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LOCATION PROBLEMS ON d-CONVEX SIMPLE PLANAR GRAPHS

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The article examines the center and median problems on graphs with special structure. In scientific works these are frequently called location problems on graphs. Although efficient solutions for graphs with some well-known properties have been found, in general case the problem remains complex. In this paper the structure of median and center in a *d*-convex simple planar graph *G* is studied. We evaluate the connection between median/center of a graph *G* with mentioned properties and median/center of a tree, determined by *G*.

Keywords: *d-convex simple graph, median, center, tree, distance, location problems.*

PROBLEME DE AMPLASARE ÎN GRAFURI d-CONVEX SIMPLE PLANARE

În articol este examinată problema centrului și problema medianei, cunoscute în literatura de specialitate ca probleme de amplasare. Fiind, în caz general, probleme dificile, acestea se rezolvă în mod eficient pe structuri matematice speciale. În lucrare se prezintă un studiu complex cu privire la structura medianei și centrului într-un graf planar *d*-convex simplu *G*. Este studiată legătura dintre mediana/centrul grafului și mediana/centrul unui arbore, determinat de *G*.

Cuvinte-cheie: *graf d-convex simplu, mediană, centru, arbore, distanță, probleme de amplasare.*

1. Introduction

Different problems involving studies of physical configurations for resource allocation are frequently met in practice. The corresponding solutions often consist in full or in part of minimizing the sum of distances between certain facilities, or of minimizing the worst-case distance to the facility. Finding medians and centers of a graph proved to be useful in this regard.

In order to mathematically model the problems described above, in most cases, undirected graphs are used (however situations that require generalized model in the form of a directed graph also do exist). In this paper all the graphs are considered to be connected. Properties of centers and medians have long been studied (for example [2,5]). Work [2] shows that a tree has one central point or two central points that are adjacent. The same property holds for tree medians [5].

In this paper, we will examine properties of central and median points in graphs with a special structure, described in [1].

Let G_1 and G_2 be two copies of a graph $G=(X;U)$, where $X=\{x_1,x_2,...,x_n\}$ is a set of vertices, and $U=\{u_1,u_2,...,u_m\}$ – the set of edges. The sets of vertices of G_1 and G_2 we denote by $X_1=\{x_1^{'},x_2^{'},...,x_n^{'}\}$ and $X_2=\{x_1^{''},x_2^{''},...,x_n^{''}\}$. The vertices $x_i^{'}\in X_1$, $x_i^{''}\in X_2$, are considered to be copies of $x_i\in X_G$, $1\le i\le n$. In this case we say that $x_i^{'}$, $x_i^{''}$ are corresponding vertices. According to [4] we will denote by $L_2(G)$ the graph obtained by joining every vertex $x_i^{'}\in X_1$ with all vertices $y\in\Gamma(x_i^{''})$, $1\le i\le n$. (Here $\Gamma(x_i^{''})$ is a set of vertices in G_2, which are adjacent to $x_i^{''}$). If vertices adjacent only to a couple of corresponding points are added, the graph will be denoted by $C(L_2(G))$. The procedure of construction of these graphs is represented in figure 1. We add that $C(L_2(G))$ graph is not constructed univocally, as opposed to $L_2(G)$.

Let *T* be a tree of order $n\ge 3$ and T_0 - a sub tree of *T* consisting of vertices of a degree not less than two. We will denote by $L(T,T_0)$ graph that is constructed form *T* and T_0 following the procedure: for every vertex $z\in T_0$, the edges incident to *z* and to every vertex from $\Gamma(\bar{z})$, were $\bar{z}\in T_0$, are added ($z,\bar{z}$ are corresponding vertices of *G*).

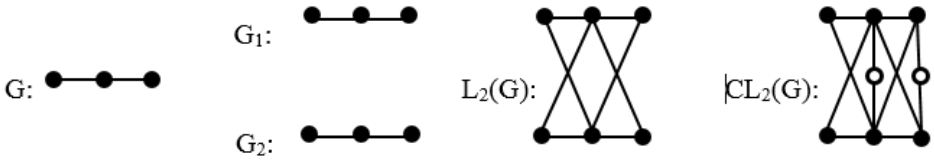

Fig.1. The process of constructing $L_2(G)$ and $CL_2(G)$.

$L_2(G)$ and $CL_2(G)$ are especially used for simple convexity examination of undirected graphs, which is having an important contribution for the development of convexity theory on discrete structures.

A subset of vertices $A \subset X_G$ is called d-convex, if for every two vertices $x, y \in A$ the following relation holds [2]:

$$< x, y >= \{ z \in X_G : d(x, y) = d(x, z) + d(z, y) \} \subset A,$$

Where graph distance between *x* and *y* is denoted by $d(x, y)$. The set $< x, y >$ is called metric segment that links *x* and *y* [2].

Definition 1.1 [1] A graph $G = (X, U)$ that does not contain d-convex sets $A \subset X_G$ of the following property

$$2 < |A| < n = |X_G|.$$

is called d-convex simple graph.

Definition 1.2 [1] Vertex $y \in X_G$ is called dominant for $x \in X_G$, if $\Gamma(x) \supseteq \Gamma(y)$. If $\Gamma(x) = \Gamma(y)$ then *x* and *y* are called copies.

Let $\mathcal{A}$ be a set of undirected graphs $G = (X, U)$ for which the following properties hold:

1. $G = (X, U)$ doesn't contain cycles of order 3;
2. Every vertex $y \in X_G$ is dominated by at least one vertex $x \in X_G$.

Lemma 1.1 [1] If $G \in \mathcal{A}$, then *G* is *d*-convex simple graph.

Theorem 1.1 If *G* is a graph that does not contain the cycles of order 3, then the following statements holds:

1. $L_2(G)$ is a d-convex simple graph;
2. Every graph of $CL_2(G)$ type is d-convex simple;

Proof. Considering Lemma 1, we will prove the theorem affirmation, by showing that $L_2(G)$ and every graph of $CL_2(G)$ type are connected and belong to $\mathcal{A}$.

If *G* is connected, then the property of connectivity of $L_2(G)$ and $CL_2(G)$ immediately results from the definition of their structure.

1) $L_2(G)$ consists of two copies of G. The procedure of adding new edges to construct $L_2(G)$ graph necessarily implies that for every vertex *x* there is at least one vertex *y*, that is a copy of *x* (is true equality $\Gamma(x) = \Gamma(y)$). Following definition 1.2, we can say that *y* dominates vertex *x*. Thus $L_2(G) \in \mathcal{A}$.

2) As it has been shown that $L_2(G) \in \mathcal{A}$, only vertices of a degree 2 in $CL_2(G)$ remain to be studied. If a vertex *x* is of a degree 2, then *x* is adjacent with *y*, and y' that is the copy of *y*. From the structure of $L_2(G)$ results that $deg(y) > 1$. Therefore, there exists a vertex *z*, that is different from *x*, adjacent to *y*. It

follows that *z* is also adjacent to y'. Conclude $\Gamma\ z\ \supseteq \Gamma\ x$. So, *z* dominates *x* and $CL_2(G)$ belongs to $\mathcal{A}$. ■

2. The Center problem

We will examine the problem of $CL_2(G)$ center determination.

Definition 2.1 [2] The eccentricity *e*(*x*) of a vertex *x* is the greatest geodesic distance between this vertex and any other vertex of graph:

$$e\ x\ = \max_{y\in X_G} d(x,y).$$

Definition 2.2 [2] The radius of a graph G is the minimum eccentricity of all vertices of graph.

Definition 2.3 [2] Graph center is the set of vertices $W_G = x_1,\ldots,x_n$ with eccentricity equal to the graph's radius:

$$e(x_i) = r(G), \forall x_i \in W_G$$

Thus, vertices in the center (central points) minimize the maximal distance from other points in the graph.

Single chain that links two vertices in a tree is denoted by x, y .

Theorem 2.1. If $x \in W_G$, then $x \in W_{L2\ G}$.

Proof. $L_2(G)$ is constructed by adding only vertex copies, therefore

$$e_G\ x\ = \max_{y\in X_G} d(x,y) = \max_{y\in X_{L_2(G)}} d(x,y) = e_{L_2(G)}\ x\ .$$

This implies that the theorem affirmation holds.■

Theorem 2.2. Central points of *T* belong to the center of $L\ T,T_0$.

Proof. The fact that *T* is a tree implies that the distance between two vertices x and y equals to the length of the chain that connects them x, y minus 1. The identity $\underset{x,y\in T}{d(x,y)} = \underset{x,y\in L(T,T_0)}{d(x,y)}$ results from the structure of $L\ T,T_0$. It follows that eccentricities of the vertices from W_T remain to be minimal in $L\ T,T_0$.

3. The Median Problem

Now let us see what properties the $CL_2(G)$-median has. The $CL_2(G)$ planar graph is of special interest. This case will be examined further.

Definition 3.1 [2] Function

$$f\ x\ = \sum_{y\in X} d(x,y)$$

is called median function of a graph $G = X,U$. Vertex $x^* \in X$ that verifies the identity

$$f\ x^*\ = \min_{x\in X} f(x)$$

is called median vertex, or median of graph.

Theorem 3.1. If *x* is median in *G*, then *x* is median in $L_2(G)$.

Proof. Suppose that x is median in *G*. Then the sum $\sum_{z\in G} d(x,z)$ has minimum value. From the structure of $L_2(G)$ it follows that $\sum_{z\in L_2(G)} d(x,z) = 2\sum_{z\in G} d(x,z) + 2$.The minimal value of $\sum_{z\in G} d(x,z)$ guarantees that median function in $L_2(G)$ reaches its lowest value in vertex *x*. Conclude that *x* is median in $L_2(G)$.

Theorem 3.2. If *x* is a vertex of a degree 2 in $CL_2(G)$, then x is not median.

Proof. Let x be a vertex of a degree 2 in $CL_2(G)$. Vertex *x* is adjacent to *y* and *z* (z is a copy of *y*). It follows that $d(x,t) = d(y,t)+1$ for every vertex $t \in G$ and $d(x,t') = d(z,t')+1$ for every t' that belongs to the copy of G.

Therefore $d(x,t) = d(y,t)+1, \forall t \in CL_2(G) \setminus z$, and $d(x,z) = 1 = d(y,z) - 1$.

Conclude that $\sum_{t \in CL_2 G} d(x,t) = \sum_{t \in CL_2 G} d(y,t) + m - 1$, where $m = |CL_2(G) \setminus \{z\}|$, and *m* is greater than 1. This implies that *x* is not median.

Lemma 3.1. Let $x_0 = x^*, x_1, \ldots, x_t$ be a chain in a tree and vertex *x** is a median. Then the function *f* has the following properties:

1. $f(x_{i+1}) > f(x_i),\ 0 < i \le t-1$;
2. $f(x_{i+1}) - f(x_i) > f(x_i) - f(x_{i-1}),\ 0 < i \le t-1$;

Proof. Let $B_1(x^*), \ldots, B_k(x^*)$ be the branches of *T* from *x**.

Assume that $x_0 = x^*, x_1, \ldots, x_t \subseteq B_i(x^*)$. We will denote by T_i a subgraph of the branch $B_i(x^*)$ with the property: $x_i \in T_i,\ x_{i-1} \notin T_i, x_{i+1} \notin T_i$.

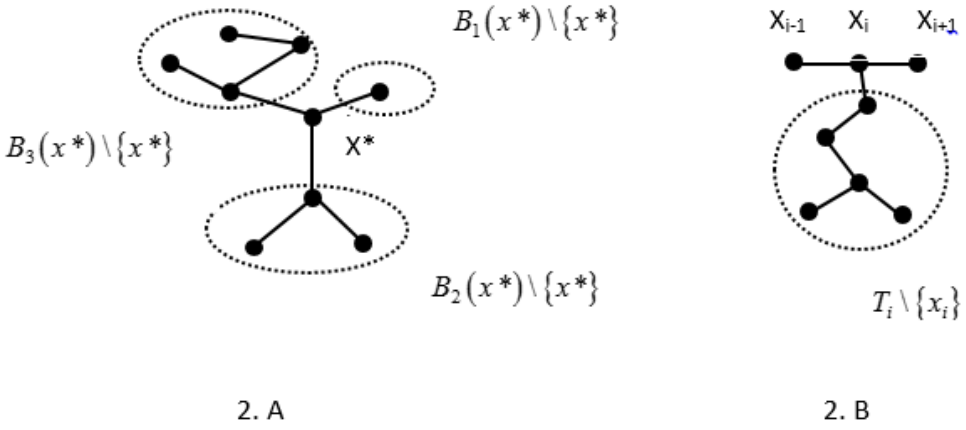

Fig.2. Subgraphs $L_2(G)$ and T_i.

We have:

$$Step\ 1.\ f(x_1) = \sum_{y \in T} d(x_1, y) = \sum_{y \in T \setminus B_i(x^*) \cup \{x^*\}} d(x_1, y) + \sum_{y \in B_i(x^*) \setminus \{x^*\}} d(x_1, y) =$$

$$\sum_{y \in T \setminus B_i(x^*) \cup \{x^*\}} (d(x^*, y) + 1) + \sum_{y \in B_i(x^*) \setminus \{x^*\}} (d(x^*, y) - 1) =$$

$$\sum_{y \in T \setminus B_i(x^*) \cup \{x^*\}} d(x^*, y) + n - n(B_i(x^*)) + 1 + \sum_{y \in B_i(x^*) \setminus \{x^*\}} d(x^*, y) - n(B_i(x^*)) + 1 =$$

$$\sum_{y \in T} d(x^*, y) + n - 2n(B_i(x^*)) + 2 = f(x^*) + n - 2n(B_i(x^*)) + 2$$

We denote the order of *T* by *n*(*T*)=*n*, and the order of $B_i(x^*)$ by $n(B_i(x^*))$.

According to [2] a tree has exactly one median, or exactly two median joined by an edge. Therefore the term $n(T)-2n\ B_i\ a\ +2$ is not less than 0.The next step shows us that the values of f are increasing.

$$Step\ 2.\ f(x_2)=\sum_{y\in T}d(x_2,y)=\sum_{y\in T\setminus B_i(x^*)\cup T_1\cup\{x^*\}}d(x_2,y)+\sum_{y\in B_i(x^*)\setminus\{x^*\}\setminus T_1}d(x_2,y)=$$

$$\sum_{y\in T\setminus B_i(x^*)\cup T_1\cup\{x^*\}}(d(x_1,y)+1)+\sum_{y\in B_i(x^*)\setminus\{x^*\}\setminus T_1}(d(x_1,y)-1)=$$

$$\sum_{y\in T\setminus B_i(x^*)\cup T_1\cup\{x^*\}}d(x_1,y)+n-n(B_i(x^*))+n_1+1+\sum_{y\in B_i(x^*)\setminus\{x^*\}\setminus T_1}d(x_1,y)-n(B_i(x^*))+n_1+1=$$

$$\sum_{y\in T}d(x_1,y)+n-2n(B_i(x^*))+2+2n_1=f(x_1)+n-2n(B_i(x^*))+2+2n_1.$$

$$Step\ j+1.\ f(x_{j+1})=\sum_{y\in T}d(x_{j+1},y)=\sum_{y\in T\setminus B_i(x^*)\cup T_1\cup\ldots\cup T_j\cup\{x^*\}}d(x_{j+1},y)+\sum_{y\in B_i(x^*)\setminus\{x^*\}\setminus T_1\setminus\ldots\setminus T_j}d(x_{j+1},y)=$$

$$=\sum_{y\in T\setminus B_i(x^*)\cup T_1\cup\ldots\cup T_j\cup\{x^*\}}(d(x_j,y)+1)+\sum_{y\in B_i(x^*)\setminus\{x^*\}\setminus T_1\setminus\ldots\setminus T_j}(d(x_j,y)-1)=\sum_{y\in T\setminus B_i(x^*)\cup T_1\cup\ldots\cup T_j\cup\{x^*\}}d(x_j,y)+$$

$$+n-n(B_i(x^*))+n_1+n_2+\ldots+n_j+1+\sum_{y\in B_i(x^*)\setminus\{x^*\}\setminus T_1\setminus\ldots\setminus T_{j1}}d(x_j,y)-n(B_i(x^*))+n_1+n_2+\ldots+n_j+1=$$

$$=f(x_j)+n-2n(B_i(x^*))+2+2(n_1+n_2+\ldots+n_j).$$

We immediately conclude that $f\ x_{j+1}>f\ x_j$ and $f\ x_{i+1}-f\ x_i\ >f\ x_i\ -f\ x_{i-1}$.■

Let $z\in T$ be a vertex for which the following relation holds

$$g\ z\ =\min_{x\in T}g\ x\ =\min_{x\in T}\sum_{\substack{y\in T\\ degy=1}}d\ x,y\ .$$

Lemma 3.2. Let $x_0=x^*,x_1,\ldots,x_t$ be a chain in a tree T. Then the following inequalities hold:

1) $g\ x_{i+1}\ \geq g\ x_i\ ,0<i,,n-1;$

2) $g\ x_{i+1}\ -g\ x_i\ \geq g\ x_i\ -g\ x_{i-1}\ \ 1,,i,,n-1;$

If $deg(x_t)=1$, then $g\ x_t\ >g\ x_{t-1}$.

Proof. Let $B_1\ x^*$, … , $B_k\ x^*$ be the branches of *T* from *z* (figure 3). Assume that *s* vertices of a degree 1 belong to $B_1\ z\ \setminus\ z$, *m* vertices of a degree 1 belong to $B_2\ z\ \setminus\ z$, and k vertices of a degree 1 belong to $\bigcup_{i=1}^{k}[B_i(z)]$.

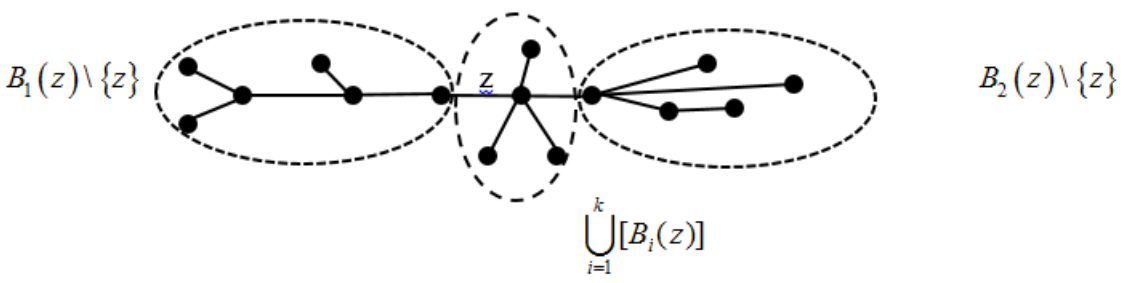

Fig.3. The structure of *T*.

Suppose that $B_1(z \setminus z) \supseteq z_0, \ldots, z_t$. We will evaluate the value of $g(z_{i+1})$.

$$\text{Step 1. } \sum_{\substack{y\in T\\ degy=1}} d(z_1,y) = \sum_{\substack{y\in B1(z\setminus z)\\ degy=1}} d(z_1,y) + \sum_{\substack{y\in\bigcup_{i=3}^{k}[Bi(z)]\\ degy=1}} d(z_1,y) + \sum_{\substack{y\in B2(z\setminus z)\\ degy=1}} d(z_1,y) =$$

$$\sum_{\substack{y\in B1(z\setminus z)\\ degy=1}} [d(z,y)-1] + \sum_{\substack{y\in\bigcup_{i=3}^{k}[Bi(z)]\\ degy=1}} [d(z,y)+1] + \sum_{\substack{y\in B2(z\setminus z)\\ degy=1}} [d(z,y)+1] =$$

$$\sum_{\substack{y\in B1(z\setminus z)\\ degy=1}} d(z,y) - s + \sum_{\substack{y\in\bigcup_{i=3}^{k}[Bi(z)]\\ degy=1}} d(z,y) + k + \sum_{\substack{y\in B2(z)\\ degy=1}} d(z,y) + m = \sum_{\substack{y\in T\\ degy=1}} d(z,y) + m + k - s$$

According to the definition of vertex z the inequality $m + k - s \geq 0$ holds.

We assume now that for every $x_i, t > i \geq 1$, there could exist vertices $T^i = t^i{}_j$ (figure 4), of a degree 1, $|T^i| = n_i$ that belong to $B_1(z \setminus z)$, and , $z \notin [z_q, t^i{}_j]$, $z_q \notin [z, t^i{}_j]$. $q > I$.

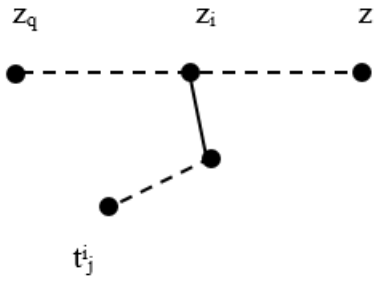

Fig.4. Vertices T^i.

$$\text{Step 2. } \sum_{\substack{y\in T\\ degy=1}} d(z_2,y) = \sum_{\substack{y\in B1(z\setminus z\setminus T1)\\ degy=1}} d(z_2,y) + \sum_{\substack{y\in\bigcup_{i=3}^{k}[Bi(z)]\\ degy=1}} d(z_2,y) + \sum_{\substack{y\in B2(z\setminus z\cup T1)\\ degy=1}} d(z_2,y) =$$

$$\sum_{\substack{y\in B1(z\setminus z\setminus T1)\\ degy=1}} [d(z_1,y)-1] + \sum_{\substack{y\in\bigcup_{i=3}^{k}[Bi(z)]\\ degy=1}} [d(z_1,y)+1] + \sum_{\substack{y\in B2(z\setminus z\cup T1)\\ degy=1}} [d(z_1,y)+1] =$$

$$\sum_{\substack{y\in B1(z\setminus z\setminus T1)\\ degy=1}} d(z_1,y) - (s-n_1) + \sum_{\substack{y\in\bigcup_{i=3}^{k}[Bi(z)]\\ degy=1}} d(z_1,y) + k + \sum_{\substack{y\in B2(z)\setminus T1\\ degy=1}} d(z_1,y) + m + n_1 =$$

$$\sum_{\substack{y\in T\\ degy=1}} d(z_1,y) + m + k - s + 2n_1.$$

In this case the value of $2n_1$ is non-negative, because it is entirely possible that a tree does not have vertices that belong to T^1, implying that $\deg(z_1) = 2$. So z_1 is only adjacent to z_0 and z_2.

The inequality $\sum_{\substack{y\in T\\ degy=1}} d\ z_2, y\ \geq \sum_{\substack{y\in T\\ degy=1}} d\ z_1, y$ holds because in other cases, however, the degree of z_1 can be greater than two, ensuring the positive value of $2n_1$.

Step k. $\sum_{\substack{y\in T\\ degy=1}} d\ z_k, y = \sum_{\substack{y\in B1\ z\ \setminus\ z\ \setminus T1\setminus T2\setminus\ldots\setminus Tk\\ degy=1}} d\ z_k, y + \sum_{\substack{y\in \bigcup_{i=3}^{k}[Bi\ z\ ]\\ degy=1}} d\ z_k, y + \sum_{\substack{y\in B2\ z\ \setminus\ z\ \cup T1\cup T2\ldots\cup Tk\\ degy=1}} d\ z_k, y =$

$\sum_{\substack{y\in B1\ z\ \setminus\ z\ \setminus T1\setminus T2\setminus\ldots\setminus Tk\\ degy=1}} \left[d\ z_{k-1}, y\ -1\right] + \sum_{\substack{y\in \bigcup_{i=3}^{k}[Bi\ z\ ]\\ degy=1}} \left[d\ z_{k-1}, y\ +1\right] + \sum_{\substack{y\in B2\ z\ \setminus\ z\ \cup T1\cup T2\ldots\cup Tk\\ degy=1}} \left[d\ z_{k-1}, y\ \right]1] =$

$\sum_{\substack{y\in B1\ z\ \setminus\ z\ \setminus T1\setminus T2\setminus\ldots\setminus Tk\\ degy=1}} d\ z_{k-1}, y\ (s-n\ -n_2-\ldots n_{k-1}) + \sum_{\substack{y\in \bigcup_{i=3}^{k}[Bi\ z\ ]\\ degy=1}} d\ z_{k-1}, y\ k + \sum_{\substack{y\in B2\ z\ \setminus\ z\ \cup T1\cup T2\ldots\cup Tk\\ degy=1}} d\ z_{k-1}, y\ )$

$+m+n_1+n_2+\ldots+n_{k-1} = \sum_{\substack{y\in T\\ degy=1}} d\ z_{k-1}, y\ m+k-s\ +2n_1+\ 2n_2+\ldots+2n_{k-1}.$

It follows that $\sum_{\substack{y\in T\\ degy=1}} d\ z_k, y\ \geq \sum_{\substack{y\in T\\ degy=1}} d\ z_{k-1}, y\ .$

So the inequalities 1 and 2 hold. Now we have to prove that $\sum_{\substack{y\in T\\ degy=1}} d\ z_t, y\ > \sum_{\substack{y\in T\\ degy=1}} d\ z_{t-1}, y\ $, if $deg(z_t)=1.$

Let $deg\ z_t\ = 1$. Then $z_{t-1}\in z_t, y\ , \forall y\in T$, evidently $z_{t-1}\in z_t, y\ , \forall y\colon deg\ y=1$. We have:

$$\sum_{\substack{y\in T\\ degy=1}} d\ z_t, y\ = \sum_{\substack{y\in T\\ degy=1}} d\ z_{t-1}, y\ +m+s-1+k\ .$$

Considering the fact, that a tree has at least two vertices of a degree 1, we have $m+s+k\geq 2$, and $m+s-1+k\geq 1$.■

Theorem 3.1. Let *M* be a set of medians in a *d*-convex simple planar graph $G=\ X,U\ $. Then exactly one of the following affirmation holds:

1. $M=\ x,x^*\colon x-is\ a\ copy\ of\ x^*\ ;$
2. $M=\{x,\ x^*,\ y,\ y^*\colon\ x\sim y,\ x^*\sim y^*;x^*\ ,\ y^*are\ copies\ of\ x,y\}$

Proof. First, assume that the graph has two medians *x* and *y*, we consider that *x* is not a copy of *y*, and the length of the shortest path between *x* and *y* is greater than 2 (figure 5). We will denote vertices that belong to $x,y\ $ by: $x_1=x\ ,\ldots,x_n=y$. Evidently *x** and *y** are medians too. We will consider *x,y* – case only, because for *x**, *y** the reasoning repeats itself.

Median function in *d*-convex simple planar graph $G\ =\ L\ T,T_0\ $ has the following form:

$$\sum_{z\in G} d\ x,z\ = 2\sum_{z\in T\,0} d\ x,z\ + \sum_{\substack{z\in T\\ \deg z=1}} d\ x,z\ = 2f_1\ x\ + f_2\ x\ ,$$

Observe that f_1 is the median function in T_0, and f_2 is the function discussed in Lemma 3.2.

$$\sum_{z\in G} d(x,z) = 2\sum_{z\in T_0} d(x,z) + \sum_{\substack{z\in T \\ \deg z=1}} d(x,z) = 2f_1(x) + f_2(x).$$

According to the properties of functions described in Lemma 3.1, and Lemma 3.2, every median belongs to the path $[x_{min}, y_{min}]$, where x_{min} is the median in T_0, and in y_{min} function f_2 reaches its lowest value.

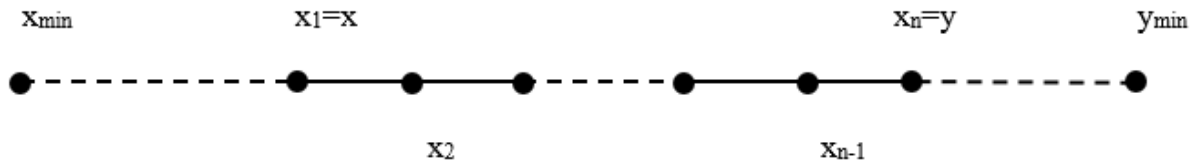

Fig.5. Medians in G.

So x_1 is a median, and x_2 is not. We have:

$$\sum_{z\in G} d(x_2,z) - \sum_{z\in G} d(x_1,z) = 2[f_1(x_2) - f_1(x_1)] + f_2(x_2) - f_2(x_1) > 0.$$

If $f_1(x_2) = f_1(x_1)$, then

- $\sum_{z\in G} d(x_2,z) - \sum_{z\in G} d(x_1,z) = 0$, if $f_2(x_2) = f_2(x_1)$ contradiction.
- $\sum_{z\in G} d(x_2,z) - \sum_{z\in G} d(x_1,z) < 0$, if $f_2(x_2) < f_2(x_1)$ contradiction.

Conclude that $f_1(x_2) - f_1(x_1) > 0$.

According to Lemma 3.1, and Lemma 3.2 we have:

$$f_1(x_{j+1} - x_j) < f1(x_{i+1} - f_1(x_i) \quad and \quad f_2(x_{j+1}) - f_2(x_j) \le f_2(x_{i+1}) - f_2(x_i) \quad for\, i > j.$$

It follows that

$$0 < f_1(x_{n-1}) - f_1(x_{n-2}) < f_1(x_n) - f_1(x_{n-1}), \; 0 \le f_2(x_{n-1}) - f_2(x_{n-2}) \le f_2(x_n) - f_2(x_{n-1}).$$

So

$$\sum_{z\in G} d(x_n,z) - \sum_{z\in G} d(x_{n-1},z) = 2[f_1(x_n) - f_1(x_{n-1})] + f_2(x_n) - f_2(x_{n-1}) > 0.$$

We obtained a contradiction.

Second, assume that G has 3 medians x, y, z, $x \sim y \sim z$. (The same property holds for their copies $x^* \sim y^* \sim z^*$).

We have:

a. $f_1(x) = f_1(y)$ (T_0 has two medians). The identity $2f_1(x) + f_2(x) = 2f_1(y) + f_2(y)$ results from x and y being medians in G. So, $f_2(x) = f_2(y)$. According to lemmas 3.1 and 3.2 we obtain that $f_1(z) > f_1(y)$, $f_2(z) \ge f_2(y)$. Contradiction: $2f_1(z) + f_2(z) > 2f_1(y) + f_2(y)$ and z is median in G.

b. $f_1(x) < f_1(y)$. We will denote $f_1(y) - f_1(x) = r_1 > 0$. Then

$$f_2(x) - f_2(y) = 2f_1(y) - 2f_1(x) = r_1.$$

The inequality $f_1(z) > f_1(y)$ results from lemma 3.1 and $r_2 = f_1(z) - f_1(y) > f_1(y) - f_1(z) = r_1$. So we have $f_2(z) < f_2(y)$ (lemma 3.2) and $r_3 = f_2(y) - f_2(z) < f_2(x) - f_2(y) = r_1$.

Contradiction:

$$2f_1(z) + f_2(z) - 2f_1(y) + f_2(y) = 2(f_1(z) - f_1(y)) + f_2(z) - f_2(y) = 2r_2 - r_3 > 0.\blacksquare$$

4. Conclusions

This paper outlines the research that completes well-known results associated with center and median problems. For the class of *d*-convex simple planar graphs, structure and properties of median and center are described. The mentioned results can lead to the elaboration of methods for median/center calculation on a *d*-convex simple planar graph *G*, using properties of a special tree-median/center.

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