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MINIMUM CONVEX COVER OF SPECIAL NONORIENTED GRAPHS

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A vertex set *S* of a graph *G* is *convex* if all vertices of every shortest path between two of its vertices are in *S*. We say that *G* has a *convex p-cover* if $X(G)$ can be covered by *p* convex sets. The *convex cover number* of *G* is the least $p \geq 2$ for which *G* has a convex *p*-cover. In particular, the *nontrivial convex cover number* of *G* is the least $p \geq 2$ for which *G* has a convex *p*-cover, where every set contains at least 3 elements. In this paper we determine convex cover number and nontrivial convex cover number of special graphs resulting from some operations. We examine graphs resulting from join of graphs, cartesian product of graphs, lexicographic product of graphs and corona of graphs.
Keywords: *nonoriented graphs, convex covers, convex number, operations, join, cartesian product, lexicographic product, corona.*

ACOPERIREA CONVEXĂ MINIMĂ A GRAFURILOR SPECIALE NEORIENTATE

Mulţimea de vârfuri *S* ale grafului *G* se numeşte *convexă* dacă pentru orice două vârfuri *x*, *y* din *S* toate vârfurile ce aparţin tuturor lanţurilor de lungime minimă cu extremităţile *x*, *y* se conţin în *S*. Se spune că *G* conţine o *p-acoperire convexă* dacă $X(G)$ poate fi acoperită cu *p* mulţimi convexe. *Numărul acoperirii convexe* al lui *G* este cel mai mic număr $p \geq 2$, pentru care *G* conţine o *p*-acoperire convexă. În particular, *numărul acoperirii convexe netriviale* al lui *G* este cel mai mic număr $p \geq 2$, pentru care *G* conţine o *p*-acoperire convexă, în care orice mulţime constă din cel puţin 3 vârfuri. În această lucrare noi determinăm numărul acoperirii convexe şi numărul acoperirii convexe netriviale al unor clase speciale de grafuri obţinute din următoarele operaţii pe grafuri: suma, produsul cartezian, produsul lexicografic, coroana.
Cuvinte-cheie: *grafuri neorientate, acoperiri convexe, numărul acoperirii convexe, operaţii, suma grafurilor, produs cartezian, produs lexicografic, coroană.*

Introduction

In this paper we consider only connected and nonoriented graphs. We denote by *G* a graph with vertex set $X(G)$ and edge set $U(G)$. An edge joining two vertices *x* and *y* in *G* is denoted by *xy*. The distance between vertices *x* and *y* in *G* is denoted by $d(x, y)$. The *diameter* of a graph is the length of the shortest path between the most distant nodes.

A set $S \subseteq X(G)$ is a *clique* if every pair of vertices of *S* is adjacent in *G*. The *neighborhood* of a vertex *x* of $X(G)$ is the set of all vertices *y* of $X(G)$ such that *x* and *y* are adjacent, and it is denoted by $\Gamma(x)$. A vertex *x* is called *simplicial* if $\Gamma(x)$ is a clique. Also, a vertex *x* is called *universal* if $\Gamma(x) = X(G) \setminus \{x\}$. Let *S* be a subset of $X(G)$. We say that $G[S]$ is the subgraph of *G induced* by *S.*

Now we remind some concepts from [1]. The *metric segment* $\langle x, y \rangle$ is the set of all vertices lying on a shortest path between vertices *x* and *y* in *G*. A set $S \subseteq X(G)$ is called *convex* if $\langle x, y \rangle \subseteq S$ for all $x, y \in S$. The *convex hull* of $S \subseteq X(G)$, denoted $d-conv(S)$, is the smallest convex set containing *S*.

A family of sets is called *convex cover* of $G = (X;U)$ and is denoted by $\mathcal{P}(G)$ if the following conditions hold:

(i) Every set of $\mathcal{P}(G)$ is convex in *G*.

(ii) $X(G) = \bigcup_{S \in \mathcal{P}(G)} S$.

(iii) $S \not\subset \bigcup_{C \in \mathcal{P}(G), C \neq S} C$, for every $S \in \mathcal{P}(G)$.

If $|\mathcal{P}(G)| = p$, then this family is called *convex p-cover* of *G* and is denoted by $\mathcal{P}_p(G)$ [2].

A convex cover $\mathcal{P}(G)$ of graph G is called *nontrivial convex cover* if every set $S \in \mathcal{P}(G)$ satisfies the inequalities: $3 \le |S| \le |X(G)| - 1$. The minimum number of cliques that cover all the vertices of a graph is known as a *clique cover number* $\theta(G)$, introduced by Berge [3]. Also, *convex cover number* $\varphi_c(G)$ was defined as the least $p \ge 2$ for which G has a convex p-cover [2]. Similarly to $\varphi_c(G)$, we introduced *nontrivial convex cover number* $\varphi_{cn}(G)$ [4].

Note that there are graphs for which there are no nontrivial convex covers. For instance, every convex simple graph has no nontrivial convex covers. A graph G is called *convex simple* if it does not contain nontrivial convex set [5]. Let us remark that if G has a nontrivial convex cover, then we have $\varphi_c(G) \le \varphi_{cn}(G)$.

The *minimum convex cover* $\mathcal{P}_{\varphi_c}(G)$ is the convex p-cover of graph G such that $p = \varphi_c(G)$. In the same way, we define *minimum nontrivial convex cover* $\mathcal{P}_{\varphi_{cn}}(G)$ and *minimum clique cover* $\mathcal{P}_\theta(G)$ of graph G.

By $P(G)$ we denote a family of convex sets, where $X(G) = \bigcup_{S \in P(G)} S$. We denote by $\mathcal{P}(P(G))$ a convex cover of G that consists of sets, which belong to $P(G)$.

A nonempty subset S of $X(G)$ is *a nonconnecting* set in G if for every pair of vertices $x, y \in X(G) \setminus S$ with $d(x, y) = 2$ we have $\Gamma(x) \bigcap \Gamma(y) \bigcap S = \varnothing$.

A map $p_G : X(G*H) \to X(G)$, $p_G((g,h)) = g$, is the *projection* onto G and $p_H : X(G*H) \to X(H)$, $p_H((g,h)) = h$, the *projection* onto H, where G and H are two graphs and * is one of two operations: cartesian product, lexicographic product.

Convex cover of a graph was studied by many mathematicians. Any latest results on graph convex covers are given in [2, 4, 6-8]. Deciding whether a graph G has a convex p-cover or a nontrivial convex p-cover for a fixed $p \ge 2$, it is known to be NP-complete [2, 4]. Besides, convexity was studied in some graph operations [9-11]. Further, there is particular interest in establishing of convex cover number and nontrivial convex cover number for special graphs resulting from graph operations, such as join of graphs, cartesian product of graphs, lexicographic product of graphs and corona of graphs.

Preliminary Results

Firs, note that for a given $P(G)$, which has no set $X(G)$, we can easily obtain $\mathcal{P}(P(G))$ by removing from $P(G)$ all sets contained in the union of other sets of the family $P(G)$. It can easily be checked that Propositions 1, 2 and 3 are true.

Proposition 1. *Let G be a connected graph of order $n \ge 2$. Then for every vertex $x \in X(G)$ there is a convex set $S \in X(G)$ such that $x \in S$ and $|S| = 2$.*

Proposition 2. *Let G be a connected graph of order $n \ge 3$. There exists $\mathcal{P}_{\varphi_c}(G)$ such that for every set $S \in \mathcal{P}_{\varphi_c}(G)$ condition $|S| \ge 2$ holds.*

Proposition 3. *Let G be a connected graph of order $n \ge 3$. There exists $\mathcal{P}_\theta(G)$ such that for every set $S \in \mathcal{P}_\theta(G)$ condition $|S| \ge 2$ holds.*

Theorem 1. *Let G be a connected graph of order $n \ge 3$ that contains a universal vertex. Then for every vertex $g \in X(G)$ there is a convex set $S \in X(G)$ such that $g \in S$ and $|S| = 3$.*

Proof. Let x be a universal vertex of G and $\Gamma(x) = X(G) \setminus \{x\}$. Suppose that $G[\Gamma(x)]$ is a disconnected graph. This means that there are two connected components $G_1[\Gamma(x)]$ and $G_2[\Gamma(x)]$. Further, for every two vertices $x_1 \in X(G_1[\Gamma(x)])$ and $x_2 \in X(G_2[\Gamma(x)])$ we get a convex set $\{x, x_1, x_2\}$, and this set is nontrivial.

Now suppose that $G[\Gamma(x)]$ is a connected graph. In this case every vertex y of $X(G) \setminus \{x\}$ has an adjacent vertex $z \in X(G) \setminus \{x\}$. Hence, set $\{x, y, z\}$ is convex and consists of three vertices. □

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Consequence 1. *Let G be a connected graph of order $n \geq 4$ that contains a universal vertex. Then, G has a nontrivial convex cover.*

Consequence 2. *Let G be a connected graph of order $n \geq 4$ that contains a universal vertex. Then, $\varphi_c(G) = \varphi_{cn}(G)$.*

Join of Graphs

The *join* of graphs *G* and *H*, denoted $G \vee H$, is a graph with $X(G \vee H) = X(G) \bigcup X(H)$ and $U(G \vee H) = U(G) \bigcup U(H) \bigcup U\{xy : x \in X(G), y \in X(H)\}$.

Theorem 2 [9]. *Let G be a connected graph and K_m the complete graph of order m. Then a proper subset $C = S_1 \bigcup S_2$ of $X(G \vee K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$, is convex in $G \vee K_m$ if and only if either*

(i) *S_1 is a clique in G, or*

(ii) *$S_1 \subseteq X(G) \setminus S$ and $S_2 = X(K_m)$ for some nonconnecting set S of G.*

Theorem 3. *Let G be a noncomplete graph on n vertices with diameter* 2 *and K_m the complete graph of order $m \geq 1$. Let $C = S_1 \bigcup S_2$ be a proper convex subset of $X(G \vee K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$. Then S_1 is convex in G.*

Proof. By Theorem 2, let us consider two cases. Firstly, if S_1 induces a complete subgraph of *G*, then evidently it is convex in *G*. Without loss of generality it can be assumed that S_1 does not induce a complete subgraph of *G*. Thus, $S_1 \subseteq X(G) \setminus S$ and $S_2 = X(K_m)$ for some nonconnecting set *S* of *G*. Assume further that S_1 is not convex in *G*. Let *x* and *y* be two vertices of S_1 such that there exists a vertex $z \in \langle x, y \rangle_G$ that does not belong to S_1. Since diameter of *G* is 2, we obtain $d_G(x, y) = d_{G \vee K_m}(x, y) = 2$ and $z \in \Gamma_G(x) \bigcap \Gamma_G(y)$. Hence, $z \in S_1$. From definition of nonconnecting set, $\Gamma_G(x) \bigcap \Gamma_G(y) \bigcap S = \varnothing$ and consequently $z \notin S$. Thus, Theorem 2 is satisfied and therefore there is a contradiction. Furthermore, *S* is convex in *G*. □

Theorem 4. *Let G be a connected graph on $n \geq 1$ vertices and K_m the complete graph of order $m \geq 1$. Then, the following statements hold.*

1) *If G is complete, then $\varphi_c(G \vee K_m) = 2$.*

2) *If G is complete and $n + m \geq 4$, then $\varphi_{cn}(G \vee K_m) = 2$.*

3) *If G is noncomplete with diameter* 2, *then $\varphi_c(G \vee K_m) = \varphi_{cn}(G \vee K_m) = \varphi_c(G)$.*

4) *If G is noncomplete with diameter at least* 3, *then $\varphi_c(G \vee K_m) = \varphi_{cn}(G \vee K_m) \leq \varphi_c(G)$.*

Proof.

1) Suppose $G = K_n$. Then, by definition of the join of two graphs, it follows that $G \vee K_m$ also is complete. Here graphs K_n and K_m are nonempty. Further, we obtain $\varphi_c(K_n \vee K_m) = 2$.

2) Suppose $G = K_n$ and $n + m \geq 4$. As before, $G \vee K_m$ is complete. Since every nontrivial convex set has at least three elements, we have $\varphi_{cn}(K_n \vee K_m) = 2$.

3) Suppose *G* is noncomplete graph with diameter 2. Let *C* be a proper convex subset of $X(G \vee K_m)$, which satisfies conditions of Theorem 2. It follows from Theorem 3 that $X(G) \bigcap C$ is convex set in *G*. Let $\mathcal{P}_{\varphi_c}(G \vee K_m)$ be a minimum convex cover of $G \vee K_m$. We get family of sets $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \vee K_m)} \{X(G) \bigcap S\}$. It is clear that $P(G)$ has no set $X(G)$. This yields that $|\mathcal{P}(P(G))| \leq \varphi_c(G \vee K_m)$. In fact, we obtain inequality $\varphi_c(G) \leq \varphi_c(G \vee K_m)$.

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By Proposition 2, a connected graph G on $n \geq 3$ vertices has a minimum convex cover $\mathcal{P}_{\varphi_c}(G)$ such that for every set $S \in \mathcal{P}_{\varphi_c}(G)$ condition $|S| \geq 2$ holds. Hence, we obtain a nontrivial convex cover $\mathcal{P}(G \vee K_m)$ of $G \vee K_m$, adding $X(K_m)$ to Y_i, where $Y_i \in \mathcal{P}_{\varphi_c}(G)$, for $1 \leq i \leq \varphi_c(G)$. Note that $|\mathcal{P}(G \vee K_m)| = \varphi_c(G)$ and $\varphi_c(G \vee K_m) \leq \varphi_{cn}(G \vee K_m) \leq \varphi_c(G)$. Continuing this line of reasoning, we see that $\varphi_c(G \vee K_m) = \varphi_{cn}(G \vee K_m) = \varphi_c(G)$.

4) Now, assume that G is noncomplete and its diameter is at least 3. As above, it is easy to prove that every minimum convex cover of G, which satisfies Proposition 2, generates a nontrivial convex cover of $G \vee K_m$. Thence, $\varphi_{cn}(G \vee K_m) \leq \varphi_c(G)$. Note also that there are noncomplete graphs W, with diameter at least 3, for which strict inequality $\varphi_{cn}(W \vee K_m) < \varphi_c(\mathrm{W})$ holds. For instance, graph represented in Figure 1 is the join of graphs W and K_1, where $X(K_1) = \{k\}$. This graph has minimum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}}(W \vee K_1) = \{\{x_1, x_7, x_9, k\}, \{x_2, x_8, x_{10}, k\}, \{x_3, x_5, k\}, \{x_4, x_6, k\}\}$, but graph W has minimum convex cover $\mathcal{P}_{\varphi_c}(W) = \{\{x_1, x_3\}, \{x_5, x_7\}, \{x_2, x_4\}, \{x_6, x_8\}, \{x_9\}, \{x_{10}\}\}$ and further $\varphi_{cn}(W \vee K_1) = 4$, but $\varphi_c(G) = 6$.

We stress that nontrivial convex cover is a particular case of convex cover. Since any vertex of $k \in X(K_m)$ is universal in $G \vee K_m$, Consequence 2 implies that the equality holds $\varphi_c(G \vee K_m) = \varphi_{cn}(G \vee K_m)$. Thus, we obtain $\varphi_c(G \vee K_m) = \varphi_{cn}(G \vee K_m) \leq \varphi_c(G)$. □

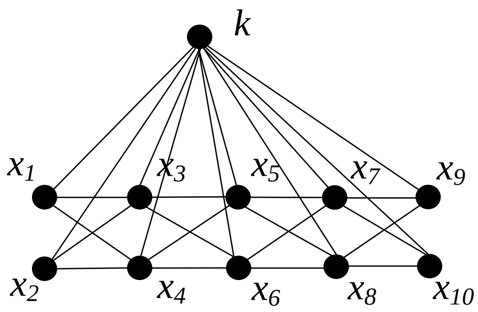

Fig.1.

Theorem 5 [9]. *Let G and H be noncomplete connected graphs. Then a proper subset* $C = S_1 \bigcup S_2$ *of* $X(G \vee H)$, *where* $S_1 \subseteq X(G)$ *and* $S_2 \subseteq X(H)$, *is convex in* $G \vee H$ *if and only if* S_1 *and* S_2 *are cliques in G and H respectively.*

Theorem 6. *Let G and H be noncomplete connected graphs. Then, the following equalities hold:* $\theta(G \vee H) = \varphi_c(G \vee H) = \varphi_{cn}(G \vee H) = \max\{\theta(G), \theta(H)\}$.

Proof. From Theorem 5, we know that every convex set of $G \vee H$ is a clique. Further, every convex cover of $G \vee H$ is a clique cover. Therefore, we have $\varphi_c(G \vee H) = \theta(G \vee H)$. Let $\mathcal{P}_{\varphi_c}(G \vee H)$ be a minimum convex cover of $G \vee H$. By Theorem 5, we obtain a family of sets $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \vee H)} \{X(G) \bigcap S\}$. It is clear that $P(G)$ has no $X(G)$ and every set of $P(G)$ is a clique. This implies inequality $|\mathcal{P}(P(G))| \leq \varphi_c(G \vee H)$. Thus, $\theta(G) \leq \varphi_c(G \vee H)$. Continuing in the same way, we see that $|\mathcal{P}(P(H))| \leq \varphi_c(G \vee H)$, where $P(H) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \vee H)} \{X(H) \bigcap S\}$, and further $\theta(H) \leq \varphi_c(G \vee H)$. Hence, $\max\{\theta(G), \theta(H)\} \leq \varphi_c(G \vee H)$.

By Proposition 3, consider minimum clique covers $\mathcal{P}_\theta(G)$ and $\mathcal{P}_\theta(H)$ of graphs G and H, such that every set of $\mathcal{P}_\theta(G)$ and $\mathcal{P}_\theta(H)$ has at least to vertices. If $\theta(G) \geq \theta(H)$ then we construct a nontrivial clique cover $\mathcal{P}(G\vee H)$, which satisfies the equality $|\mathcal{P}(G\vee H)|=\theta(G)$. Since every convex set of $G\vee H$ is a clique, we unify sets X_i and Y_i, where $X_i\in\mathcal{P}_\theta(G)$ and $Y_i\in\mathcal{P}_\theta(H)$, for $1\le i\le\theta(H)$, and after X_i unify with Y_1, for $\theta(H)+1\le i\le\theta(G)$. Similarly, if $\theta(G)<\theta(H)$, then it can be constructed a nontrivial clique cover $\mathcal{P}(G\vee H)$, where $|\mathcal{P}(G\vee H)|=\theta(H)$. We obtain $\varphi_c(G\vee H)\le\varphi_{cn}(G\vee H)\le\max\{\theta(G),\theta(H)\}$. So, $\varphi_c(G\vee H)=\varphi_{cn}(G\vee H)=\max\{\theta(G),\theta(H)\}$. □

Cartesian Product of Graphs

The *cartesian product* of graphs G and H is a graph $G\square H$ on vertex set $X(G)\times X(H)$ in which vertices (g_1,h_1) and (g_2,h_2) are adjacent if and only if either $g_1=g_2$ and $h_1h_2\in U(H)$ or $h_1=h_2$ *and* $g_1g_2\in U(G)$.

Theorem 7 [9]. *Let G and H be two connected graphs. A set* $C\subseteq X(G\square H)$ *is convex in* $G\square H$ *if and only if* $p_G(C)$ *is convex set in G,* $p_H(C)$ *is convex set in H, and* $C=p_G(C)\times p_H(C)$.

Theorem 8. *Let G be a connected graph on* $n\ge1$ *vertices and* K_m *the complete graph of order* $m\ge1$ *such that* $n+m\ge3$. *Then, the following statements hold.*

1) *If* $m=1$, *then* $\varphi_c(G\square K_m)=\varphi_c(G)$.

2) *If* $m=1$ *and* $n\ge4$, *then* $\varphi_{cn}(G\square K_m)=\varphi_{cn}(G)$.

3) *If* $m\ge2$, *then* $\varphi_c(G\square K_m)=2$.

4) *If* $m\ge2$ *and* $n\ge3$ or $m\ge3$ *and* $n\ge2$, *then* $\varphi_{cn}(G\square K_m)=2$.

Proof.

1) Suppose m = 1. Here we see that $G=G\square K_1$. Since $n+m\ge3$, it is obvious that $\varphi_c(G\square K_k)=\varphi_c(G)$. Further assume that $n\ge4$. In this case $G\square K_1$ has a nontrivial convex cover if and only if graph G has a nontrivial convex cover. Consequently, we have $\varphi_{cn}(G\square K_m)=\varphi_{cn}(G)$. So, statement 2) also holds.

3) Suppose $m\ge2$. We choose two different vertices $k_1,k_2\in X(K_m)$ and obtain two sets:

$$C_1=\{(g,k):g\in X(G),k\in X(K_m)\setminus\{k_1\}\}\text{ and }C_2=\{(g,k):g\in X(G),k\in X(K_m)\setminus\{k_2\}\}.$$

Since K_m is a complete graph, both sets C_1 and C_2 satisfy Theorem 7. Furthermore, sets C_1 and C_2 form a convex 2-cover of graph $G\square K_m$ and $\varphi_c(G\square K_m)=2$. If $n\ge3$, then we see that C_1 and C_2 form a nontrivial convex 2-cover of $G\square K_m$ and further $\varphi_{cn}(G\square K_m)=2$. Similarly, if $m\ge3$ and $n\ge2$, then we also get $\varphi_{cn}(G\square K_m)=2$. Thus, statement 4) also holds. □

Theorem 9. *Let G and H be two noncomplete connected graphs and* $P(G)=\bigcup_{S\in\mathcal{P}_{\varphi_c}(G\square H)}\{p_G(S)\}$. *Then* $|P(G)|=1$ *or* $|P(G)\setminus\{X(G)\}|\ge2$.

Proof. Let $\mathcal{P}_{\varphi_c}(G\square H)$ be minimum convex cover of $G\square H$. Let $|P(G)|=1$ and $C\in P(G)$. It means that $C=X(G)$. Now, assume that $|P(G)\setminus\{X(G)\}|=1$. Further, for a set $S\in P(G)\setminus\{X(G)\}$ there is $S'\in\mathcal{P}_{\varphi_c}(G\square H)$ such that $p_G(S')=S$. If $\{X(G)\}\notin P(G)$, then we obtain a contradiction, because $X(G)\setminus S\ne\varnothing$, which means that $G\square H$ is not covered by convex sets. Suppose further $\{X(G)\}\in P(G)$. From definition of convex cover, we know that every set of $\mathcal{P}_{\varphi_c}(G\square H)$ has at least one vertex that belongs only to this set. Hence, there is $h\in X(H)$ for which there is a vertex (g,h) of $G\square H$ that belongs to S' and does not belong to $S''\in\mathcal{P}_{\varphi_c}(G\square H)$, where $p_G(S'')=X(G)$. By Theorem 7, for h that we fixed before, and $g\in X(G)\setminus S$, vertices (g,h) remains uncovered in $G\square H$. It is a contradiction. □

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Consequence 3. *Let G and H be two connected noncomplete graphs and* $P(H)=\bigcup_{S\in\mathcal{P}_{\varphi_c}(G\square H)}\{p_H(S)\}$. *Then* $|P(H)|=1$ *or* $|P(H)\setminus\{X(H)\}|\geq 2$.

Theorem 10. *Let G and H be two connected noncomplete graphs. Then, the following equalities hold:* $\varphi_c(G\square H)=\varphi_{cn}(G\square H)=\min\{\varphi_c(G),\varphi_c(H)\}$.

Proof. First, note that $|G|\geq 3$ and $|H|\geq 3$. By Proposition 2, there is a minimum convex cover $\mathcal{P}_{\varphi_c}(G)$ of *G* such that every set of $\mathcal{P}_{\varphi_c}(G)$ has at least two elements. Further, by Theorem 7, we obtain a nontrivial convex cover $\mathcal{P}(G\square H)$, which consists of sets $C_i=\{(g,h): g\in S_i, h\in X(H)\}$, where $S_i\in\mathcal{P}_{\varphi_c}(G)$, $1\leq i\leq\varphi_c(G)$. Note that $|\mathcal{P}(G\square H)|=\varphi_c(G)$. Thus, $\varphi_{cn}(G\square H)\leq\varphi_c(G)$. For the same reason, if $\mathcal{P}_{\varphi_c}(H)$ is a minimum convex cover of *H*, then we obtain a nontrivial convex cover $\mathcal{P}(G\square H)$ of $G\square H$ such that $|\mathcal{P}(G\square H)|=\varphi_c(H)$ and further $\varphi_{cn}(G\square H)\leq\varphi_c(H)$. We have $\varphi_c(G\square H)\leq\varphi_{cn}(G\square H)\leq\min\{\varphi_c(G),\varphi_c(H)\}$.

Let $\mathcal{P}_{\varphi_c}(G\square H)$ be a minimum convex cover of graph $G\square H$. Using Theorem 7, we get $P(G)=\bigcup_{S\in\mathcal{P}_{\varphi_c}(G\square H)}\{p_G(S)\}$, $P(H)=\bigcup_{S\in\mathcal{P}_{\varphi_c}(G\square H)}\{p_H(S)\}$. Evidently, equalities $|P(G)|=1$ and $|P(H)|=1$ do not hold at the same time. By Theorem 9 and Consequence 3, let us consider three cases:

Suppose $|P(G)|=1$. In this case inequality $|P(H)\setminus\{X(H)\}|\geq 2$ holds. Consequently, for convex cover $\mathcal{P}(P(H))$ of *G* we get $|\mathcal{P}(P(H))|\leq\varphi_c(G\square H)$ and $\varphi_c(H)\leq\varphi_c(G\square H)$. Now, suppose $|P(H)|=1$. As above, we have $|\mathcal{P}(P(G))|\leq\varphi_c(G\square H)$ and $\varphi_c(G)\leq\varphi_c(G\square H)$. Similarly, if $|P(G)\setminus\{X(G)\}|\geq 2$ and $|P(H)\setminus\{X(H)\}|\geq 2$, we have $\varphi_c(G)\leq\varphi_c(G\square H)$ and $\varphi_c(H)\leq\varphi_c(G\square H)$. Combining these three cases, we obtain that $\min\{\varphi_c(G),\varphi_c(H)\}\leq\varphi_c(G\square H)$. Finally, we have $\varphi_c(G\square H)=\varphi_{cn}(G\square H)=\min\{\varphi_c(G),\varphi_c(H)\}$.□

Lexicographic Product of Graphs

The *lexicographic product* of graphs *G* and *H*, denoted $G\circ H$, is a graph on vertex set $X(G\circ H)=X(G)\times X(H)$, where vertices (g_1,h_1) and (g_2,h_2) are adjacent if and only if either $g_1g_2\in U(G)$ or $g_1=g_2$ and $h_1h_2\in U(H)$. The graph $G\circ H$ is called nontrivial if both graphs have at least two vertices.

Theorem 11 [11]. *Let C be a proper subset of a nontrivial connected lexicographic product* $G\circ H$. *If C induces a noncomplete subgraph of* $G\circ H$, *then C is convex if and only if the following conditions hold:*

(i) $p_G(C)$ *is convex in G,*

(ii) $\{g\}\times X(H)\subseteq C$ *for every nonsimplicial vertex* $g\in p_G(C)$,

(iii) *H is complete.*

Consequence 4. *Let C be a proper subset of a nontrivial connected lexicographic product* $G\circ H$, *where H is noncomplete. Then C is convex if and only if it induces a complete subgraph of* $G\circ H$ *and the following conditions hold:*

(i) $p_G(C)$ *induces a complete subgraph of G,*

(ii) *For every* $g\in p_G(C)$, *set* $p_H(C^g)$ *induces a complete subgraph of H, where*

$$C^g=\{(g,h)\in C: \text{for any } h\in H\}.$$

Theorem 12. *Let G be a connected graph on* $n\geq 1$ *vertices and* K_m *the complete graph of order* $m\geq 1$ *such that* $n+m\geq 3$. *Then, the following statements hold.*

1) *If G is complete, then* $\varphi_c(G\circ K_m)=\varphi_c(K_m\circ G)=2$.

2) *If G is complete and* $n+m\geq 5$, *or* $n=2$ *and* $m=2$, *then* $\varphi_{cn}(G\circ K_m)=\varphi_{cn}(K_m\circ G)=2$.

3) *If G is noncomplete and* $m=1$, *then* $\varphi_c(G\circ K_m)=\varphi_c(K_m\circ G)=\varphi_c(G)$.

4) *If G is noncomplete,* $n \geq 4$ *and* $m = 1$, *then* $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = \varphi_{cn}(G)$.

5) *If G is noncomplete, it has a simplicial vertex and* $m \geq 2$, *then* $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = 2$.

6) *If G is noncomplete, it has no simplicial vertices and* $m \geq 2$, *then* $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = \varphi_c(G)$.

7) *If G is noncomplete and* $m \geq 2$, *then* $\varphi_c(K_m \circ G) = \varphi_{cn}(K_m \circ G) = \theta(G)$.

Proof.

1) Suppose G is complete. Then, it is obvious that obtained graph is complete and we have $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = 2$. In addition, suppose $n + m \geq 5$, or $n = 2$ and $m = 2$. Obtained complete graph with at least 4 vertices has a nontrivial convex 2-cover. Whence, $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = 2$. Statement 2) also holds.

3) Suppose G is noncomplete. If $m = 1$, then graphs $G \circ K_m$ and $K_m \circ G$ are equal to G and further we have $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = \varphi_c(G)$. In the same way, with condition $n \geq 4$, statement 4) holds. In other words $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = \varphi_c(G)$. Assume that $m \geq 2$. If G has a simplicial vertex g', then we choose two different vertices $k_1, k_2 \in X(K_m)$ and obtain two sets:

$$C_1 = (X(G) \setminus \{g'\} \times X(K_m)) \bigcup \{(g', k) : k \in X(K_m) \setminus \{k_1\}\} \text{ and}$$
$$C_2 = (X(G) \setminus \{g'\} \times X(K_m)) \bigcup \{(g', k) : k \in X(K_m) \setminus \{k_2\}\}.$$

Evidently, sets C_1 and C_2 satisfy Theorem 11 and these sets form a nontrivial convex 2-cover of $G \circ K_m$. Further, we have $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = 2$. Statement 5) is satisfied.

Now assume that G has no simplicial vertices. We know from Theorem 11 that for every convex set C of $G \circ K_m$ the projection $p_G(C)$ must be convex in G. Let $\mathcal{P}_{\varphi_c}(G \circ K_m)$ be a minimum convex cover of $G \circ K_m$. We get family $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \circ K_m)} \{p_G(S)\}$. Since noncomplete graph G has no simplicial vertices, it follows that $P(G)$ has no set $X(G)$. Obviously, for convex cover $\mathcal{P}(P(G))$ of graph G we have $|\mathcal{P}(P(G))| \leq \varphi_c(G \circ H)$. Consequently, $\varphi_c(G) \leq \varphi_c(G \circ K_m)$.

Let $\mathcal{P}_{\varphi_c}(G)$ be a minimum convex cover of G. Then, sets $S_i = C_i \times X(K_m)$, $1 \leq i \leq \varphi_c(G)$, form a convex cover of $G \circ K_m$, where $C_i \in \mathcal{P}_{\varphi_c}(G)$, $1 \leq i \leq \varphi_c(G)$, and further we get $\varphi_c(G \circ K_m) \leq \varphi_c(G)$. We have $\varphi_c(G \circ K_m) = \varphi_c(G)$. From Proposition 2 we obtain $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = \varphi_c(G)$. So, statement 6) also holds.

It follows from Consequence 4 that every proper convex subset of $K_m \circ G$ is a clique and further by Proposition 2 and Proposition 3 we have $\varphi_c(K_m \circ G) = \varphi_{cn}(K_m \circ G) = \theta(G)$. Furthermore, statement 7) also holds. □

Theorem 13. *Let G and H be two connected noncomplete graphs. Then, the following equalities hold:* $\varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \theta_c(G \circ H) = \theta_c(G)\theta_c(H)$.

Proof. From Consequence 4 we know that every convex set of $G \circ H$ is a clique. Further, we have $\varphi_c(G \circ H) = \theta_c(G \circ H)$. Moreover, it can be checked that $\theta_c(G \circ H) = \theta_c(G)\theta_c(H)$. Taking into account Proposition 2 and Proposition 3, we get $\varphi_c(G \circ H) = \varphi_{cn}(G \circ H)$. Finally, we have inequalities $\varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \theta_c(G \circ H) = \theta_c(G)\theta_c(H)$. □

Corona of Graphs

The *corona* of graphs *G* and *H* is the graph $G \odot H$ obtained by taking one copy of *G* and *n* copies of *H*, where $|X(G)| = n$, and then joining by an edge the *i*th vertex of *G* to every vertex in the *i*th copy of *H*.

We consider a general version of corona of graphs. Let G be a connected graph on n vertices. Let $\{g_1, g_2, ..., g_k\} \in X(G)$ and $H_{g_1}, H_{g_2}, ..., H_{g_k}$, where $1 \le k \le n$, be connected graphs of order at least one. Then by $(G;\{g_1, g_2, ..., g_k\}) \circ (H_{g_1}, H_{g_2}, ..., H_{g_k})$ is denoted a graph obtained by taking one copy of G and after joining every vertex g_i to every vertex of H_{g_i}, where $1 \le i \le k$. If $H_{g_1} = H_{g_2} = \cdots = H_{g_k} = H$, then we simply denote $(G;\{g_1, g_2, ..., g_k\}) \circ H$. If also $k = n$, then $(G;\{g_1, g_2, ..., g_k\}) \circ H$ is the corona $G \circ H$.

Theorem 14 [10]. *Let G be a connected graph and H be any graph, with* $\{g_1, g_2, ..., g_k\} \subseteq X(G)$ *and* $H_{g_1}, H_{g_2}, ..., H_{g_k}$ *being the corresponding copies of H. A nonempty set* $C \subseteq X(G \circ H)$ *is convex in* $G \circ H$ *if and only if it satisfies one of the following conditions*:

(i) *C is a convex set in G.*

(ii) *C induces a complete subgraph of* H_g *for a vertex* $g \in X(G)$.

(iii) $G[C] = (G[S];\{s_1, s_2, ..., s_l\}) \circ (H^*_{s_1}, H^*_{s_2}, ..., H^*_{s_l})$, *S is convex in graph G,* $\{s_1, s_2, ..., s_l\} \subseteq S$, $\{s_1, s_2, ..., s_l\} \subseteq \{g_1, g_2, ..., g_k\}$ *and* $X(s_i \vee H^*_{s_i})$ *is convex in* $s_i \vee H_{s_i}$ *for each* $i = 1, 2, ..., l$.

Theorem 15. *Let G and H be two connected graph on* $n \ge 1$ *and* $m \ge 1$ *vertices, with* $\{g_1, g_2, ..., g_k\} \subseteq X(G)$, where $1 \le k \le n$. *Then, the following statements hold.*

1) *If* $n = 1$ *and H is complete, then* $\varphi_c(G \circ H) = 2$.

2) *If* $n = 1$, *H is complete and* $m \ge 3$, *then* $\varphi_{cn}(G \circ H) = 2$.

3) *If* $n = 1$, *H is noncomplete with diameter* 2, then $\varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \varphi_c(G)$.

4) *If* $n = 1$, *H is noncomplete with diameter at least* 3, then $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) \le \varphi_c(G)$.

5) *If* $n \ge 2$, *then* $\varphi_c((G;\{g_1, g_2, ..., g_k\}) \circ H) = 2$.

6) *If* $n \ge 2$ *and* $k * m + n \ge 4$, *then* $\varphi_{cn}((G;\{g_1, g_2, ..., g_k\}) \circ H) = 2$.

Proof. Suppose $n = 1$. In fact, $\varphi_c(K_1 \circ H) = \varphi_c(K_1 \vee H)$. Consequently, statements 1), 2), 3), 4) follow from Theorem 4.

5) Suppose $n \ge 2$. It can easily be checked that sets $X(H_{g_1})$ and $X(G) \cup \bigcup_{i=2}^{k} X(H_{g_i})$ satisfy conditions of Theorem 14 and further form a convex 2-cover of graph $(G;\{g_1, g_2, ..., g_k\}) \circ H$. This implies that $\varphi_c((G;\{g_1, g_2, ..., g_k\}) \circ H) = 2$.

6) Now suppose that $k * m + n \ge 4$. In other words, the cardinality of set $X((G;\{g_1, g_2, ..., g_k\}) \circ H)$ must be at least 4. Taking into account Theorem 14, we show nontrivial convex 2-covers of $(G;\{g_1, g_2, ..., g_k\}) \circ H$ in two cases:

a) If $m = 1$, then we choose a vertex $g' \in \Gamma(g) \setminus X(H_g)$ for a vertex $g \in \{g_1, g_2, ..., g_k\}$, that yields a nontrivial convex 2-cover:

$$\mathcal{P}_2((G;\{g_1, g_2, ..., g_k\}) \circ H) = \{\{g, g'\} \cup X(H_g), X(G) \cup \bigcup_{g'' \in \{g_1, g_2, ..., g_k\}, g'' \ne g} X(H_{g''})\}.$$

b) If $m \ge 2$, then we choose a vertex $h \in H_g$ for a vertex $g \in \{g_1, g_2, ..., g_k\}$ and obtain a nontrivial convex 2-cover:

$$\mathcal{P}_2((G;\{g_1, g_2, ..., g_k\}) \circ H) = \{\{g\} \cup X(H_g), \{h\} \cup X(G) \cup \bigcup_{g' \in \{g_1, g_2, ..., g_k\}, g' \ne g} X(H_{g'})\}.$$

The theorem is proved. □

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