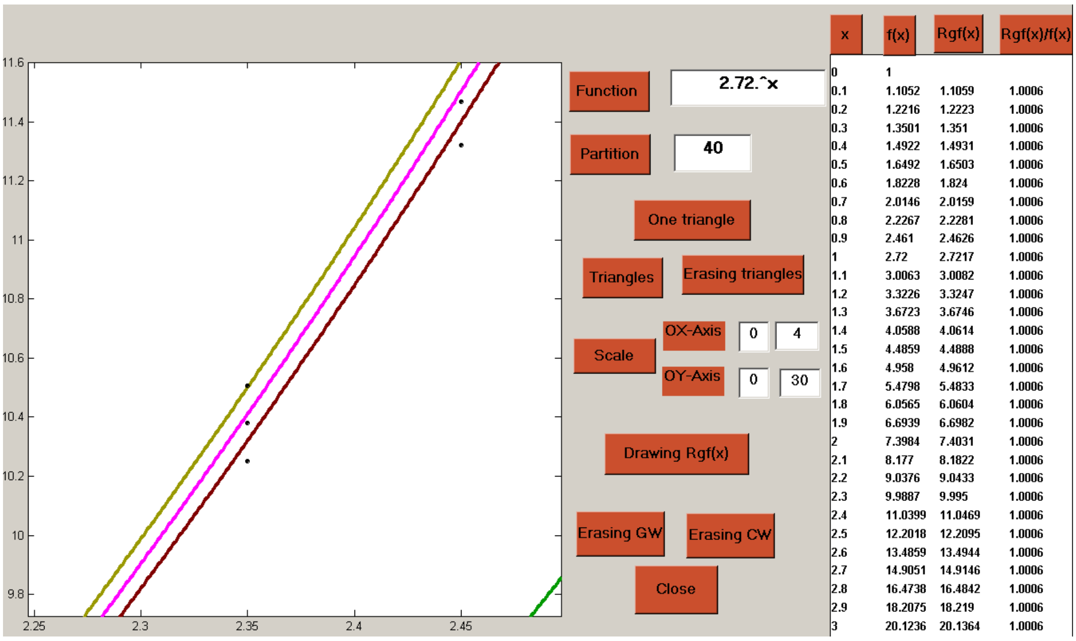


Figure 29: 2.72^x second from above

Now is the time to rest in our inquiry and inform him that he has discovered the first three digits of a very special number, called '*e*' in mathematics, which will play a very significant role in the further development of Calculus in the classroom: *e* lets you take a simple growth rate for a phenomenon (where all change happens at the end of a unit, say a year) and find the impact of compound, continuous growth, where every tiny interval (say a nanosecond) it is growing just a little bit. *e* shows up whenever systems grow exponentially and continuously such as radioactive decay, interest calculations, and more. Even jagged systems that don't grow smoothly can be approximated by *e*. Just like every number can be considered a "scaled" version of 1 (the base unit), every rate of growth can be considered a "scaled" version of *e* (the "unit" rate of growth). So *e* is not a random number**:** *e* represents the idea that all continually growing systems are scaled versions of a common rate.



x	f(x)	Rgf(x)	Rgf(x)/f(x)
0	1		
0.1	1.1052	1.1059	1.0006
0.2	1.2216	1.2223	1.0006
0.3	1.3501	1.351	1.0006
0.4	1.4922	1.4931	1.0006
0.5	1.6492	1.6503	1.0006
0.6	1.8228	1.824	1.0006
0.7	2.0146	2.0159	1.0006
0.8	2.2267	2.2281	1.0006
0.9	2.461	2.4626	1.0006
1	2.72	2.7217	1.0006
1.1	3.0063	3.0082	1.0006
1.2	3.3226	3.3247	1.0006
1.3	3.6723	3.6746	1.0006
1.4	4.0588	4.0614	1.0006
1.5	4.4859	4.4888	1.0006
1.6	4.958	4.9612	1.0006
1.7	5.4798	5.4833	1.0006
1.8	6.0565	6.0604	1.0006
1.9	6.6939	6.6982	1.0006
2	7.3984	7.4031	1.0006
2.1	8.177	8.1822	1.0006
2.2	9.0376	9.0433	1.0006
2.3	9.9887	9.995	1.0006
2.4	11.0399	11.0469	1.0006
2.5	12.2018	12.2095	1.0006
2.6	13.4859	13.4944	1.0006
2.7	14.9051	14.9146	1.0006
2.8	16.4738	16.4842	1.0006
2.9	18.2075	18.219	1.0006
3	20.1236	20.1364	1.0006

Figure 30: Zooming the graphs.

4. Final words

It has been our aim to explore growth via functional thinking in our students by interesting them in variability, how to measure it and the search for regularities. We devised our experience as a journey with an explorer mentality in an environment where challenges occur, allowing better focalization in mathematics as a process by encouraging the development of qualities as imagination, rationality, critical analysis and a bettering of communication skills and it was designed in a way to turn an interview into a learning process.

A suitable concept image of functional relationship had to be developed through a carefully planned and orderly implemented Socratic interview allowing for feedback and assessment stepping thoughtfully through the micro-development of it. Replicating the overall behavior of exponential functions, the experience proceeded very slowly at the beginning, taking time where periods of incubation were deemed necessary, mainly when trying to reconcile their previous notion of functional relationship with what appeared as new information. Once this stage was completed and in order to investigate different types of growth and their rates of growth, a computer-generated tool based on a powerful mathematical assistant was designed as a Graphic User Interface (GUI) allowing the combination of multiple windows (graphic and computational) with interactive capabilities to show different representations of the topics treated and simulate dynamism. The introduction of the GUI triggered quick responses and better performances on the interviewees and a gratifying sense of mental acceleration was palpable. At the end, the number e made its appearance when connecting (exponential) functions with their 'almost' local rate of growth.

It seems to us that this line of presentation allows this remarkable number to appear more naturally than through the traditional classroom approaches.

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