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Global stability of a system of exponential difference equations

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Abstract

We investigate the qualitative behavior of a second-order system of exponential difference equations. Particularly, we study boundedness and persistence, existence and uniqueness of positive steady-state, parametric conditions for local and global asymptotic stability of the unique positive equilibrium point and the rate of convergence of positive solutions of this system. Numerical simulations are provided to illustrate theoretical discussions.

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1 Introduction

Discrete dynamical systems in exponential form have rich dynamics. Such systems can be used to discuss population models. Due to adequate computational consequences discrete dynamical system are much better than allied systems in differential equations. Particularly, in case of non-overlapping generations difference equations are more apposite to study the behavior of population models [1, 18, 21, 22, 24]. For more detail of some amiable population models both in differential equations as well as in difference equations, we refer the interested reader to [2, 3, 17, 19]. Moreover, for some basic properties of nonlinear difference equations one can see [23, 29]. It is very curios to investigate the dynamics of solutions of systems of nonlinear difference equations and to discourse about the local and global asymptotic stability of their equilibrium points. For some notable results related to the qualitative behavior of difference equations, we refer the reader to [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

El-Metwally et al. [20] discussed the qualitative behavior of the following two-dimensional population model:

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}.$$

Papaschinopoulos et al. [25] discussed the qualitative behavior of the following two-dimensional interactive and invasive species model:

$$x_{n+1} = a + b x_{n-1} e^{-y_n}, \ y_{n+1} = c + d y_{n-1} e^{-x_n}.$$

Papaschinopoulos et al. [26] discussed the qualitative behavior of the following three systems of difference equations of exponential form:

$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \ y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + x_{n-1}},$$

$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, \ y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}},$$

and

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, \ y_{n+1} = \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}}.$$

Papaschinopoulos et al. [27] discussed the qualitative behavior of the following two difference equations:

$$x_{n+1} = a + by_{n-1}e^{-y_n},\ y_{n+1} = c + dx_{n-1}e^{-x_n},$$

and

$$x_{n+1} = a + by_{n-1}e^{-x_n},\ y_{n+1} = c + dx_{n-1}e^{-y_n}.$$

Moreover, Din [5] investigated the global asymptotic stability of the following discrete-time population model:

$$x_{n+1} = \alpha x_n e^{-y_n} + \beta,\ y_{n+1} = \alpha x_n \left(1 - e^{-y_n}\right).$$

Furthermore, Din [7] discussed the qualitative behavior of the following two-dimensional plant-herbivore system:

$$x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}},\ y_{n+1} = \gamma(x_n + 1)y_n.$$

In [9] the author studied the qualitative behavior of the following modified host-parasitoid system:

$$\begin{aligned} H_{n+1} &= rN_0 + r(H_n - N_0)\exp(-aP_n), \\ P_{n+1} &= e(H_n - N_0)\left(1 - \exp(-aP_n)\right). \end{aligned}$$

In [16] the authors investigated the qualitative behavior of the following generalized Beddington model:

$$\begin{cases} N_{t+1} = N_t \exp\left[r\left(1 - \frac{N_t}{k}\right) - aP_t\right], \\ P_{t+1} = \lambda N_t\left[1 - \exp(-bP_t)\right]. \end{cases}$$

Motivated by above study, our aim in this paper is to investigate the qualitative behavior the following two-dimensional discreet dynamical system of exponential form:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-y_n} + \gamma_1 e^{-y_{n-1}}}{a_1 + b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-x_n} + \gamma_2 e^{-x_{n-1}}}{a_2 + b_2 x_n + c_2 x_{n-1}}, \tag{1}$$

where the parameters $\alpha_i,\ \beta_i,\ \gamma_i,\ a_i,\ b_i,\ c_i$ for $i \in \{1, 2\}$ and initial conditions $x_0,\ x_{-1},\ y_0,\ y_{-1}$ are positive real numbers.

Particularly, we study boundedness, existence and uniqueness of nontrivial steady-state, parametric conditions for local and global asymptotic stability of the unique positive equilibrium point and the rate of convergence of positive solutions of system (1) which converge to its unique positive equilibrium point. In system (1), if we take $\gamma_1 = \gamma_2 = b_1 = b_2 = 0$ and $c_1 = c_2 = 1$, then it reduces to a population model of two species which has been investigated in [26].

2 Existence and stability of positive equilibrium

Following theorem guarantees about the boundedness and persistence of every solution of (1).

Theorem 1. *Every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded and persists.*

Proof. For any positive solution $\{(x_n, y_n)\}$ of system (1), one has

$$x_{n+1} \leq \frac{\alpha_1 + \beta_1 + \gamma_1}{a_1} = U_1, \ y_{n+1} \leq \frac{\alpha_2 + \beta_2 + \gamma_2}{a_2} = U_2, \ n = 0, 1, 2, \cdots . \quad (2)$$

Furthermore, from system (1) and (2) we obtain that

$$x_{n+1} \geq \frac{\alpha_1 + \beta_1 e^{-U_2} + \gamma_1 e^{-U_2}}{a_1 + b_1 U_2 + c_1 U_2} = L_1, \ y_{n+1} \geq \frac{\alpha_2 + \beta_2 e^{-U_1} + \gamma_2 e^{-U_1}}{a_2 + b_2 U_1 + c_2 U_1} = L_2, \ n = 2, 3, \cdots . \quad (3)$$

From (2) and (3), it follows that

$$L_1 \leq x_n \leq U_1, \ L_2 \leq y_n \leq U_2, \ n = 3, 4, \cdots .$$

Hence, theorem is proved. □

Lemma 1. *Let $\{(x_n, y_n)\}$ be a positive solution of system (1). Then, $[L_1, U_1] \times [L_2, U_2]$ is invariant set for system (1).*

Proof. The proof follows by induction. □

Next, we consider the following general systems of two difference equations

$$x_{n+1} = f(y_n, y_{n-1}), \ y_{n+1} = g(x_n, x_{n-1}), \quad (4)$$

where f, g are continuous functions and the initial conditions x_i, y_i for $i \in \{-1, 0\}$ are positive real numbers. Arguing as in [26], we have following result for global behavior of (1).

Lemma 2. *Assume that $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ and $g : (0,\infty) \times (0,\infty) \to (0,\infty)$ be continuous functions and $a,\ b,\ c,\ d$ are positive real numbers with $a < b,\ c < d$. Moreover, suppose that $f : [c,d] \times [c,d] \to [a,b]$ and $g : [a,b] \times [a,b] \to [c,d]$ such that following conditions are satisfied:*

(i) $f(y_1, y_2)$ is decreasing in both y_1 and y_2, $g(x_1, x_2)$ is decreasing in both x_1 and x_2.

(ii) Let $m_1,\ M_1,\ m_2,\ M_2$ are real numbers such that

$m_1 = f(M_2, M_2)$, $M_1 = f(m_2, m_2)$, $m_2 = g(M_1, M_1)$ and $M_2 = g(m_1, m_1)$, then $m_1 = M_1$ and $m_2 = M_2$.

Then the system of difference equation (4) has a unique positive equilibrium point $(\bar{x}, \bar{y})$ and every positive solution of system of difference equations (4) which satisfies

$$x_{n_0} \in [a,b],\ x_{n_0+1} \in [a,b],\ y_{n_0} \in [c,d],\ y_{n_0+1} \in [c,d],\ n_0 \in \mathbb{N} \tag{5}$$

converges to the unique positive equilibrium of system (4).

Theorem 2. *System (1) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of system (1) converges to the unique positive equilibrium $(\bar{x}, \bar{y})$ as $n \to \infty$, if the following holds true:*

$$\beta_2 + \gamma_2 < a_1,\ \beta_1 + \gamma_1 < a_2,\ b_1 + c_1 = b_2 + c_2. \tag{6}$$

Proof. Consider the following functions:

$$f(u,v) = \frac{\alpha_1 + \beta_1 e^{-u} + \gamma_1 e^{-v}}{a_1 + b_1 u + c_1 v},\ g(z,w) = \frac{\alpha_2 + \beta_2 e^{-z} + \gamma_2 e^{-w}}{a_2 + b_2 z + c_2 w},$$

where $z, w \in [L_1, U_1] = I_1$ and $u, v \in [L_2, U_2] = I_2$ which implies that $f(u,v) \in I_1$ and $g(z,w) \in I_2$ so that $f : I_2 \times I_2 \to I_1$ and $g : I_1 \times I_1 \to I_2$. Assume that $\{(x_n, y_n)\}$ be any positive solution of (1), then by Lemma 2 we have $x_n \in I_1$ and $y_n \in I_2$. Next, we assume that m_1, M_1, m_2, M_2 be the positive real numbers such that

$$\begin{aligned} M_1 &= \frac{\alpha_1 + \beta_1 e^{-m_2} + \gamma_1 e^{-m_2}}{a_1 + b_1 m_2 + c_1 m_2},\ m_1 = \frac{\alpha_1 + \beta_1 e^{-M_2} + \gamma_1 e^{-M_2}}{a_1 + b_1 M_2 + c_1 M_2}, \\ M_2 &= \frac{\alpha_2 + \beta_2 e^{-m_1} + \gamma_2 e^{-m_1}}{a_2 + b_2 m_1 + c_2 m_1},\ m_2 = \frac{\alpha_2 + \beta_2 e^{-M_1} + \gamma_2 e^{-M_1}}{a_2 + b_2 M_1 + c_2 M_1}. \end{aligned} \tag{7}$$

Let

$$F(x) = \frac{\alpha_1 + (\beta_1 + \gamma_1)\, e^{-f(x)}}{a_1 + (b_1 + c_1) f(x)} - x,$$

where

$$f(x) = \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-x}}{a_2 + (b_2 + c_2) x},\ x \in I_1.$$

Then F maps the interval I_1 into itself. In order to show that the equation $F(x) = 0$ has unique solution in I_1 we have:

$$F'(x) = \\ -f'(x)\left[\frac{(a_1 + (b_1 + c_1)f(x))(\beta_1 + \gamma_1)e^{-f(x)} + (\alpha_1 + (\beta_1 + \gamma_1)e^{-f(x)})(b_1 + c_1)}{(a_1 + (b_1 + c_1)f(x))^2}\right] - 1, \tag{8}$$

where

$$f'(x) = -\left[\frac{(\beta_2 + \gamma_2)(a_2 + (b_2 + c_2)x)e^{-x} - (b_2 + c_2)(\alpha_2 + (\beta_2 + \gamma_2))e^{-x}}{(a_2 + (b_2 + c_2)x)^2}\right]. \tag{9}$$

By using the fact that $F(\bar{x}) = 0$ together with (8) and (9) we obtain

$$F'(\bar{x}) = \left[\frac{(\beta_2 + \gamma_2) + (b_2 + c_2)f(\bar{x})}{a_1 + (b_1 + c_1)f(\bar{x})}\right] \times \left[\frac{(\beta_1 + \gamma_1)e^{-f(\bar{x})} + (b_1 + c_1)\bar{x}}{a_2 + (b_2 + c_2)\bar{x}}\right] - 1. \tag{10}$$

By using the condition define in (6) we have $F'(\bar{x}) < 0$ and this implies that the equation $F(x) = 0$ has unique positive solution in I_1. Furthermore, from (7) we see that M_1 and m_1 satisfy equation $F(x) = 0$ which shows that $M_1 = m_1$. Therefore from equation (7) it is clear that $M_2 = m_2$. From Lemma 2 it follows that system (1) has unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of system equation (1) converges to the unique positive equilibrium point as $n \to \infty$. This completes the proof of the theorem. $\square$

In the following result we study the conditions for global asymptotic stability of unique positive equilibrium of system (1).

Theorem 3. *The unique positive equilibrium of system (1) is globally asymptotically stable if condition (6) of Theorem 2 is satisfied.*

Proof. As we know that $(\bar{x}, \bar{y})$ is nontrivial equilibrium of system (1), so one has

$$\bar{x} = \frac{\alpha_1 + (\beta_1 + \gamma_1)e^{-\bar{y}}}{a_1 + (b_1 + c_1)\bar{y}}, \ \bar{y} = \frac{\alpha_2 + (\beta_2 + \gamma_2)e^{-\bar{x}}}{a2 + (b_2 + c_2)\bar{x}}. \tag{11}$$

The linearized system of (1) evaluated at unique positive equilibrium $(\bar{x}, \bar{y})$ together with equation (11) is given by

$$\begin{aligned} x_{n+1} &= -\frac{\beta_1 e^{-\bar{y}} + b_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} y_n - \frac{\gamma_1 e^{-\bar{y}} + c_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} y_{n-1}, \\ y_{n+1} &= -\frac{\beta_2 e^{-\bar{x}} + b_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} x_n - \frac{\gamma_2 e^{-\bar{x}} + c_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} x_{n-1}, \end{aligned}$$

which is equivalent to the following matrix form

$$Z_{n+1} = F_J(\bar{x}, \bar{y}) Z_n,$$

where $Z_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ and the Jacobian matrix $F_J(\bar{x}, \bar{y})$ evaluated at nontrivial equilibrium $(\bar{x}, \bar{y})$ of system (1) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & A_1 & 0 & A_2 \\ B_1 & 0 & B_2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= -\frac{b_1\bar{x} + \beta_1 e^{-\bar{y}}}{a_1 + (b_1 + c_1)\bar{y}}, \ A_2 = -\frac{c_1\bar{x} + \gamma_1 e^{-\bar{y}}}{a_1 + (b_1 + c_1)\bar{y}}, \\ B_1 &= -\frac{b_2\bar{y} + \beta_2 e^{-\bar{x}}}{a_2 + (b_2 + c_2)\bar{x}}, \ B_2 = -\frac{c_2\bar{y} + \gamma_2 e^{-\bar{x}}}{a_2 + (b_2 + c_2)\bar{x}}. \end{aligned}$$

The characteristic equation of Jacobian matrix $F_J(\bar{x}, \bar{y})$ is given by

$$\lambda^4 - A_1B_1\lambda^2 - (A_1B_2 + A_2B_1)\lambda - A_2B_2 = 0. \tag{12}$$

Assume that condition (6) holds true and taking $\Omega = |A_1B_1| + |A_1B_2| + |A_2B_1| + |A_2B_2|$, we have

$$\begin{aligned}\Omega &= \frac{\beta_1 e^{-\bar{y}} + b_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\beta_2 e^{-\bar{x}} + b_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} + \frac{\beta_1 e^{-\bar{y}} + b_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\gamma_2 e^{-\bar{x}} + c_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} \\ &\quad + \frac{\gamma_1 e^{-\bar{y}} + c_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\beta_2 e^{-\bar{x}} + b_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} + \frac{\gamma_1 e^{-\bar{y}} + c_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\gamma_2 e^{-\bar{x}} + c_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} \\ &\leq \frac{\beta_1 + b_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\beta_2 + b_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} + \frac{\beta_1 + b_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\gamma_2 + c_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} \\ &\quad + \frac{\gamma_1 + c_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\beta_2 + b_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} + \frac{\gamma_1 + c_1\bar{x}}{a_1 + (b_1 + c_1)\bar{y}} \times \frac{\gamma_2 + c_2\bar{y}}{a_2 + (b_2 + c_2)\bar{x}} \\ &= \left[\frac{(\beta_2 + \gamma_2) + (b_2 + c_2)\bar{y}}{a_1 + (b_1 + c_1)\bar{y}}\right] \times \left[\frac{(\beta_1 + \gamma_1) + (b_1 + c_1)\bar{x}}{a_2 + (b_2 + c_2)\bar{x}}\right] < 1.\end{aligned}$$

Then it follows that all the roots of equation (12) are of absolute less than one which sure that $(\bar{x}, \bar{y})$ is locally asymptotically stable. Using Theorem 2, we obtain that $(\bar{x}, \bar{y})$ is globally asymptotically stable. This completes the proof of the theorem. □

3 Rate of convergence

In this section, we investigate the rate at which the nontrivial solution of system (1) converges to positive constant solution of (1).

The following results are fundamental in difference equations for the rate of convergence of solutions. First, we consider the following system of difference equations:

$$T_{n+1} = (C + D(n))\, T_n, \tag{13}$$

where T_n is an m-dimensional vector, $C \in \mathbb{R}^{m\times m}$ is a constant matrix, and $D : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ is a matrix function which holds

$$\|D(n)\| \to 0 \tag{14}$$

as $n \to \infty$ where $\|\cdot\|$, indicate any arbitrary matrix norm which is associated with the vector norm

$$\|(u, v)\| = \sqrt{u^2 + v^2}.$$

Proposition 1. *(Perron's Theorem)[28] Suppose that condition* (14) *holds. If T_n is a* *solution of* (13)*, then either* $T_n = 0$ *for all large* n *or*

$$\tau = \lim_{n\to\infty} (\|T_n\|)^{1/n} \tag{15}$$

exists and is equal to the absolute of one the eigenvalues of matrix C*.*

Proposition 2. *[28] Suppose that condition (14) holds. If* T_n *is a solution of (13), then either* $T_n = 0$ *for all large* n *or*

$$\tau = \lim_{n\to\infty} \frac{\|T_{n+1}\|}{\|T_n\|} \tag{16}$$

exists and is equal to the absolute of one the eigenvalues of matrix C*.*

Let $\{(x_n, y_n)\}$ be any solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [L_1, U_1]$ and $\bar{y} \in [L_2, U_2]$. To evaluate the error terms, one has from the system (1)

$$\begin{aligned}
x_{n+1} - \bar{x} &= \frac{\alpha_1 + \beta_1 e^{-y_n} + \gamma_1 e^{-y_{n-1}}}{a_1 + b_1 y_n + c_1 y_{n-1}} - \frac{\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}}}{a_1 + (b_1 + c_1)\bar{y}} \\
&= \frac{\beta_1 \left(e^{-y_n} - e^{-\bar{y}}\right)}{(a_1 + b_1 y_n + c_1 y_{n-1})(y_n - \bar{y})}(y_n - \bar{y}) \\
&- \frac{b_1 \left(\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}}\right)}{(a_1 + (b_1 + c_1)\bar{y})\,(a_1 + b_1 y_n + c_1 y_{n-1})}(y_n - \bar{y}) \\
&+ \frac{\gamma_1 \left(e^{-y_{n-1}} - e^{-\bar{y}}\right)}{(a_1 + b_1 y_n + c_1 y_{n-1})(y_{n-1} - \bar{y})}(y_{n-1} - \bar{y}) \\
&- \frac{c_1 \left(\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}}\right)}{(a_1 + (b_1 + c_1)\bar{y})\,(a_1 + b_1 y_n + c_1 y_{n-1})}(y_{n-1} - \bar{y}),
\end{aligned}$$

and

$$\begin{aligned}
y_{n+1} - \bar{y} &= \frac{\alpha_2 + \beta_2 e^{-x_n} + \gamma_2 e^{-x_{n-1}}}{a_2 + b_2 x_n + c_2 x_{n-1}} - \frac{\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{x}}}{a_2 + (b_2 + c_2)\bar{x}} \\
&= \frac{\beta_2 \left(e^{-x_n} - e^{-\bar{x}}\right)}{(a_2 + b_2 x_n + c_2 x_{n-1})(x_n - \bar{x})}(x_n - \bar{x}) \\
&- \frac{b_2 \left(\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{x}}\right)}{(a_2 + (b_2 + c_2)\bar{x})\,(a_2 + b_2 x_n + c_2 x_{n-1})}(x_n - \bar{x}) \\
&+ \frac{\gamma_2 \left(e^{-x_{n-1}} - e^{-\bar{x}}\right)}{(a_2 + b_2 x_n + c_2 x_{n-1})(x_{n-1} - \bar{x})}(x_{n-1} - \bar{x}) \\
&- \frac{c_2 \left(\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{x}}\right)}{(a_2 + (b_2 + c_2)\bar{x})\,(a_2 + b_2 x_n + c_2 x_{n-1})}(x_{n-1} - \bar{x}).
\end{aligned}$$

Let $e_n^1 = x_n - \bar{x}$, and $e_n^2 = y_n - \bar{y}$, then one has

$$e_{n+1}^1 = a_n e_n^2 + b_n e_{n-1}^2,$$

and

$$e_{n+1}^2 = c_n e_n^1 + d_n e_{n-1}^1,$$

where

$$a_n = \frac{\beta_1 \left(e^{-y_n} - e^{-\bar{y}}\right)}{(a_1 + b_1 y_n + c_1 y_{n-1})(y_n - \bar{y})} - \frac{b_1 \left(\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}}\right)}{\left(a_1 + (b_1 + c_1)\bar{y}\right)\left(a_1 + b_1 y_n + c_1 y_{n-1}\right)},$$

$$b_n = \frac{\gamma_1 \left(e^{-y_{n-1}} - e^{-\bar{y}}\right)}{(a_1 + b_1 y_n + c_1 y_{n-1})(y_{n-1} - \bar{y})} - \frac{c_1 \left(\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}}\right)}{\left(a_1 + (b_1 + c_1)\bar{y}\right)\left(a_1 + b_1 y_n + c_1 y_{n-1}\right)},$$

$$c_n = \frac{\beta_2 \left(e^{-x_n} - e^{-\bar{x}}\right)}{(a_2 + b_2 x_n + c_2 x_{n-1})(x_n - \bar{x})} - \frac{b_2 \left(\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{x}}\right)}{\left(a_2 + (b_2 + c_2)\bar{x}\right)\left(a_2 + b_2 x_n + c_2 x_{n-1}\right)},$$

$$d_n = \frac{\gamma_2 \left(e^{-x_{n-1}} - e^{-\bar{x}}\right)}{(a_2 + b_2 x_n + c_2 x_{n-1})(x_{n-1} - \bar{x})} - \frac{c_2 \left(\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{x}}\right)}{\left(a_2 + (b_2 + c_2)\bar{x}\right)\left(a_2 + b_2 x_n + c_2 x_{n-1}\right)}.$$

Moreover,

$$\lim_{n\to\infty} a_n = -\frac{\beta_1 e^{-\bar{y}}}{a_1 + (b_1 + c_1)\bar{y}} - \frac{b_1 \left(\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}}\right)}{\left(a_1 + (b_1 + c_1)\bar{y}\right)^2} = A_1,$$

$$\lim_{n\to\infty} b_n = -\frac{\gamma_1 e^{-\bar{y}}}{a_1 + (b_1 + c_1)\bar{y}} - \frac{c_1 \left(\alpha_1 + (\beta_1 + \gamma_1) e^{-\bar{y}}\right)}{\left(a_1 + (b_1 + c_1)\bar{y}\right)^2} = A_2,$$

$$\lim_{n\to\infty} c_n = -\frac{\beta_2 e^{-\bar{x}}}{a_2 + (b_2 + c_2)\bar{x}} - \frac{b_2 \left(\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{x}}\right)}{\left(a_2 + (b_2 + c_2)\bar{x}\right)^2} = B_1,$$

$$\lim_{n\to\infty} d_n = -\frac{\gamma_2 e^{-\bar{x}}}{a_2 + (b_2 + c_2)\bar{x}} - \frac{c_2 \left(\alpha_2 + (\beta_2 + \gamma_2) e^{-\bar{x}}\right)}{\left(a_2 + (b_2 + c_2)\bar{x}\right)^2} = B_2.$$

Now the limiting system of error terms can be written as

$$\begin{bmatrix} e_{n+1}^1 \\ e_{n+1}^2 \\ e_n^1 \\ e_n^2 \end{bmatrix} = \begin{bmatrix} 0 & A_1 & 0 & A_2 \\ B_1 & 0 & B_2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{bmatrix},$$

which is same as the linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y})$.

Using Proposition 1, one has following result.

Theorem 4. *Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (1) such that*

$\lim_{n\to\infty} x_n = \bar{x}$, *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where* $\bar{x} \in [L_1, U_1]$ *and* $\bar{y} \in [L_2, U_2]$. *Then, the error vector* $e_n = \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{pmatrix}$ *of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty} (\|e_n\|)^{\frac{1}{n}} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|, \quad \lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|,$$

where $\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})$ *are the characteristic roots of Jacobian matrix* $F_J(\bar{x}, \bar{y})$.

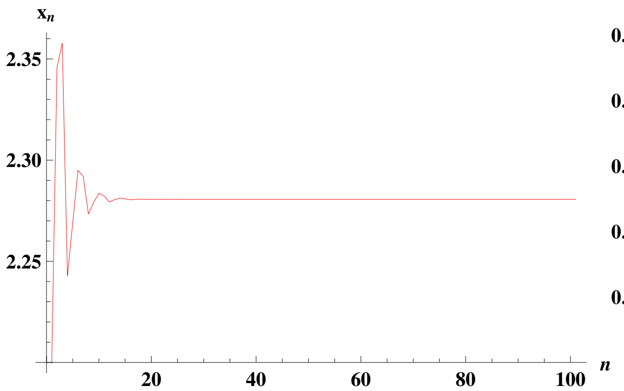
4 Numerical simulations and discussion

Example 1. *Let* $\alpha_1 = 0.002$, $\beta_1 = 0.2$, $\gamma_1 = 1.6$, $a_1 = 0.4$, $b_1 = 0.0008$, $c_1 = 0.07$, $\alpha_2 = 0.5$, $\beta_2 = 0.1$, $\gamma_2 = 0.7$, $a_2 = 0.99$, $b_2 = 0.0002$, $c_2 = 0.003$. *Then, system (1) can be written as*

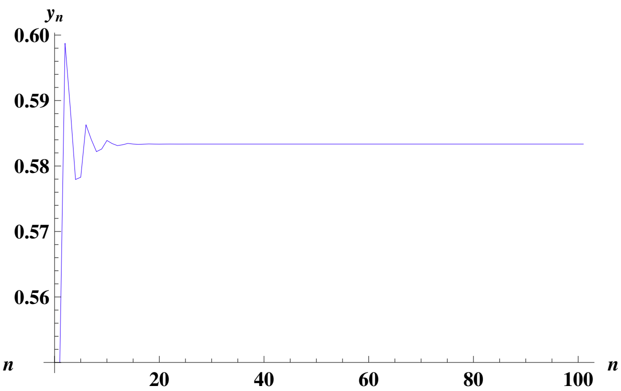
$$x_{n+1} = \frac{0.002 + 0.2e^{-y_n} + 1.6e^{-y_{n-1}}}{0.4 + 0.0008y_n + 0.07y_{n-1}}, \quad y_{n+1} = \frac{0.5 + 0.1e^{-x_n} + 0.7e^{-x_{n-1}}}{0.99 + 0.0002x_n + 0.003x_{n-1}}, \tag{17}$$

with initial conditions $x_{-1} = 2.2$, $x_0 = 2.1$, $y_{-1} = 0.56$, $y_0 = 0.55$.

In this case the unique positive equilibrium point of system (17) is given by $(\bar{x}, \bar{y}) = (2.28063, 0.583352)$. *Moreover, in Fig. 1 the plot of* x_n *is shown in Fig. 1a and the plot of* y_n *is shown in Fig. 1b for system (17).*



(a) Plot of x_n for the system (17)



(b) Plot of y_n for the system (17)

Figure 1: Plots for the system (17)

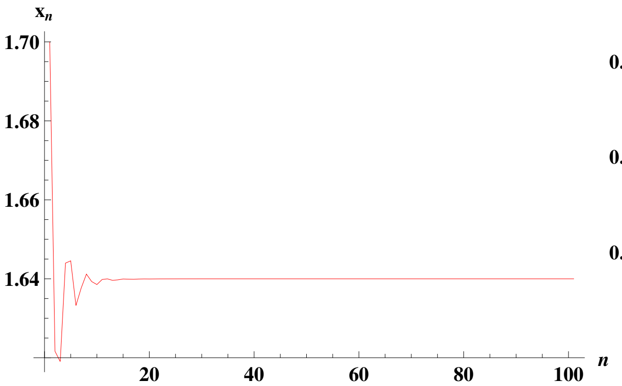
Example 2. *Let* $\alpha_1 = 2$, $\beta_1 = 12$, $\gamma_1 = 16$, $a_1 = 14$, $b_1 = 8$, $c_1 = 7$, $\alpha_2 = 5$, $\beta_2 = 9$,

$\gamma_2 = 11,\ a_2 = 13,\ b_2 = 1.1,\ c_2 = 30.$ *Then, system (1) can be written as*

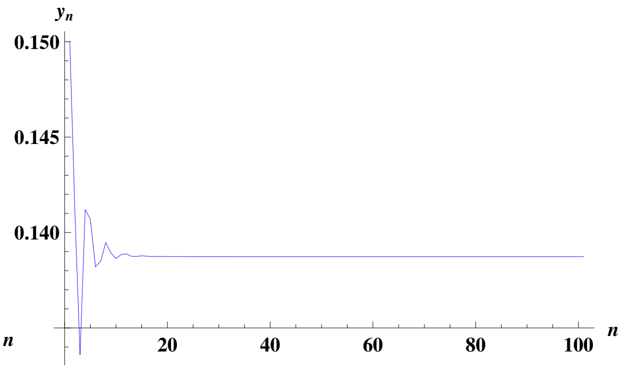
$$x_{n+1} = \frac{2 + 12e^{-y_n} + 16e^{-y_{n-1}}}{14 + 8y_n + 7y_{n-1}}, \quad y_{n+1} = \frac{5 + 9e^{-x_n} + 11e^{-x_{n-1}}}{13 + 1.1x_n + 30x_{n-1}}, \tag{18}$$

with initial conditions $x_{-1} = 1.6,\ x_0 = 1.7,\ y_{-1} = 0.14,\ y_0 = 0.15.$

In this case the unique positive equilibrium point of system (18) is given by $(\bar{x}, \bar{y}) = (1.64, 0.138736)$. *Moreover, in Fig. 2 the plot of* x_n *is shown in Fig. 2a an the plot of* y_n *is shown in Fig. 2b of the system (18).*



(a) Plot of x_n for the system (18)



(b) Plot of y_n for the system (18)

Figure 2: Plots for the system (18)

Concluding remarks

This work is related to the qualitative behavior of a discrete-time dynamical system, which may be considered as generalized population model discussed in [26]. Thus our results, considerably extend some previous investigations in literature. First it is investigated that system (1) is bounded and persists and then existence and uniqueness of positive equilibrium point is proved. We proved that the system (1) has a unique positive equilibrium point, which is locally asymptotically stable. The method of linearization is used to prove the local asymptotic stability of unique equilibrium point. Linear stability analysis shows that the positive steady-state of the system (1) is asymptotically stable under certain parametric conditions. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which parametric conditions lead to these long-term behaviors. In case of nonlinear dynamical systems, it is very crucial to discuss global behavior of the system. Moreover, we investigated the rate of convergence of a solution that converges to the unique positive equilibrium point of system (1).

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