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A Number Theoretic Aspect of Cancellative Principal Subgroup Near-Ring

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Abstract

In this paper, our attempt is to present a result on commutative cancellative principal subgroup near-ring. We show if N is a commutative cancellative principal subgroup near-ring in which sum of two N-subgroups is again an N-subgroup, then for every pair of non-zero elements a and b of N,

$$ab = (a,b)[a,b]$$

Where (a,b) denotes the greatest common divisor (g.c.d.) of a and b & [a,b] denotes the least common multiple (l.c.m.) of a and b.

Key Words: Cancellative principal subgroup near-ring, Commutative near-ring, N-subgroup, g.c.d. & l.c.m.

1. **Introduction:** A triple (N, +,•), where N is a non-empty set, + and • are two binary operations in N, is called a right near-ring if
   (i) (N,+) is a group (not necessarily Abelian)
   (ii) (N, •) is a semi group and
   (iii) $(a+b)\bullet c = a\bullet c+b\bullet c$ for $a,b,c \in N$

[If (iii) is replaced by (IV) $a \bullet(b +c) = a\bullet b +a\bullet c$, then the corresponding triple is called a left near- ring.]

By ab, we will mean $a\bullet b$ for $a, b \in N$.

Obviously, every ring is a left as well as a right near-ring. So, a near-ring can be called a generalised ring.

Example 1. The set M (G) of all mappings of an additive group G into itself with addition and multiplication defined by

$(f + g) (a) = f (a) + g (a)$ and

$(f g) (a) = f (g (a))$, for all $a\in G$ and $f, g\in M (G)$

Forms a right near-ring. Here, the distributive law $(f +g) h=f h +g h$ is always satisfied, while the other $f(g +h) = f g +f h$ is not –even if G is Abelian. Hence, near-rings have distinct existence.
In this paper, we confine our discussion on right near-ring only. By a near-ring (nr.) we will mean a right near-ring.

The additive identity of the group (N, +) of a near-ring N is called the zero element and it is denoted by 0.

Lemma 1. In a near-ring N, $0\bullet a = 0$, for all $a \in N$

A near-ring N is called zero-symmetric if $a\bullet 0 = 0$, for all $a \in N$. If the semi group (N, •) of a near-ring N possesses an element 1 such that $a\bullet 1 = a = 1\bullet a$, for all $a \in N$, then 1 is called the identity or unity of N. An element $x \in N$ is called idempotent if $x^2 = x$, Moreover, an idempotent x is called central if ax = xa for all $a \in N$. It is to be noted that the identity (if exists) of N is always central idempotent. If (N, •) is commutative, we call N itself a commutative near-ring.

If N is a near-ring with unity 1, then the group (E, +) is called an N-group (near-ring group) when there exists a map $N\times E \to E$, $(n, e) \to ne$ such that

(i) $(n_1+n_2)e = n_1e+ n_2e$
(ii) $(n_1n_2)e = n_1(n_2e)$
(iii) $1.e = e$, for all $n_1, n_2 \in N$, $e \in E$

In what follows, E will stand for the near-ring group N^E. Clearly, near-ring N can always be considered as an N-group. We shall write N^N to denote N as an N-group.

Example 2. Let G be an additive group and M(G) be a near-ring defined in Example 1., then G is a M (G)-group when $M(G)\times G \to G$ such that $(f ,x) \to f(x)$, for all $x \in G$, $f \in M(G)$.

Example 3. Every left module M over a ring R is an R-group over the near-ring R.

If N is a near-ring and H is a subgroup of (N, +), then H is called

(i) a left N-subgroup of N if $NH \subseteq H$
(ii) a right N-subgroup of N if $HN \subseteq H$
(iii) a subnear-ring of N if $HH \subseteq H$ and
(iv) an invariant subnear-ring of N if $NH \subseteq H$ and $HN \subseteq H$.

A subgroup M of an N-group E over the near-ring N is called an N-subgroup of E if $NM \subseteq M$.

A left N-subgroup H of N is an N-subgroup of N^N and conversely.

If I is an additive normal subgroup of a near-ring N, then I is called

(i) a right ideal of N if $in \in I$, for all $i \in I$, $n \in N$.
(ii) a left ideal of N if $n_1(i+n_2) - n_1n_2 \in I$, for all $i \in I$, $n_1, n_2 \in N$.
(iii) an ideal of N if I is a right as well as a left ideal of N.

Lemma 2. Every left ideal of a near-ring N is a left N-subgroup of N.
Lemma 3. Intersection of two ideals of an N-group E is again an ideal of E.
Lemma 4. Intersection of two N-subgroups of a near-ring N is again an N-subgroup of N.

If X be a non-empty subset of an N-group E, then the intersection of all N-subgroups (ideals) of E that contain X is the N-subgroup (ideal) generated by X.

Lemma 5. The sum of two left ideals of a near-ring N is also a left ideal of N.

If $a \in N$, then the N-subgroup (left ideal, ideal) of N generated by a is denoted by $\langle a\rangle$ ((a), ((a)))

Since (a) is the intersection of all left ideals of N containing a and ((a)) is a left ideal of N containing a, we note that $(a) \subseteq ((a))$.

2. **Principal Subgroup Near-Ring:** The N-subgroup Na ($a \in N$) is called a principal N-subgroup of N.

N is a principal subgroup near-ring (PSNR) if every N-subgroup of it is principal, i.e., of the form Na, $a \in N$.

Let a, b, g $\in N, \neq 0$. Then g is a greatest common divisor (g.c.d.) of a, b if
(i) $g \mid a$, $g \mid b$
(ii) If $d \mid a$, $d \mid b$ for some $d \in N$, then $d \mid g$.

A g.c.d. of a, b is denoted by (a, b).

Let a, b, m $\in N, \neq 0$. Then m is a least common multiple (l.c.m.) of a, b if
(i) $a \mid m$, $b \mid m$
(ii) If $a \mid d$, $b \mid d$ for some $d \in N$, then $m \mid d$
An l.c.m. of a, b is denoted by [a, b].

Proposition 1. Given any $a \in N$, the set $Na=\{xa \mid x \in N\}$ is an N-subgroup of N such that $Na \subseteq \langle a \rangle$ and if $1 \in N$, then $\langle a \rangle = Na$.

N is a cancellative near-ring if in it ax = ay implies x = y for $a \neq 0$.
We immediately get the

Proposition 2. Let N be a cancellative near- ring. Then
(i) N is a zero symmetric near-ring,
(ii) N has no proper zero-divisors,
(iii) N allows right cancellation, i.e., xa = ya implies x=y in N for $a \neq 0$.

Proposition 3. If N is cancellative, every left ideal of N is an N-subgroup of N.
Lemma 6. If N is cancellative and $a \in N$, then $Na \subseteq \langle a \rangle \subseteq (a) \subseteq ((a))$.
Proposition 4. If N is cancellative and e is a non-zero idempotent of N, then e = 1.

3. Cancellative Principal Subgroup Near-Ring

A principal subgroup near-ring N is called a cancellative principal subgroup near-ring if in it ax = ay implies x = y for $a \neq 0$

Theorem 1. Let N be a cancellative PSNR. Every left ideal of N is a principal N-subgroup of N.
Proof: Since N is cancellative, it is a zero symmetric near-ring by proposition 2 and hence every left ideal of N is an N-subgroup of N. The fact that N is a PSNR completes the proof.
Theorem 2. If N is a cancellative PSNR, then $1 \in N$.
Proof : Since N is a PSNR and N is an N-subgroup of itself, we have N = Na for some $a \in N, \neq 0$. Hence there exists $e \in N, \neq 0$ such that a = ea. We note that eea = ea. As N is cancellative, it follows that ee = e. Thus e is a non-zero idempotent of N and hence e = 1 by proposition 4.

Theorem 3. Let N be a cancellative PSNR in which sum of two N-subgroups is again an N-subgroup. Then every pair of non-zero elements a and b of N has a g.c.d and an l.c.m. Further if c = g.c.d. (a, b), then
c = xa + yb, for some $x, y \in N$.
Proof: Since N is a cancellative PSNR, so by theorem 2, we have $1 \in N$. Also, then we have $\langle a \rangle = Na$ and $\langle b \rangle = Nb$.

Again, since sum of two N-subgroups of N is an N-subgroup and since N is a PSNR, so we have,

$$\langle a \rangle + \langle b \rangle = Na + Nb = Nc = \langle c \rangle \text{ for some } c \in N.$$

Let $x \in \langle a \rangle$. Then $x = n_1 a$ for some $n_1 \in N$

$\Rightarrow x = n_1 a + 0b \in Na + Nb = Nc = \langle c \rangle$

Hence $\langle a \rangle \subseteq \langle c \rangle$

Similarly, $\langle b \rangle \subseteq \langle c \rangle$

Hence $a \in \langle c \rangle = Nc$ and $b \in \langle c \rangle = Nc$.

$\Rightarrow a = uc$ and $b = vc$ for some $u,v \in N$

$\Rightarrow c \mid a$ and $c \mid b$.

Let $d \in N, \neq 0$ be such that $d \mid a$ and $d \mid b$. Then

$a = md$ and $b = nd$ for some $m,n \in N$.

$\Rightarrow a \in Nd$ and $b \in Nd$.

$\Rightarrow a \in \langle d \rangle$ and $b \in \langle d \rangle$. [as $1 \in N$, so $\langle d \rangle = Nd$]

$\Rightarrow \langle a \rangle \subseteq \langle d \rangle$ and $\langle b \rangle \subseteq \langle d \rangle$ [as $\langle x \rangle$ is the smallest N-subgroup of N containing x]

$\Rightarrow \langle a \rangle + \langle b \rangle \subseteq \langle d \rangle$

$\Rightarrow \langle c \rangle \subseteq \langle d \rangle$

$\Rightarrow c \in \langle d \rangle = Nd$

$\Rightarrow c = qd$ for some $q \in N$.

So $d \mid c$

Hence c is a g.c.d. of a and b.

Now, $c \in \langle c \rangle = \langle a \rangle + \langle b \rangle = Na + Nb$

$\Rightarrow c = xa + yb$ for some $x,y \in N$.

Let us now consider $\langle a \rangle \cap \langle b \rangle$. Since intersection of two N-subgroups of N is again an N-subgroup of N and since every N-subgroup of N is a principal N-subgroup of N, so we have,

$\langle a \rangle \cap \langle b \rangle = Nz$ for some $z \in N$ [as N is a PSNR]

$\Rightarrow \langle a \rangle \cap \langle b \rangle = \langle z \rangle$ [as $1 \in N$]

Hence $\langle z \rangle \subseteq \langle a \rangle$ and $\langle z \rangle \subseteq \langle b \rangle$

$\Rightarrow a \mid z$ and $b \mid z$ (as above).

Let $h \in N, \neq 0$ be such that $a \mid h$ and $b \mid h$.

Then , $\langle h \rangle \subseteq \langle a \rangle$ and $\langle h \rangle \subseteq \langle b \rangle$ [as $h \in Na = \langle a \rangle$ and $h \in Nb = \langle b \rangle$]

$\Rightarrow \langle h \rangle \subseteq \langle a \rangle \cap \langle b \rangle$

$\Rightarrow \langle h \rangle \subseteq \langle z \rangle$

$\Rightarrow h \in Nz$ [as $h \in \langle h \rangle \subseteq \langle z \rangle = Nz$]

$\Rightarrow h = kz$ for some $k \in N$.

$\Rightarrow z \mid h$.

Hence z is an l.c.m of a and b.

Theorem 4. If N is a commutative cancellative PSNR in which sum of two N-subgroups is again an N-subgroup, then for every pair of non-zero elements a and b of N,

$$ab = (a, b)\,[a, b]$$

Proof: since N is a cancellative PSNR in which sum of two N-subgroups is again an N-subgroup, therefore by theorem 3, every pair of non-zero elements a and b of N has a g.c.d. and an l.c.m.

Let $d = (a, b)$. Then

$d \mid a$ and $d \mid b$

$\Rightarrow$ a = dr and b = ds for some r, s $\in$ N
Hence ab = (dr) (ds) = d (rds) = dm……… (i) where m = rds
Now, m = rds = as as dr = a
and m = rds = br as ds = b
Hence a | m and b | m.
So, m is a common multiple of a and b.
Let c be any common multiple of a and b.
Then a | c and b | c
$\Rightarrow$ c = au = bv for some u, v $\in$ N
Now, d = (a,b) $\Rightarrow$ d = ax + by for some x,y, $\in$ N [by theorem 3]
Now, cd = c (ax + by) = acx + bcy = abvx + bauy = ab (vx + uy)
= dm (vx + uy) [by (i)]
Hence c = m (vx + uy) [by cancellation law]
$\Rightarrow$ m | c
Thus m = [a, b]
So, from (i), we have ab = (a, b) [a, b]
This completes the proof.

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