Algorithms in Solving Polynomial Inequalities

Asia Pacific Journal of Multidisciplinary Research
Vol. 3 No. 4, 113-116
November 2015 Part IV
P-ISSN 2350-7756
E-ISSN 2350-8442
www.apjmr.com

Christopher M. Cordero[1] and Neil M. Mame[2]
College of Arts & Sciences, Batangas State University, Rizal Avenue, Batangas City, Philippines
[1]Princehon_17@yahoo.com,[2]mameneil10@yahoo.com

Date Received: September 29, 2015; Date Revised: October 29, 2015

Abstract - *A new method to solve the solution set of polynomial inequalities was conducted. When* $(x-r_1)(x-r_2) > 0$ *where* $r_1, r_2 \in \mathbb{R}$ *and* $r_1 < r_2$, *the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (-\infty,\ r_2) \cup (r_1, \infty\,)\,\}$. *Thus, when the inequality is* $(x-r_1)(x-r_2) \geq 0$, *then the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (-\infty,\ r_1] \cup [r_2, \infty)$. *If* $(x-r_1)(x-r_2) < 0$, *then the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (r_1,\ r_2)\,\}$. *Thus when* $(x-r_1)(x-r_2) \leq 0$, *the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in [r_1,\ r_2]\,\}$. *Let* $f(x) = ax^2 + bx + c$ *where* $a \neq 0, b$ *and* $c \in \mathbb{R}$. *If* $b^2 - 4ac < 0$, *then the solution of quadratic inequalities is* $\{\mathbb{R}\}$ *when, by substitution of a particular real number, the inequality is true. Otherwise, the solution of the inequality is* $\emptyset$. *Let* $r_1 < r_2 < \ldots < r_n \in \mathbb{R}$ *and* $n \geq 3$. *Let* $(x-r_1)(x-r_2)\ldots(x-r_n) > 0$ *if n is even. Then, the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (-\infty,\ r_1) \cup (r_n\ ,+\infty\,) \cup (\,r_i,\ r_{i+1}) \colon i \text{ is even}\,\}$. *Thus, when (x − r1x−r2...x−rn≥0, the solution is x∈ℝ x∈−∞, r1∪rn ,+∞∪ri, ri+1: i is even}. If n is odd, then the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (r_n\ ,+\infty\,) \cup (\,r_i,\ r_{i+1}) \colon i \text{ is odd}\,\}$. *Thus, when* $(x-r_1)(x-$ *r2...x−rn≥0, the solution set is x∈ℝ x∈rn ,+∞∪ri, ri+1:i is odd}. Let x−r1x−r2...x−rn<0 if n is even. Then, the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (\,r_i,\ r_{i+1}) : i \text{ is odd}\,\}$. *Thus, when* $(x-r_1)(x-$ *r2...x−rn≤0, then the solution set is x∈ℝ x∈ri, ri+1:i is odd}. If n is an odd, then the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (-\infty,\ r_1\,) \cup (\,r_i,\ r_{i+1}) \colon i \text{ is even}\,\}$. *Thus, when* $(x-r_1)(x-r_2)\ldots(x-r_n) \leq 0$, *the solution set is* $\{\, x \in \mathbb{R} \,|\, x \in (-\infty,\ r_1] \cup [r_i,\ r_{i+1}] \colon i \text{ is even}\,\}$. *This research provides a novel method in solving the solution set of polynomial inequalities, in addition to other existing methods.*

Keywords: *polynomial, polynomial inequality, solution, solution set, quadratic inequality, real roots and imaginary roots*

INTRODUCTION

A *polynomial*, with degree *n*, denoted by $P_n(x)$, is an algebraic expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, where *n* is an integer and $n \geq 0$, x is a variable and a_i's are the constant coefficients with $a_0 \neq 0$[1]. *Polynomial inequality* is any inequality that can be put in one of the forms of $P(x) > 0$, $P(x) < 0, P(x) \geq 0$ and $P(x) \leq 0$ where $P(x)$ is a polynomial. A real number *r* is a *solution* of the polynomial inequality if, upon the substitution of *r* for the *x* in the inequality, the inequality is true. The set of all solution is called a *solution set*. The objective of this study is to provide alternative methods to solve for the solution set of polynomial inequalities. In this research, the researchers presents general solutions to quadratic inequality and develops algorithms in solving polynomial inequality. Quadratic inequality is inequality where roots of quadratic equation are real and imaginary is considered[2]. The polynomial inequalities that will be considered are the polynomial inequalities where $n \geq 3$ and roots of polynomial equation are real and distinct. Thus, polynomial inequalities where $n \geq 3$ and some roots of the polynomial are imaginary are not covered in this study. Presentation and derivation of a shorter method in solving polynomial inequality is a contribution to the development of mathematics particularly in the branch of college algebra. Everybody knows, algebra is a tool in solving problems in other branches of mathematics. Thus, using the results of the study, the solution to other mathematics problems that involves inequalities will be shortened. Therefore, in a way, the study contributes significantly to the development of other branches of mathematics.

For the students taking the subject of College Algebra this will serve as alternative way of finding solution set of polynomial inequalities. On the part of teachers, this study serves as reference and supplementary material in their continuous study on the solutions of polynomial inequalities and other related topics. In 2008, Ikenaga [3] of Millersville University conducted the study "Quadratic Inequality" and disclosed the solution set of quadratic inequality using the graph of the quadratic function. A quadratic function is a function of the form f(*x*) = $ax^2 + bx + c$. This study is related in a way that it is another method on finding the solution set of quadratic inequality. This concept used the researchers in proving the theorem of finding the solution set of a quadratic inequality where the roots of quadratic equation are imaginary. This study is related in a way that it is another method on finding the solution set of quadratic inequality.

METHODS

The methodology employed in the current study is basically the development of algorithms using expository method. The data or significant information needed in the study of solving polynomial inequalities was gathered through research from the library and the world wide web.

Since the purpose of the study is to introduce new a concept of solving polynomial inequalities, the researchers conducted a trial and error method to find a pattern in order to make a conjecture in certain cases. Then make a proof of the conjecture to assure the generalization of the claim and illustration of a provided theorem. Finally, the researchers made a comparison of the solution set of an inequality using the obtained theorem and the existing method.

Preliminary Results

Theorem 4.1.1. Let r_1 and $r_2 \in \mathbb{R}$ where $r_1 < r_2$. If $(x - r_1)(x - r_2) > 0$, then the solution is $\{ x \in$ *ℝ x∈−∞, r1∪r2,+∞ }*.

Corollary 4.1.1.1. Let r_1 and r_2 are real numbers where $r_1 < r_2$. If $(x - r_1)(x - r_2) \geq 0$, then the solutions is $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_2, +\infty) \}$.

Theorem 4.1.2. Let r_1 and r_2 are real numbers where $r_1 < r_2$ .If $(x - r_1)(x - r_2) < 0$, then the solution set is $\{ x \in \mathbb{R} \mid x \in (r_1, r_2)\}$.

Corollary 4.1.1.2. Let r_1 and r_2 are real numbers where $r_1 < r_2$.If $(x - r_1)(x - r_2) \leq 0$, then the solution set is $\{ x \in \mathbb{R} \mid x \in [r_1, r_2] \}$.

Remark 4.1.1. In a quadratic inequality where $b^2 - 4ac < 0$, the solution set is the set of all real numbers when, by substitution of a particular real number, the inequality is true. Otherwise, the solution set of the inequality is empty set.

Theorem 4.2.1. Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < . . . < r_n$. Let $n \geq 3$ be an even integer. Then $(x - r_1)\ (x - r_2) . . .(x - r_n) > 0$ if and only if $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_n , +\infty) \cup (r_i, r_{i+1}),$ i is even $\}$.

Corollary 4.2.1.1. Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < . . . < r_n$. Let $n \geq 3$ be an even integer. Then the solution set of $(x - r_1)(x - r_2)\ldots(x - r_n) \geq 0$ is the set $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_n , +\infty) \cup [r_i, r_{i+1}],$ i is even $\}$.

Theorem 4.2.2. Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < . . . < r_n$. Let $n \geq 3$ be an odd integer. Then $(x - r_1)\ (x - r_2) . . .(x - r_n) > 0$ if and only if $\{ x \in \mathbb{R} \mid x \in (r_n , +\infty) \cup (r_i, r_{i+1}),$ i is odd $\}$.

Corollary 4.2.2.1. Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < . . . < r_n$. Let $n \geq 3$ be odd integer. Then the solution set is $(x - r_1)\ (x - r_2) . . .(x - r_n) \geq 0$ is the set $\{ x \in \mathbb{R} \mid x \in [r_n , +\infty) \cup [r_i, r_{i+1}]$: i is odd $\}$.

Theorem 4.3.1. Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < . . . < r_n$. Let $n \geq 3$ is an even integer. Then $(x - r_1)\ (x - r_2) . . .(x - r_n) < 0$, if and only if $\{ x \in \mathbb{R} \mid x \in (r_i, r_{i+1})$, i is odd $\}$.

Corollary 4.3.1.1. Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < . . . < r_n$. Let $n \geq 3$ is even integer. Then the solution set is $(x - r_1)\ (x - r_2) . . .(x - r_n) \leq 0$ is the set $\{ x \in \mathbb{R} \mid x \in [r_i, r_{i+1}]$, i is odd $\}$.

Theorem 4.3.2. Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < \ldots < r_n$. Let $n \geq 3$ is odd integer. Then $(x - r_1)(x - r_2) \ldots (x - r_n) < 0$, if and only if $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_i, r_{i+1}), i \text{ is even} \}$.

Corollary 4.3.2.1 Let $r_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $r_1 < r_2 < \ldots < r_n$. Let $n \geq 3$ is odd integer. Then the solution set is $(x - r_1)(x - r_2) \ldots (x - r_n) \leq 0$ is the set $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_i, r_{i+1}], i \text{ is even} \}$.

FINDINGS

The findings of the study were as follows:

1. Quadratic inequality where the roots of quadratic equation are real and imaginary.

1.a.1 When the inequality is $(x - r_1)(x - r_2) > 0$, where r_1 and $r_2 \in \mathbb{R}$ and $r_1 < r_2$, then the solution set is $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_2, +\infty) \}$. Thus, if the inequality is $(x - r_1)(x - r_2) \geq 0$, then the solution set is $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_2, \infty) \}$.

1.a.2. When the inequality is $(x - r_1)(x - r_2) < 0$, where r_1 and $r_2 \in \mathbb{R}$ and $r_1 < r_2$, then the solution set is $\{ x \in \mathbb{R} \; x \in r1, r2 \}$. Thus, if the inequality is $(x-r1)x-r2 \leq 0$, then the solution set is $x \in \mathbb{R} \; x \in [r1, r2] \}$.

1.b. Let $f(x) = ax^2 + bx + c$, where $a \neq 0, b$ and c $\in \mathbb{R}$. If $b^2 - 4ac < 0$, then the solution set of quadratic inequalities is $\{\mathbb{R}\}$ when, by substitution of a particular real number, the inequality is true. Otherwise, the solution set of the inequality is $\emptyset$.

2. Let $(x - r_1)(x - r_2) \ldots (x - r_n) > 0$, where $r_1 < r_2 < \ldots < r_n \in \mathbb{R}$ and $n \geq 3$. If n is even then the solution set is $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_n, +\infty) \cup (r_i, r_{i+1}): i \text{ is even} \}$ thus, when $(x - r_1)(x - r_2) \ldots (x - r_n) \geq 0$ then, the solution is $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_n, +\infty) \cup [r_i, r_{i+1}]: i \text{ is even} \}$; if n is odd then the solution set is $\{ x \in \mathbb{R} \mid x \in (r_n, +\infty) \cup (r_i, r_{i+1}): i \text{ is odd} \}$ thus, when $(x - r_1)(x - r_2) \ldots (x - r_n) \geq 0$ then, the solution set is $\{ x \in \mathbb{R} \mid x \in [r_n, +\infty) \cup [r_i, r_{i+1}]: i \text{ is odd} \}$.

3. Let $(x - r_1)(x - r_2) \ldots (x - r_n) < 0$ where $r_1 < r_2 < \ldots < r_n \in \mathbb{R}$ and $n \geq 3$. If n is even then the solution set is $\{ x \in \mathbb{R} \mid x \in (r_i, r_{i+1}) : i \text{ is odd} \}$ thus, when $(x - r_1)(x - r_2) \ldots (x - r_n) \leq 0$ then the solution set is $\{ x \in \mathbb{R} \mid x \in [r_i, r_{i+1}]: i \text{ is odd} \}$; if n is an odd then the solution set is $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_i,$ $ri+1:i$ is even $\}$ thus, when $x-r1x-r2 \ldots x-rn \leq 0,$ then the solution set is $x \in \mathbb{R} \; x \in -\infty, r1 \cup ri, ri+1:i$ is even $\}$.

CONCLUSIONS

This paper provided the general solution of quadratic inequality where the roots of quadratic equation are real and imaginary. When the roots of quadratic equation are real, then the solution set of $ax^2 + bx + c > 0$ is the set of $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_2, \infty) \}$ thus, when the quadratic inequality $ax^2 + bx + c \geq 0$ the solution is the set $\{ x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_2, +\infty) \}$; if $ax^2 + bx + c < 0$ the solution set is the set is $\{ x \in \mathbb{R} \mid x \in (r_1, r_2) \}$ thus, when the quadratic inequality $ax^2 + bx + c \leq 0$ the solution set is the set $\{ x \in \mathbb{R} \mid x \in [r_1, r_2] \}$. If $b^2 - 4ac < 0$, then the solution set of quadratic inequalities is $\{\mathbb{R}\}$ when, by substitution of a particular real number, the inequality is true. Otherwise, the solution set of the inequality is $\emptyset$.

This study also provided algorithms solving the solution set of polynomial inequality where $n \geq 3$ is an integer. The first steps of the two algorithms is to let $P(x) = (x - r_1)(x - r_2) \ldots (x - r_n)$then, determine $r_1, r_2, \ldots r_n$,where

P-ISSN 2350-7756 | E-ISSN 2350-8442 | www.apjmr.com

$r_1 < r_2 < \ldots < r_n$. Using algorithm1 the solution set of polynomial inequality $(x - r_1)(x - r_2)\ldots(x - r_n) > 0$ where $r_i \in \mathbb{R}$ and $1 \leq i \leq n$, if n is even the solution set is the set of $\{x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_n, +\infty) \cup (r_i, r_{i+1}) : i \text{ is even}\}$ by applying theorems 4.2.1 thus, when the inequality $(x - r_1)(x - r_2)\ldots(x - r_n) \geq 0$ also n is even then the solution is the set of $\{x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_n, +\infty) \cup [r_i, r_{i+1}] : i \text{ is even}\}$ by applying corollary 4.2.1.1; if n is odd solution the solution set is the set $\{x \in \mathbb{R} \mid x \in (r_n, +\infty) \cup (r_i, r_{i+1}), i \text{ is odd}\}$ by applying theorem 4.2.2 thus, when the inequality is $(x - r_1)(x - r_2)\ldots(x - r_n) \geq 0$ and n is also odd the solution set is the set $\{x \in \mathbb{R} \mid x \in [r_n, +\infty) \cup [r_i, r_{i+1}] : i \text{ is odd}\}$ by applying Corollary 4.2.2.1. Using algorithm2 when the polynomial inequality $(x - r_1)(x - r_2)\ldots(x - r_n) < 0$ where $r_i \in \mathbb{R}$ and $1 \leq i \leq n$, if n is even then the solution set is the set $\{x \in \mathbb{R} \mid x \in (r_i, r_{i+1}) : i \text{ is odd}\}$ by applying Theorem 4.3.1, thus when the polynomial is $(x - r_1)(x - r_2)\ldots(x - r_n) \leq 0$ and n is also even then the solution set is the set $\{x \in \mathbb{R} \mid x \in [r_i, r_{i+1}] : i \text{ is odd}\}$; if n is odd then the solution set is the set $\{x \in \mathbb{R} \mid x \in (-\infty, r_1) \cup (r_i, r_{i+1}) i \text{ is even}\}$ by applying theorem 4.3.2 thus, when the polynomial inequality $(x - r_1)(x - r_2)\ldots(x - r_n) \leq 0$ and n is also odd then the solution set is the set $\{x \in \mathbb{R} \mid x \in (-\infty, r_1] \cup [r_i, r_{i+1}] : i \text{ is even}\}$ by applying Corollary 4.3.2.1

RECOMMENDATION

The results of the study may be used as an instructional reference material for instructors of mathematics. The results of the study may be introduced to the students taking up College Algebra and other mathematics courses where inequality is a prerequisite topic. Study the solution set of the polynomial inequality whose roots of the polynomial contain imaginary except to quadratic inequality. This study may serve as a basis for a new program that is yet to be developed by IT or computer experts. The problem on solving polynomial inequalities whose roots polynomial contain multiplicity is still open.

REFERENCES

[1] Gustafson, R. D. & Frisk, P. D. (2006). College Algebra, 9th edition, Learning.

[2] Leithold, L. (2002). College Algebra and Trigonometry, Addison-Wesley Publishing Company, Inc.

[3] Ikenaga, B. (2008) Quadratic Inequality, Undergraduate Thesis, Millersville University, 2008.

[4] Benton, W. & Hemingway Benton, H. (1997). Encyclopaedia Britannica.15th Edition, International Copyright Union.

Copyrights

Copyright of this article is retained by the author/s, with first publication rights granted to APJMR. This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creative commons.org/licenses/by/4.0/)