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ANALYSIS OF STUDENTS' MENTAL STRUCTURES WHEN INCORRECTLY CALCULATING THE LIMIT OF FUNCTIONS

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Abstract. In this paper, the authors attempt to make the analysis of unacceptable and missed student responses when calculating a limit of functions. Based on the collected data and derived conclusions, relying on *APOS* theory and the *SOLO* taxonomy, we estimate that the concept of limit values and processes with that concept (but not procedurally using this concept) for the majority of the student population is acceptable with considerable difficulty. A reconstruction is offered of student's mental images using categorical terms *RBC + C* theory of abstraction and Sfard's theoretical model for the learning of mathematical concepts.

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1. Introduction

In this paper, the authors offer their reflections on mental structures constructed in students' cognitive planes in solving a standard nonlinear complex task of calculating the limit of function. When students are faced with the need to calculate the limit of function, a certain number of these students is not thinking about using the applicable technologies on the calculation. This point out the need that lecturers of Calculus should be more engaged in the learning process in order to better approach the students with the concept of limit of functions and some of applicable algorithms for calculating these values. The research questions of this study were:

- Genetic decomposition algorithm for calculating the limit of functions using 'Logarithmization' and l'Hôpital's rule.
- What can *APOS* theory do in this case?
- Reconstruction of student thinking in the process of finding the limit.
- Which parameters can be deduced from the analysis of student responses relying on *RBC + C* theory of abstraction?

2. The goal of the paper

In this paper, we use research format similar to the procedures set forth in the well-known text "*Understanding the limit concept: Beginning with a coordinated process schema*" [10], described in more detail in the initial Eduard Dubinsky's text "*A learning theory approach to calculus*" [15] and effectively applied in Lynn Heather Bowie's dissertation "*A learning theory approach to students' misconceptions in Calculus*" [5].

In essence, our steps are as follows:

(1) Using the framework of theoretical perspectives *(APOS* theory), we expose students to the concept of the limit of function in the example of one such task that seeks to resolve non-linear access (calling at several different properties of mathematical objects and the establishment of their unity, thus establishing a correlation between them - relational level task within the *SOLO* taxonomy);

(2) We observe and evaluate a student's success in solving the given task;

(3) Analyzing students' mistakes, we are trying to determine when these errors occur in the procedures and also the type of errors;

(4) We are trying to understand why errors occur relying on elements of the *RBC + C* theory of abstraction;

(5) Reviewing the genetic decomposition of instructional teaching in this particular example, by repeating the procedure.

3. Background / Theoretical foundation

3.1. Literature Review

There are many international studies on the quality of student understanding of the concept of limit of function (for example, see [2], [8], [11], [18], [29], [31], [32], [34], [37], [38], [41], [46], [48], [49], [51]). In these studies the authors present the findings that the vast majority of tested students have significant difficulty in understanding the concept of limit of function. Furthermore, students have difficulties in understanding the process in which this concept occurs. They have difficulty with understanding and accepting of the characteristics of this concept as well as its use as a tool or as an object in the construction of other mathematical concepts. It also points out that the difficulties encountered in understanding this concept considerably complicate the understanding and mathematical acceptance of other very important concepts such as continuity, differentiability and integrability of functions (see, for example, [39], [48]). Some researchers in mathematics education, as, for example, Bernard Cornu [8] and Anna Sierpinska [46] present their observations in the form of a statement that a high percentage of students have a static view of the processes with mathematical objects. Students with such commitment only develop their procedural skills. A significant number of students with such a view of the interrelationships of mathematical objects have considerable difficulty in perceiving, understanding and accepting processes in which is incorporated the concept of mathematical objects and is resulting in abstraction (see [31]). Widely accepted substantial part of the international community of researchers in mathematics education can be illustrated as follows: student perception of the limit of function as a process that is something that never ends and therefore does not reach (for example: Jim Cottrill, Devilyna Nichols, Keith Schwingendorf, Karen Thomas and Draga Vidakovic (see [10]), and Eduard Dubinsky [16]. Even once Anna Sfard (for example, in the article [43]) pointed out that the limit of function should be seen as a dynamic structure. Although, some authors initially (for example, David

Tall in the article [48]) considered that the conceptualization of the dynamic structure is easy and natural for students, it seems that this is not so (for example, see [29], [34]). It is estimated that the problem in students' minds is the failure to realize the connection between the formal definition of the concept of the limit of function and its dynamic structure (see, for example [11], [18], [51]).

In reviewing the available international literature we did not find any reports of success in helping students overcome difficulties in understanding the dynamics and formally defined the concept of deterministic limit of functions. In this report, we will not offer our thoughts on the model that would allow a higher success in our efforts. What we do in this paper is to focus more attention on the before mentioned problems using the categories *APOS* theory by analyzing a specific task on the limit of function *(non-linear complex task*, or speaking the language of the *SOLO* taxonomy, task-level *'SOLO 4'* - relational level), which seeks an understanding of several aspects in which the results of applying these different approaches should be treated as independent of each other.

Defended master thesis (for example, [25], [28]), doctoral dissertations (for example [4], [7], [26], [35], [36], [42]) and a large number of published articles (for example, [3], [8], [11], [12], [22], [24], [27], [33], [34], [37], [38], [40], [46], [47], [49], [51]) represent the actuality of the problem.

3.2. *APOS* theory

Authors of *APOS* theory is Eduard Dubinsky and colleagues (see, for example, [9], [14], [17]). The name of this theory consists of the first letters of the following terms: Action, Processes, Objects and Schemes. We will explain these terms to the available literature: [1], [6], [19] and [50].

Action is the transformation of objects perceived by an individual, essentially foreign and as a requirement, either explicitly or from memory, step-by-step instructions on how to perform the operation. When the action is repeated and individual reflection in relation to the act to occur, then we can make the internal mental construction, which we call *the process,* a person who carries out activities is able to think of it as an exercise of the observed processes.

The objects were constructed during the process when an individual becomes aware of the process as a category and accept that some transformations can act on an object.

Finally, the *scheme* for a particular mathematical concept of a single collection made up of actions, processes, facilities, and other schemes that are connected through some form of general principles and framework in the mind of the individual in that situation can be treated fit. This framework must be coherent in the sense that it provides, explicitly or implicitly, the method of determining which phenomena are within the scheme and which are not. This approach implies that all subjects in it can represent in terms of actions, processes, facilities and schemes.

These four components - an action, process, object and scheme, here are presented in a hierarchical sequence. The authors (Jim Cottrill, Devilyna Nichols, Keith Schwingendorf, Karen Thomas, Draga Vidakovic) of this theoretical approach believe (for example, see [10]) that each individual constructs in this series must be constructed prior to the next. In reality the situation is not exactly linear.

In fact, when considering the conceptualization of an idea, but its development and its use in some processes, i.e., linking with other categorical concepts, the situation is much more complex since there may be a reverse impact.

3.3. *RBC + C* theory of abstraction

RBC (Recognizing *Building-with-Constructing) model* emphasizes the need for *abstraction* and stimulation in the process of abstraction. Activities under this theory are: *Recognition*, the *use of tools* (with building) and *construction.* Abstraction is one of the basic concepts used in this study, is defined in a simple way as "the process of moving from *recognizable* to *abstract"* [20]. Three mentioned epistemological activity will further explain:

Identifying and the use of pre-formed structures. Thus, under the 'recognition' known mathematical structure, in terms of the thought occurs when the mind of him who learns meet, identify and understand (as a whole but also by parts) this structure in a mathematical context in which that forests moves [20].

In the process *'use of elements and tools'* student’s mind still is not enriched with new complex knowledge is already used existing knowledge to understand and solve new paradigm within the mathematical concept which is moving in search of new knowledge [21].

The process of *'designing'*, also known as (re)arranging for upgrading, is the process of constructing new knowledge. It is the process by which existing components of mathematical knowledge possibly supplemented with other components, the new way of establishing a mutually regulating complex arrangement in the previous grid knowledge and thus gain new meanings.

In an effort to *RBC* through these theories offer the all acceptable justification in the formation of new structures of knowledge, Dreyfus [13] suggested its expansion process *'consolidation'* forming a *C + RBC* theory of abstraction.

3.4. Sfard's theoretical model for the learning of mathematical concepts

A mathematical concept is a complex web of ideas developed from mathematical definitions and mental constructs [44, 45]). We use Sfard's model of learning mathematics process. Sfard's [44, 45] theoretical model for the learning of mathematical concepts encompasses both operational (procedural, algorithmic) understanding and structural (conceptual, abstract) understanding, characterizing both as necessary and complementary. According to Sfard [44], when learning a new concept, a natural starting point is through a definition. Some mathematical definitions treat concepts as objects that exist and are components of a larger system. This is considered a structural conceptualization. On the other hand, concepts can also be defined in terms of processes, algorithms, or actions leading to an operational conception. A structural conception requires the ability to visualize the mathematical concept as a "real thing" that exists as part of an abstract mathematical structure, whereas an operational conception implies more of a potential that requires some action or procedure to be realized. Sfard emphasizes that the operational and structural conceptions are not mutually exclusive; they are complementary. The two aspects of conception can be considered as two sides of the same coin; both are critical to building a deep understanding of mathematics.

4. The task

Task: Calculate

$$\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} .$$

4.1. Preliminary genetic decomposition

1. We should observe the object $f(x) = (ctgx)^{\frac{1}{lnx}}$ and specify the domain of the function $x \to f(x)$: The domain of this function is determined by the following conditions: (1) ctg $x > 0$, (2) $ln\ x \neq 0$, (3) $x > 0$. Solutions of the first two inequalities are:

(1) ctg$x > 0 \Leftrightarrow x \in (0 + k\pi, \frac{\pi}{2} + k\pi)$, where is $k \in \boldsymbol{Z}$.

(2) $lnx \neq 0 \Leftrightarrow x \in (0,1) \cup (1,+\infty)$.

So, if we want to assess $\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}}$ observe the boundary process 'behavior functions $(ctgx)^{\frac{1}{lnx}}$ for $x \to 0^+$'. To this end, we choose the variable *x* from the interval with (0,1). For selected values variable worth *ctgx* $x > 0$ and $lnx < 0$. This part of the activities in solving this task we recognize as 'action'.

2. The equality $f(x) = (ctgx)^{\frac{1}{lnx}} = M$ wherein $x \in (0, 1)$, we apply natural logarithm *(ln):*

$$ln\text{M} = \frac{1}{lnx} ln(ctgx).$$

It follows

$$\lim_{x \to 0^+} lnM = \lim_{x \to 0^+} \frac{1}{lnx} ln(ctgx).$$

This part of the activities recognized as preliminary activities at the object. The second part of the activities at the object is determined in the following point.

3. Since the function $x \to lnx$ is a continual function on the interval $\langle 0,+\infty \rangle$, we conclude that it is worth

$$\lim_{x\to 0^+} lnM = ln(\lim_{x\to 0^+} M) = \lim_{x\to 0^+} \frac{1}{lnx} ln(ctgx).$$

4. How $lnx \to -\infty$ and $ln(ctgx) \to +\infty$ when $x \to 0^+$, and how the conditions of the l'Hôpital's rule[1] are fulfilled, we have

$$ln(\lim_{x\to 0^+} M) = \lim_{x\to 0^+} \frac{1}{lnx} ln(ctgx)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{ctgx}\cdot(\frac{-1}{sin^2x})}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{-x}{cosx\ sinx}$$

$$= \lim_{x\to 0^+} \frac{x}{sinx}\cdot\frac{-1}{cosx}$$

$$= -1$$

because $\lim_{x\to 0^+} \frac{x}{sinx} = 1$ and $cos\ 0 = 1$. Application l'Hôpital's rule in this decomposition can be regarded as a separate and pre-designed scheme but its application in this case we recognize as part of the process. The second part of the process is anti-logarithm.

5. Finally, we obtain

$$\lim_{x\to 0^+} (ctgx)^{\frac{1}{lnx}} = e^{-1}. \quad \square$$

4.2. Experiences in Teaching

Tested were 48 candidates - first-year students study program for the education of mechanical engineers at the Faculty of Mechanical Engineering, University of Banja Luka.

The test results / Distribution of student successfulness (N = 48) are represented in the following table:

Efficacy	∅	0	Acceptable with mistakes	Without mistakes	Σ
Number	19	20	4	5	**48**
Frequency	39.58	41.67	8.33	10.42	**100.00**

Table 1

Legend: The symbol '∅' denotes students who did not offer any answers to the task. Symbol '0' indicates completely unacceptable information that students offered as a solution to the problem.

Illustration of some students' mistakes

Example 1 Despite the strongly efforts, the authors of this text were not in position to reconstruct student's considerations in the design equation

$$\lim_{x\to 0^+} (ctgx)^{\frac{1}{lnx}} = \lim_{x\to 0^+} (1 - \frac{cosx}{sinx} + 1)^{\frac{1}{lnx}}.$$

We tend to judge it comes to 'slip of the tongue' so that the next row, row (2), we get a correct transformation of the previous hypothesized that if -1 instead of +1. Line (3) and line (4) gave the

[1] Guillaume François Antoine, Marquis de l'Hôpital (1661 -1704), a French mathematician.

correct algebraic transformations: (2) → (3) and (3) → (4). The error is observed in the transformation (4) → (5). It is unacceptable transformation

$$\lim_{x\to 0^+}\left[(1-\frac{cosx+sinx}{sinx})^{\frac{1}{lnx}}\right]^{\frac{sinx}{cosx+sinx}\frac{cosx+sinx}{sinx}} = e^{\lim_{x\to 0^+}\frac{(cosx+sinx)}{sinx}\frac{1}{lnx}}$$

estimating transform unscrews it in the following way:

$$\lim_{x\to 0^+}\left[(1-\frac{cosx+sinx}{sinx})^{\frac{1}{lnx}}\right]^{\frac{sinx}{cosx+sinx}\frac{cosx+sinx}{sinx}} = \lim_{x\to 0^+}\left[(1-\frac{cosx+sinx}{sinx})^{\frac{sinx}{cosx+sinx}}\right]^{\frac{1}{lnx}\frac{cosx+sinx}{sinx}}$$

Thus, the inadmissibility of the offered solutions rectify this task stems from the false identification

$$\lim_{x\to 0^+}\left(1-\frac{cosx+sinx}{sinx}\right)^{\frac{sinx}{cosx+sinx}} = \text{e}.$$

$$\lim_{x\to 0^+}(ctgx)^{\frac{1}{lnx}} = \lim_{x\to 0^+}(1-\frac{cosx}{sinx}+1)^{\frac{1}{lnx}} \qquad \text{line (1)}$$

$$= \lim_{x\to 0^+}(1-\frac{cosx+sinx}{sinx})^{\frac{1}{lnx}} \qquad \text{line (2)}$$

$$= \lim_{x\to 0^+}\left[(1-\frac{cosx+sinx}{sinx})^{\frac{1}{lnx}}\right]^{\frac{sinx}{cosx+sinx}\frac{cosx+sinx}{sinx}} \qquad \text{line (3)}$$

$$= \lim_{x\to 0^+}\left[(1-\frac{cosx+sinx}{sinx})^{\frac{sinx}{cosx+sinx}}\right]^{\frac{1}{lnx}\frac{cosx+sinx}{sinx}} \qquad \text{line (4)}$$

$$= e^{\lim_{x\to 0^+}\frac{(cosx+sinx)}{sinx}\frac{1}{lnx}} \qquad \text{line (5)}$$

$$= \text{e}^0 = 1. \qquad \text{line (6)}$$

The following example is analogous to the preceding can be observed similarly as erroneous identification

$$\lim_{x\to 0^+}(1+(ctgx-1))^{\frac{1}{ctgx-1}} = \text{e}.$$

Example 2 $\lim_{x\to 0^+}(ctgx)^{\frac{1}{lnx}} = \lim_{x\to 0^+}(1+\frac{cosx}{sinx}-1)^{\frac{1}{lnx}}$

$$= \lim_{x\to 0^+}\left[(1+(ctgx-1))^{\frac{1}{ctgx-1}}\right]^{(ctgx-1)\frac{1}{lnx}}$$

$$= e^{\lim_{x\to 0^+}\frac{ctgx-1}{lnx}} = |L.P.| = e^{\lim_{x\to 0^+}\frac{-\frac{1}{1+x^2}}{\frac{1}{x}}}$$

$$= \text{e}^0 = 1. \square$$

The following two examples of student errors are not demanding reconstructed.

Example 3 $\lim_{x\to 0^+}(ctgx)^{\frac{1}{lnx}} = (+\infty)^{\frac{1}{1}} = +\infty.$ □

Example 4 $\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = \lim_{x\to 0^+}\left(\frac{1}{tgx}\right)^{\frac{1}{lnx}} = \lim_{x\to 0^+}\left(\frac{1^{\frac{1}{lnx}}}{tgx^{\frac{1}{lnx}}}\right) = \left(\frac{1}{0^+}\right) = +\infty$. □

The example that follows, in the fifth step in identifying $\lim_{x \to 0^+}\frac{1}{lnx}=0$, can be used for demonstration didactic situations when wrongly executed steps from school mathematics lead to unacceptable solutions:

Example 5 Let be $\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = L$. Applying the natural logarithm to the left- and right-side of this equality, we obtain

$$lnL = ln\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = \lim_{x \to 0^+} ln(ctgx)^{\frac{1}{lnx}} = \lim_{x \to 0^+}\frac{1}{lnx}ln(ctgx) = \lim_{x \to 0^+}ln(ctgx)$$

because $\lim_{x \to 0^+}\frac{1}{lnx} = 0$. Furthermore, since we have

$$\lim_{x \to 0^+}ln(ctgx) = \lim_{x \to 0^+}ln\left(\frac{cosx}{sinx}\right) = \lim_{x \to 0^+}ln\left(\frac{\sqrt{1-sinx^2}}{sinx}\right) = ln\frac{0}{1} = 1$$

we conclude that $L = e^1$. □

The last example is a demonstration of how to deduce the correct start of a strange conclusion.

Example 6 Let be $\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = C$. Using *ln* on this equality, we obtain

$$lnC = ln\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = \lim_{x \to 0^+}ln(ctgx)^{\frac{1}{lnx}} = \lim_{x \to 0^+}\frac{1}{lnx}ln(ctgx).$$

From here, further, followed by

$$\lim_{x \to 0^+}\frac{1}{lnx}\cdot\lim_{x \to 0^+}ln(ctgx) = (0 - \varepsilon).$$

Therefore, it is

$$C = e^{0-\varepsilon} = \frac{e^0}{e^\varepsilon} = e^{-\varepsilon} \approx 1.\ □$$

4.3. The second model of genetic decomposition. We use a simple algebraic transformation

$$(ctgx)^{\frac{1}{lnx}} = e^{\frac{1}{lnx}\ln(ctgx)}.$$

From here, because the function *Exp*: $x \to e^x$ is a continus function, it follows

$$\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = e^{\lim_{x\to 0^+}\frac{1}{lnx}\ln(ctgx)}$$

Here, as in the first model, we have

$$\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = e^{-1}.\ □$$

4.4. The third model of genetic decomposition. If we denote

$$\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = \lim_{x\to 0^+}\left(\frac{cosx}{sinx}\right)^{\frac{1}{lnx}} = L,$$

we have

$$lnL = ln\lim_{x \to 0^+}\left(\frac{cosx}{sinx}\right)^{\frac{1}{lnx}} = \lim_{x \to 0^+}ln\left(\frac{cosx}{sinx}\right)^{\frac{1}{lnx}} = \lim_{x \to 0^+}\frac{1}{lnx}ln\left(\frac{cosx}{sinx}\right) = \lim_{x \to 0^+}\frac{1}{lnx}(lncosx - lnsinx)$$

$$= \lim_{x \to 0^+} \frac{1}{lnx}(lncosx) - \lim_{x \to 0^+} \frac{1}{lnx}(lnsinx) = - \lim_{x \to 0^+} \frac{1}{lnx}(lnsinx)$$

because $\lim_{x \to 0^+} \frac{1}{lnx}(lncosx) = 0$. Other limes, since $\lim_{x \to 0^+} lnx = -\infty$ and $\lim_{x \to 0^+}(lnsinx) = -\infty$, we calculate by applying l'Hôpital's rule:

$$- \lim_{x \to 0^+} \frac{lnsinx}{lnx} = - \lim_{x \to 0^+} \frac{\frac{1}{sinx} \cdot cosx}{\frac{1}{x}} = \quad - \lim_{x \to 0^+} \frac{x}{sinx} \cdot cosx = -1.$$

From here we obtain $\lim_{x \to 0^+}(ctgx)^{\frac{1}{lnx}} = e^{-1}$. □

4.5. The general model of genetic decomposition. Suppose you had a requirement to calculate $\lim_{x \to c}(f(x))^{g(x)}$, where *f* and *g* are functions such that *f*(*x*) > 0 in the vicinity of the point *c*. Proceed in the following manner:

***First step**:* If the L mark $f(x)^{g(x)}$, we have: $lnf(x)^{g(x)} = g(x)lnf(x) = lnL$.
Possible failures and possible errors:

1. *Do not determine the scope of the function* $x \to f(x)^{g(x)}$*;*
2. *Unnoticeably of necessity of applying the procedure described above*;
3. *Incorrect using of the properties of the logarithm*.

Note that the application procedure

$$(f(x))^{g(x)} = e^{g(x)\ln f(x)},$$

although it is correct, strongly complicates recording which also opens up the possibility of occurrence of some other flaws.

***Second step**:* Since the $x \to lnx$ is a continuous function where it is defined, followed by

$$ln(\lim_{x \to c} L) = \lim_{x \to c} lnL = \lim_{x \to c} g(x)lnf(x).$$

Suppose we set up an additional requirement: *'Describe the reasoning methods to solve the task'*, this step allowed us to determine the student's understanding of the concept of continuity of the function *ln*: $x \to lnx$

***Step Three**:* If there is a natural way, we can calculate the limit $\lim_{x \to c} g(x)lnf(x) = a$ and after that we get anti-logarithm

$$L = e^a.$$

***The fourth step** (optional):* If in the calculation of limit values have difficulty $\lim_{x \to c} g(x)lnf(x)$ forms

$$\lim_{x \to c} g(x) = 0 \wedge \lim_{x \to c} lnf(x) = \infty$$

or forms

$$\lim_{x \to c} g(x) = \infty \wedge \lim_{x \to c} lnf(x) = 0 \,,$$

we can apply l'Hôpital's rule on $\lim_{x \to c} g(x)lnf(x) = \lim_{x \to c} \frac{lnf(x)}{\frac{1}{g(x)}}$ after prior transformation

$$\lim_{x \to c} \frac{lnf(x)}{\frac{1}{g(x)}} = \lim_{x \to c} \frac{\frac{1}{f(x)}f'(x)}{-\frac{g'(x)}{g(x)^2}} \,.$$

Possible failures and possible errors:

4. *Ignoring the conditions for the application of l'Hôpital's rule*;
5. *Improper use l'Hôpital's rule*;
6. *Incorrectly calculating the deduction*.

Step Five: If exists $\lim_{x \to c} \frac{\frac{1}{f(x)} f'(x)}{-\frac{g'(x)}{g(x)^2}} = a$, then $ln(\lim_{x \to c} L) = a$, and make anti-logarithm of the last equality yields the desired result

$$\lim_{x \to c}(f(x))^{g(x)} = e^a. \ \square$$

According to the applied procedure we estimate this task is a nonlinear complex task and it is a relational type (speaking in *the SOLO* taxonomy language).

Possible failures and possible errors:

7. *Unacceptable procedure in calculating limes*;
8. *Wrong calculated the value*;
9. *Omission of anti-logarithmization*.

Distribution of students' mistakes (N = 24):

Type of mistake	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**	**9.**
Number	23	1	0	22	17	19	19	20	7
Frequency	95.83	4.17	0.00	91.67	70.83	79.17	79.17	83.33	29.17

Table 2 (Results of the written assessment.)

For identifying problems that may indicate a misconception, standard quantitative analysis was used. First, we summarize student performance on the written assessment and our process for identifying problems that indicated a persistent error. Second, we investigate the persistent errors through analysis of student responses obtained during individual interviews. Results of the written assessment are presented in Table 1. The percentage of students who correctly answered the problems across the test is indicated misunderstanding of limit concept and unsatisfactorily accepted used algorithms in this calculating. Distribution of students mistakes by types are presented in Table 2.

5. Concluding Observations

To make an insight into the mental structures that are generated in the minds of students in finding a solution of the given task, we rely on two theoretical bases: 'chunk-by-chunk' analysis and elements of *APOS* theory. The collected data received from this test suggest the conclusion that the understanding of the concept of limit of functions and fluency of work with this concept is one of the most demanding proficiencies. By deducing from this task and also from a significant number of other examples of limits of functions, form the hypothesis that the students of technical faculties construct a mental image of this abstract idea with much difficulty. Our many years of experience suggest that this problem is to be presented to student with extreme carefulness.

However, our knowledge about the design of learning suggests that it should be planned with more time in the realization of both lectures and excesses with students in an effort to help them develop their mental structures at the level of processes, facilities and schemes (speaking in the language of *APOS* theory). Speaking in the language of *RBC + C* theory of abstraction implies that teaching and student's learning should focus on:

1. Type recognition of function and consideration of accumulation points of function domain observed functions in which should claim the limit;
2. The choice of tools applicable to the process of computing;
3. The process of computation;
4. Modeling using a number of variations of the scheme (consolidation).

Graphic access to a significant number of cases allows the developing of mental structures on the 'process level' and also at the 'level of the building'. The focus on the symbolic structures should help us in understanding the concept of constructed object. If the scheme allows understanding and connection procedures: 'Action → Process → Object → Actions' in an acceptable manner in a particular case, then it should be part of the course. Estimating the impact of such a focus in the implementation in teaching requires further research.

Following Sfard's emphasis that the operational and structural conceptions are complementary, we have to recognize these conceptions in student's calculations. The transition from operational understanding to structural understanding occurs in stages and it is a long and "inherently difficult" process. As students move from operational to structural understanding, they go through the three Sfard's stages: interiorization, condensation, and reification. During the first stage, interiorization, the student becomes skilled at performing processes involving the concept until these processes can be carried out mentally and with ease. During the second stage, condensation, the learner is able to think about a complicated process as a whole without need to carry out the details. The student is able to break the process into manageable units without loosing sight of the whole. In this stage, there is also a growing facility with moving between different representations, recognizing similarities, and making connections. For example, students may recognize problems with calculating the limit $\lim_{x \to c} g(x) lnf(x)$ in the step three. By recognizing the similarities between this situations and conditions in the l'Hôpital's rule, students begin to see both as specific examples of the concept of the limit. Reification represents a significant shift in thinking, one in which the concept is suddenly seen as part of a larger mathematical structure. It is at this stage that students begin to operate with a concept as an object and as the input into new processes.

Here we should point out that a significant part of misunderstanding the concept of limit of functions derives from the trinity of this concept: The symbol $\lim_{x\to c} f(x)$ *is also* a number (i.e, the object), the process - the behavior of the function in the neighborhood of the point c (c is a point of accumulation of domain of function f) and the procedure for calculating the limit. However, in contrast to the concepts of the so-called School mathematics (where the algorithm associated with a concept used to calculate the specific value of this concept), the limit of functions has no universal algorithm that works in all cases. Furthermore, the concept of the limit of function is not limited to computation in a finite number of steps to give a definitive response to the request. This is exactly the place, in the language of *APOS* theory, where begins the difference between 'actions' and 'processes'.

Useful insight into the relevant mental structures towards which teaching should focus, was revealed by the APOS genetic decomposition of the limit of a function concept. The findings of this study confirmed that the limit of a function concept is one that students find difficult to understand, and suggests that this is possibly the result of many students not having appropriate mental structures at the process, object, and schema levels. Many researchers (for example, see [31]) consider that APOS genetic decomposition is adequate. However, general reflections on the teaching design indicated that more time needs to be devoted to helping students develop the mental structures at the process, object, and schema levels.

Based on our analysis of the students' reasoning, we are able to build on the APOS framework by addressing the question of how students correlate processes described from Step 1 to Step 5. We estimate that our genetic decomposition of limit could include the first three actions of the APOS framework, along with the following three activities that address the process of coming to accept and understand the concept and procedure of informal calculation of the limit:

$$L = f(x)^{g(x)} \Leftrightarrow lnL = g(x) lnf(x),$$

$$\lim_{x \to c} L = e^a \Leftrightarrow ln(\lim_{x \to c} L) = \lim_{x \to c} g(x) lnf(x)$$

and

$$\lim_{x \to c} g(x) lnf(x) = \lim_{x \to c} \frac{lnf(x)}{\frac{1}{g(x)}} = \lim_{x \to c} \frac{lnf(x)}{\frac{1}{g(x)}} = \lim_{x \to c} \frac{\frac{1}{f(x)} f'(x)}{-\frac{g'(x)}{g(x)^2}} = a.$$

In this study we identified and characterized persistent errors made when university students are calculating the limit of function. We identify nine persistent errors. Upon analyzing these errors, we conjectured that all of them arise from an underdeveloped conception of function and processes with it.

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