# Generalized Monotone Method for Sequential Caputo Fractional Boundary Value Problems

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**Abstract** Generalized monotone method together with coupled lower and upper solutions yield monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of the nonlinear problem under consideration. In this work, we have developed generalized monotone method for sequential Caputo fractional boundary value problem with mixed boundary conditions which are in terms of Caputo fractional derivative. For that purpose, we have obtained a representation form for the corresponding linear Caputo sequential boundary value problem in terms of the Green's function. In addition, we have obtained a linear comparison result for the linear sequential differential inequalities with linear mixed boundary conditions. The comparison result is useful in proving the monotonicity of the iterates as well as the uniqueness of the solution of the nonlinear sequential boundary value problem. Our method yields the integer results as a special case. Some numerical examples for the linear sequential Caputo fractional boundary value problems have also been presented.

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## 1 Introduction

It is well known that fractional dynamic systems represent better mathematical models than its counter part of dynamic systems with integer derivatives. See [1–7, 9–13, 16–18, 22] and the references therein for initial and/or for boundary value problems and its applications. Among the different definitions of fractional derivatives, dynamic systems involving Caputo derivative are closer to integer derivative. The reason for that is, when the fractional order $q$ is an integer, the Caputo fractional dynamic systems reduces to a dynamic systems with integer derivative. In addition, the initial conditions and/or the boundary conditions of Caputo fractional dynamic systems are the same as that of the integer derivative. However, the solution of the Caputo initial value problem of order $nq$ such that say $(n-1) < nq < n$, does not reduce to the solution of the corresponding integer dynamic systems of order $n$ when $q = 1$. See section 4.1.3 of [2] where the explicit solution for Caputo initial value problem of order $q$ when $n-1 < q < n$, has been obtained. If $q = n$, one cannot obtain the corresponding integer result as a special case. The reason for this is the fact that the Caputo derivative is not sequential, where as the integer derivative is sequential. In this work, we seek solutions of nonlinear Caputo boundary value problem, when the Caputo derivative of order $2q$ is sequential of order $q$. That is ${}^cD^{2q}u(x) = {}^cD^q({}^cD^q(u(x)))$. We assume that both the left and the right derivatives are sequential for boundary value problem under consideration. The sequentiality of the Caputo derivative has an added advantage in modeling by using $q$ as a parameter not only in the dynamic equation but also in the initial, and/or boundary conditions.

In literature most of the existence and uniqueness result for Caputo boundary value problems has been obtained using some kind of fixed point theorem. See [1, 2, 14, 15, 17, 19–25]. In this paper, we have developed generalized monotone iterative technique combined with coupled lower and upper solutions to obtain the existence of coupled minimal and maximal solutions. For that purpose, consider the nonlinear sequential Caputo fractional boundary value problem with mixed boundary conditions of the form

$$-{}^cD^{2q}u(x) = f(x, u, {}^cD^qu(x)) + g(x, u, {}^cD^qu(x))$$

 

for each $x \in J := [a, b]$, with boundary conditions,

$$\begin{aligned} \alpha_0 u(a) - \beta_0 {}^c D^q u(a) = b_0, \\ \alpha_1 u(b) + \beta_1 {}^c D^q u(b) = b_1, \end{aligned} \tag{1.1}$$

where $f, g \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $u \in C^2[J, \mathbb{R}]$. However, in this paper, $f$ and $g$ above depends only on $x, u$, with suitable conditions $\alpha_i, \beta_i$ for $i = 1, 2$. Furthermore, note that the boundary conditions do involve the value ${}^c D^q u(x)$ on the boundary also. This means $q$ plays a role as a parameter in the boundary condition as well. Also, generalized monotone method is a choice method when the nonlinear function is the sum of an increasing and decreasing function in the nonlinear term. Further, it is also a constructive method. We have obtained the Green's function with the corresponding homogeneous boundary conditions to represent the solution of the linear problem in terms of its nonhomogeneous term. We have also developed a linear comparison result which is useful in proving the monotonicity of the iterates and the uniqueness of the solution of the linear and nonlinear boundary value problem. In addition, we have presented some numerical results for the linear sequential Caputo fractional boundary value problem with homogeneous boundary conditions. All our results yield the known integer results as a special case.

## 2 Preliminary Results

In this section, we recall some basic definitions which are needed in our main results.

**Definition 2.1.** *The Caputo (left-sided) fractional derivative of $u(x)$ of order $q$, when $n-1 < q < n$, is given by equation*

$${}^c D^q_{a^+} u(x) = \frac{1}{\Gamma(n-q)} \int_a^x (x-s)^{n-q-1} u^{(n)}(s) \mathrm{d}s, \ x > a,$$

*and (right-sided) fractional derivative is given by*

$${}^c D^q_{b^-} u(x) = \frac{(-1)^n}{\Gamma(n-q)} \int_x^b (s-x)^{n-q-1} u^{(n)}(s) \mathrm{d}s, \ x < b,$$

*where* $u^{(n)}(t) = \dfrac{d^n(u)}{dt^n}$.

In particular, $q = n$, an integer, then ${}^c D^q u = u^{(n)}(x)$ and ${}^c D^q u = u'(x)$ if $q = 1$.

**Definition 2.2.** *The Riemann–Liouville (left-sided) fractional derivative of order $q$, when $(n-1) < q < n$ is defined as,*

$$D^q_{a^+} u(x) = \frac{1}{\Gamma(n-q)} (\frac{d}{dx})^n \int_a^x (x-s)^{n-q-1} u(s) \mathrm{d}s, \ x > a,$$

*and (right-sided) is given by*

$$D^q_{b^-} u(x) = -\frac{1}{\Gamma(n-q)} (\frac{d}{dx})^n \int_x^b (s-x)^{n-q-1} u(s) \mathrm{d}s, \ x < b.$$

The relation between Riemann-Lioville and Caputo fractional derivatives have been established in Lemma 2.2 of [2], when $(n-1) < q < n$. In this paper, we need this relation, for $n = 1$ and $2$ only. Here, we state the relation between the Liouville and Caputo fractional derivative, when $0 < q < 1$, only. If $0 \leq \mathbb{R}(q) < 1$ and $u(x) \in [a, b]$ then

$${}^c D^q_{a^+} u(x) = D^q_{a^+} u(x) - \frac{1}{\Gamma(1-q)} \frac{u(a)}{(x-a)^q}, \tag{2.1}$$

$${}^c D^q_{b^-} u(x) = \frac{1}{\Gamma(1-q)} \frac{u(b)}{(b-x)^q} - D^q_{b^-} u(x). \tag{2.2}$$

This relation will be useful in our linear comparison theorem related to sequential Caputo boundary value problem. See [2] Corollary 2.2 for more details.

 

Next we present the sequential Caputo fractional derivative. See [1] for the definition of Riemann–Liouville sequential derivative of order $q$, when $q = q_1 + q_2 + ......q_n$.

In particular, if $q_1 = q_2 = q_3 = ... = q$, then

$$D^{nq}f(x) = D^q(D^q...D^q(f(x))).$$

We extend this definition to Caputo fractional derivative of order $nq$. In general, this need not be true for Caputo fractional derivative and it is always true for integer derivatives. However, in this work, we seek solutions of sequential Caputo fractional boundary value problem, so that integer results will be special cases of our developed results.

The left sequential Caputo fractional derivative of order $nq$ when, $(n-1) < nq < n$, is given by,

$${}^cD^{nq}_{a^+}f(x) = {}^cD^q_{a^+}({}^cD^q_{a^+}(...{}^cD^q_{a^+}))f(x),$$

and the right sequential Caputo fractional derivative of order $nq$ when, $(n-1) < nq < n$, is given by

$${}^cD^{nq}_{b^-}f(x) = {}^cD^q_{b^-}({}^cD^q_{b^-}(...{}^cD^q_{b^-}))f(x).$$

If we do not assume the Caputo derivative to be sequential, then the basis will be $1, (x-a), (x-a)^2, (x-a)^3, ....(x-a)^{n-1}$. If the Caputo fractional derivative of order, $n-1 < nq < n$, is sequential, then the basis will be $1, (x-a)^q, (x-a)^{2q}, ...(x-a)^{(n-1)q}$ and/or $1, (b-x)^q, (b-x)^{2q}, ...(b-x)^{(n-1)q}$. For example, let us consider $(x-a)^q$, then ${}^cD^{2q}_a(x-a)^q \neq 0$, whereas ${}^cD^q_a({}^cD^q_a(x-a)^q) = 0$. Throughout this work, we assume ${}^cD^{2q}_a u(x) = {}^cD^q_a({}^cD^q_a u(x))$ and so on for any $nq$. In this case the basis will be $1, (x-a)^q$, etc depending on $n$. Similar result is true for ${}^cD^{2q}_b(b-x)^q$ also. The next two definitions are related to Riemann-Liouville derivative, when $p + q = 1$. It is relatively easy to prove the very basic comparison results using the Riemann-Liouville derivative. Further using the relation between Caputo derivative and the Riemann-Liouville derivative, one can obtain the basic comparison result for Caputo derivative also. For that purpose, we recall the next two definitions.

**Definition 2.3.** *We say that $m(x)$ is a $C^a_p$ continuous function on $[a, b], R$ if $m(x)$ is continuous on $[(a, b], \mathbb{R}]$ and $(x-a)^{1-q}m(x)$ is continuous on $[a, b], \mathbb{R}]$.*

**Definition 2.4.** *We say $m(x)$ is a $C^b_p$ continuous function on $[[a, b, R]$ if $m(x)$ is continuous on $[[a, b), \mathbb{R}]$ and $(b-x)^{1-q}m(x)$ is continuous on $[[a, b], \mathbb{R}]$.*

Remark: In this work, we are discussing the solutions of sequential Caputo fractional boundary value problem which are known to be $C^2$ functions on $[a, b]$. Hence the solutions we are seeking are automatically $C_p$ continuous functions. See [1–3] for details. We use this information in our auxiliary result.

## 3 Auxiliary Results

In this section, we develop some auxiliary results which are useful in our main results. Our first auxiliary result is a comparison result related to linear sequential Caputo boundary value problem. Consider the linear sequential boundary value problem

$$\begin{aligned} -{}^cD^{2q}u + p(x){}^cD^q u + q(x)u &= F(x) \\ \alpha_0 u(0) - \beta_0 {}^cD^q u(0) &= b_0 \\ \alpha_1 u(1) + \beta_1 {}^cD^q u(1) &= b_1, \end{aligned} \tag{3.1}$$

where $F(x) \in C[[0, 1], \mathbb{R}]$. Here and throughout this paper, we assume that $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$ such that $\alpha_0\beta_1 + \alpha_1\beta_0 \neq 0$. Our aim is to develop a linear comparison result, in such a way that the linear sequential Caputo boundary value problem (3.1) has a unique solution. Our linear comparison result yields the integer comparison result as a special case for $q = 1$. In the integer derivative, we use the fact that the left and the right first and the second derivatives are the same at any given point. Similarly, here and throughout this work, we assume that the left and the right Caputo derivative of order $q$ and $2q$ are the same at any given point on $[a, b]$. That is, for any $x_1 \in [a, b]$, we assume

$${}^cD^q_{a^+}f(x)|_{x=x_1} = {}^cD^q_{b^-}f(x)|_{x=x_1}$$

and

$$^{c}D^{q}_{a^{+}}(^{c}D^{q}_{a^{+}})f(x)|_{x=x_1} = {}^{c}D^{q}_{b^{-}}(^{c}D^{q}_{b^{-}})f(x)|_{x=x_1}.$$

This follows from the fact that $^{c}D^{q}f(x)$, and $^{c}D^{q}(^{c}D^{q}f(x))$ are continuous functions of $x$ on $[a,b]$. We need this in order to develop the basic calculus result for fractional derivative at a point of maxima.

We recall the first basic result relative to left Riemann–Liouville derivative.

**Lemma 3.1.** *Let $m$ in $C^{a}_{p}(J,\mathbb{R})(J=[a,b])$ be such that for some $x_1 \in J$, we have $m(x_1)=0$ and $m(x)\leq 0$ for $x\in(a,x_1]$. Then $D^{q}_{a^{+}}m(x)|_{x=x_1}\leq 0$.*

See [8] for detailed proof. Note that this result is true for Caputo left derivative also, using the relation between the left Riemann–Liouville derivative and the left Caputo derivative of order $q$.
The next lemma provides a similar result as Lemma 3.1 for the right Riemann–Liouville derivative. Although the proof is similar to the left Riemann–Liouville derivative, we provide the proof here for completeness.

**Lemma 3.2.** *Let $m\in C^{b}_{p}(J,\mathbb{R})(J=[a,b])$ be such that for some $x_1\in J$, we have $m(x_1)=0$ and $m(x)\leq 0$ for $x\in[x_1,b)$. Then $D^{q}_{b^{-}}m(x)|_{x=x_1}\geq 0$.*

*Proof.* Note that the right Riemann–Liouville derivative at any $x$ in $(a,b)$ is given by

$$D^{q}_{b^{-}}m(x) = -\frac{1}{\Gamma(p)}\frac{d}{dx}\int_{x}^{b}(s-x)^{-q}m(s)\mathrm{d}s.$$

Let $H(x) = -\int_{x}^{b}(s-x)^{-q}m(s)\mathrm{d}s$. Then, using the fact $m(x)\leq 0$ on $[x_1,b)$ and $(s-x_1)^{-q}<(s-x_1-h)^{-q}$ for $h>0$, we get

$$H(x_1+h)-H(x_1)\geq \int_{x_1}^{x_1+h}(s-x_1)^{-q}m(s)\mathrm{d}s.$$

Note that the function $(b-x)^{p}m(x)$ is uniformly continuous on $[a,b]$, since $m(x)$ is $C^{b}_{p}$, continuous on $[a,b)$. From this it follows that,

$$|(b-s)^{p}m(s)-(b-x_1)^{p}m(x_1)|<h\epsilon_h,$$

whenever $|s-x_1|<h$. Since $m(x_1)=0$ and $m(s)\leq 0$, we get $m(s)>-\dfrac{h\epsilon_h}{(b-s)^{1-q}}$.

This implies, that

$$\int_{x_1}^{x_1+h}(s-x_1)^{-q}m(s)\mathrm{d}s > -\frac{h^{2-q}\epsilon_h(b-x_1-h)^{-p}}{1-q}.$$

Thus we get

$$\frac{H(x_1+h)-H(x_1)}{h} > \frac{h^{1-q}\epsilon_h(b-x_1-h)^{-p}}{1-q}.$$

Now taking the limit as $h\to 0$, we have

$$D^{q}_{b^{-}}m(x)|_{x=x_1}\geq 0.$$

□

Using the relation between the right Riemann–Liouville derivative and the right Caputo derivative of order $q$, we get $^{c}D^{q}_{b^{-}}m(x)|_{x=x_1}\leq 0$. This is precisely the next result.

**Lemma 3.3.** *Let $m$ in $C_p(J,\mathbb{R})$ be such that for some $x_1\in J$, we have $m(x_1)=0$ and $m(x)\leq 0$ for $x\in(a,b)$. Then $^{c}D^{q}m(x)|_{x=x_1}=0$ and $^{c}D^{2q}m(x)|_{x=x_1}\leq 0$.*

 

*Proof.* Using the relation between the left(right) Riemann–Liouville derivative and the left(right) Caputo derivative as in (2.1), (2.2) and the Lemmas 3.1, 3.2, it easily follows that ${}^cD^q_{a^+}m(x)|_{x=x_1} \geq 0$ and ${}^cD^q_{b^-}m(x)|_{x=x_1} \leq 0$. Since, ${}^cD^q_{a^+}m(x)$ is a continuous function, we get ${}^cD^q m(x)|_{x=x_1} = 0$. Next we prove that

$$ {}^cD^{2q}m(x)|_{x=x_1} \leq 0. $$

If our claim is not true, then

$$ {}^cD^{2q}m(x)|_{x=x_1} = {}^cD^q({}^cD^q)m(x)|_{x=x_1} > 0. $$

This means that by the definition of the sequential derivative, we have ${}^cD^{2q}m(x) = {}^cD^q({}^cD^q)m(x) > 0$ on $x_1 - h < x < x_1 + h,\ h > 0$. This means by the continuity of the function, that

$$ \frac{d}{dx}\left({}^cD^q_{a^+}m(x)\right) > 0, $$

on $x_1 - h < x < x_1 + h$, for small $h > 0$.
However, using the Lemma 3.1 and 3.2, we have ${}^cD^q_{a^+}m(x) \geq 0$ on the interval $x_1 - h < x < x_1,\ h > 0$ and ${}^cD^q_{b^-}m(x) \leq 0$ on $x_1 < x < x_1 + h,\ h > 0$. Since ${}^cD^q m(x_1) = 0$, we can get that $\frac{d}{dx}\left({}^cD^q_{a^+}m(x)\right) \leq 0$, on $x_1 - h < x < x_1 + h,\ h > 0$. This leads to a contradiction. This completes the proof. □

Next we prove an analogous result of an integer result namely, Corollary 2.1.1 of [4]. This result is the linear Caputo fractional comparison result for linear boundary value problems. Thus, it is very useful in our main result.

**Lemma 3.4.** *Let $q, r \in C[J, \mathbb{R}]$ with $r(x) \geq 0$ on $J$. Suppose further that $p \in C^2[J, \mathbb{R}]$ and*

$$ -{}^cD^{2q}p(x) \leq q(x)|{}^cD^q p| - r(x)p, $$

*with*

$$ \begin{aligned} \alpha_0 p(a) - \beta_0 {}^cD^q p(a) &\leq 0 \\ \alpha_1 p(b) + \beta_1 {}^cD^q p(b) &\leq 0. \end{aligned} $$

*Then $p(x) \leq 0$ on $J = [a, b]$. If the inequalities are reversed then $p(x) \geq 0$ on $J$, where $J = [a, b]$.*

*Proof.* Initially, we prove the result when all the inequalities in the hypotheses are strict. That is, we assume

$$ \begin{aligned} -{}^cD^{2q}p(x) - q(x)|{}^cD^q p| + r(x)p &< 0 \\ \alpha_0 p(a) - \beta_0 {}^cD^q p(a) &< 0 \\ \alpha_1 p(b) + \beta_1 {}^cD^q p(b) &< 0. \end{aligned} \tag{3.2} $$

Note that $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$. In this case, we will prove that $p(x) < 0$, on $[a, b]$. If the conclusion is not true, then there exists an $x_1 \in J$ such that $p(x_1) = 0$. If $x_1 = a$, then $p(a + h) < 0$. This implies $\frac{d}{dx}(p(x)) \leq 0$, on $[a, a + h)$. This means ${}^cD^q p(a) \leq 0$. From the boundary conditions at $x_1 = a$, we get $\alpha_0 p(a) - \beta_0 {}^cD^q p(a) \geq 0$, a contradiction. Similarly, we can get a contradiction, if $x_1 = b$, using the hypotheses $\alpha_1 p(b) + \beta_1 {}^cD^q p(b) < 0$.
Now, if $x_1 \in (a, b)$ then using Lemma 3.3, we have ${}^cD^q p(x_1) = 0$, and ${}^cD^{2q}p(x_1) \leq 0$. This implies that,

$$ 0 \leq -{}^cD^{2q}p(x_1) - q(x_1)|{}^cD^q p(x_1)| + r(x_1)p(x_1) < 0. $$

This leads to a contradiction. Next we consider the case when the inequalities are not strict. We have

$$ \begin{aligned} -{}^cD^{2q}p(x) - q(x)|{}^cD^q p| + r(x)p &\leq 0 \\ \alpha_0 p(a) - \beta_0 {}^cD^q p(a) &\leq 0 \\ \alpha_1 p(b) + \beta_1 {}^cD^q p(b) &\leq 0, \end{aligned} \tag{3.3} $$

with $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$. We construct a function $m(x)$ such that

$$ p(x) = m(x) - \epsilon E_{q,1}(k(x - a)^q), $$

for some appropriate $k > 0$. Here $k$ will be chosen to make the above inequalities strict with the function $m(x)$ in place of $p(x)$. Then (3.3) becomes,

$$-{}^cD^{2q}m(x) - q(x){}^cD^q m(x) + r(x)m(x) + \epsilon E_{q,1}(k(x-a)^q)(k^2 + q(x)k - r(x)) \leq 0.$$

This implies,

$$-{}^c D^{2q}m(x) - q(x){}^cD^q m(x) + r(x)m(x) \leq -\epsilon E_{q,1}(k(x-a)^q)(k^2 + q(x)k - r(x)) < 0, \tag{3.4}$$

when we choose $k$ such that $(k^2 + q(x)k - r(x)) > 0$.
Note that the above results hold true if $k > 0$, is replaced by $-k$, for an appropriate $k > 0$. Basically, it is enough to choose $k$ such that $(k^2 + |q(x)|k - r(x)) > 0$.

Initially, we consider the case when $\alpha_0, \alpha_1 \neq 0$. In this case, replacing $k$ by $-k$, we get

$$\alpha_0(m(a) - \epsilon)) - \beta_0({}^cD^q m(a) + \epsilon k) \leq 0.$$

This implies,

$$\alpha_0 m(a) - \beta_0{}^cD^q m(a) \leq \epsilon(\alpha_0 - \beta_0 k) < 0, \tag{3.5}$$

when we choose $k$ such that $k > \dfrac{\alpha_0}{\beta_0}$.

Similarly, at $x = b$, we get,

$$\alpha_1(m(b) - \epsilon E_{q,1}(k(b-a)^q)) + \beta_1({}^cD^q m(b) + \epsilon k E_{q,1}(k(b-a)^q)) \leq 0.$$

From this it follows that

$$\alpha_1 m(b) + \beta_1{}^cD^q m(b) \leq \epsilon E_{q,1}(k(b-a)^q)(\alpha_1 - k\beta_1) < 0, \tag{3.6}$$

by choosing $k$ such that $k > \dfrac{\alpha_1}{\beta_1}$.

When $\alpha_0 = 0$ and $\alpha_1 \neq 0$, we choose $k > 0$ then the boundary conditions in (3.3) simplifies to

$$-\beta_0({}^cD^q m(a) - \epsilon k) \leq 0.$$

From this we get

$$-\beta_0{}^cD^q m(a) \leq -\epsilon(k\beta_0) < 0.$$

Similarly we can prove for

$$\alpha_1 m(b) + \beta_1{}^cD^q m(b) < 0,$$

by choosing the appropriate k.
Using similar proof we can obtain strict inequalities for the function $m(x)$ on the boundary when $\alpha_0 \neq 0$, and $\alpha_1 = 0$.

Now using the strict inequality result, we get $m(x) < 0$. This implies $p(x) < -\epsilon E_{q,1}(k(x-a)^q)) < 0$. Now taking the limit as $\epsilon \to 0$, we get $p(x) \leq 0$. This completes our proof. □

Now we consider the Caputo fractional boundary value problem with mixed boundary conditions of the form

$$\begin{gathered} -{}^cD^{2q}u = f(x,u) + g(x,u) \\ \alpha_0 u(a) - \beta_0{}^cD^q u(a) = b_0 \\ \alpha_1 u(b) + \beta_1{}^cD^q u(b) = b_1, \end{gathered} \tag{3.7}$$

on $J = [a,b]$, $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$, $u \in C^2[J, \mathbb{R}]$, $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$ provided $\alpha_0\beta_1 + \alpha_1\beta_0 \neq 0$.

**Definition 3.1.** *Let $v \in C^2[J, \mathbb{R}]$, $w \in C^2[J, \mathbb{R}]$ be the lower and upper solutions of the boundary value problem, if*

$$\begin{gathered} -{}^cD^{2q}v \leq f(x,v) + g(x,v) \\ \alpha_0 v(a) - \beta_0{}^cD^q v(a) \leq b_0 \\ \alpha_1 v(b) + \beta_1{}^cD^q v(b) \leq b_1, \end{gathered}$$

 

*on $J$, and*

$$\begin{aligned} -{}^cD^{2q}w &\geq f(x,w)+g(x,w) \\ \alpha_0 w(a) - \beta_0 {}^cD^q w(a) &\geq b_0 \\ \alpha_1 w(b) + \beta_1 {}^cD^q w(b) &\geq b_1. \end{aligned}$$

*on $J$.*

From now onwards, in (3.7) we are assuming $f$ is increasing in $u$ and $g$ is decreasing in $u$ for $x \in J = [a, b]$.

Next we provide the definition of coupled lower and upper solution of Type I of boundary value problem.

**Definition 3.2.** *Let $v_0$ and $w_0 \in C^2[J,\mathbb{R}]$ be the coupled lower and upper solution of the boundary value problem, if*

$$\begin{aligned} -{}^cD^{2q}v_0 &\leq f(x,v_0)+g(x,w_0) \\ \alpha_0 v(a) - \beta_0 {}^cD^q v(a) &\leq b_0 \\ \alpha_1 v(b) + \beta_1 {}^cD^q v(b) &\leq b_1, \end{aligned}$$

*and*

$$\begin{aligned} -{}^cD^{2q}w_0 &\geq f(x,w_0)+g(x,v_0) \\ \alpha_0 w(a) - \beta_0 {}^cD^q w(a) &\geq b_0 \\ \alpha_1 w(b) + \beta_1 {}^cD^q w(b) &\geq b_1. \end{aligned}$$

Next we present the Green's function representation for a Caputo fractional boundary value problem with mixed boundary conditions which is used to construct the solution of (3.7).

Consider the Caputo fractional boundary value problem with mixed nonhomogeneous boundary condition which is given by

$$\begin{aligned} -{}^cD^{2q}u &= f(x,u)+g(x,u) \\ \alpha_0 u(a) - \beta_0 {}^cD^q u(a) &= b_0 \\ \alpha_1 u(b) + \beta_1 {}^cD^q u(b) &= b_1, \end{aligned} \tag{3.8}$$

on $J=[a,b]$, $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$, $u \in C^2[J,\mathbb{R}]$, $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$, provided $\alpha_0\beta_1 + \alpha_1\beta_0 \neq 0$ and $b_0, b_1$ are constants.

The unique solution of (3.8) in terms of Green's function is given by

$$u(x) = C_1(x-a)^q + C_2(b-x)^q + \int_a^b G(x,s)F(s)\mathrm{d}s, \tag{3.9}$$

where $C_1$ and $C_2$ are constants and which can be found by the boundary conditions of (3.8) to be

$$C_1 = \frac{b_1(\alpha_0(b-a)^q - \beta_0(\Gamma(q+1)) - b_0(\Gamma(q+1)\beta_1)}{\alpha_0\alpha_1(b-a)^{2q} + 2\beta_0\beta_1(\Gamma(q+1))^2 - (b-a)^q\Gamma(q+1)(\alpha_0\beta_1+\alpha_1\beta_0)}$$

$$C_2 = \frac{b_0(\alpha_1(b-a)^q - \Gamma(q+1)\beta_1) + b_1\beta_0\Gamma(q+1)}{\alpha_0\alpha_1(b-a)^{2q} + 2\beta_0\beta_1(\Gamma(q+1))^2 - (b-a)^q\Gamma(q+1)(\alpha_0\beta_1+\alpha_1\beta_0)},$$

where $F(s) = f(s,u(s)) + g(s,u(s))$, and $G(x,s)$ is the Green's function satisfying

$$-{}^c\, D^{2q}G(x,s) = \delta(x-s). \tag{3.10}$$

The Green's function satisfies the related homogeneous boundary conditions,

$$\begin{aligned} \alpha_0 G(a,s) - \beta_0 {}^cD^q G(a,s) &= 0 \\ \alpha_1 G(a,s) + \beta_1 {}^cD^q G(b,s) &= 0, \end{aligned} \tag{3.11}$$

where $\delta(x-s) = {}^cD^qH(x-s)$, and $\delta(x-s)$ is a Dirac delta function, and $H(x-s)$ is the Heavyside unit step function.

 

From (3.9) we have

$$^{c}D^{2q}u(x)=\int_a^b F(s)\,^{c}D^{2q}G(x,s)\mathrm{d}s,$$

from (3.10) we get,

$$\int_a^b F(s)\delta(x-s)\mathrm{d}s=F(s),$$

where $F(s)=f(s)+g(s)$ and Green's function satisfies the related homogeneous boundary conditions,

$$\begin{aligned}\alpha_0 G(a,s)-\beta_0\,^{c}D^{q}G(a,s)=0,\\ \alpha_1 G(b,s)+\beta_1\,^{c}D^{q}G(b,s)=0.\end{aligned}\tag{3.12}$$

The Green's function $G(x,s)$ is given by,

$$G(x,s)=\begin{cases}A(x-a)^q+B,\ x<s\\ C(b-x)^q+D,\ x>s,\end{cases}$$

where $A,B,C,D$ are constants. By applying the Green's function boundary condition (3.12) in the above equation we get

$$G(x,s)=\begin{cases}A((x-a)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1)),\ x<s\\ C((b-x)^q+\frac{\beta_1}{\alpha_1}\Gamma(q+1)),\ x>s,\end{cases}\tag{3.13}$$

The constants $A$ and $C$ can be found by continuity and the jump condition of $G(x,s)$ is as follows:

At $x=s$, $G(x,s)$ must be continuous, $G(s^-,s)=G(s^+,s)$, and we have

$$A((s-a)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1))=C((b-s)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1)).\tag{3.14}$$

By the jump condition of $G(x,s)$ we get

$$^{c}D^{q}G|_{x=s^+}-\,^{c}D^{q}G|_{x=s^-}=1,$$

we have

$$-C\Gamma(q+1)-A\Gamma(q+1)=1.\tag{3.15}$$

By solving the equations (3.14) and (3.15), we get

$$C=\frac{1}{\Gamma(q+1)}\left[\frac{(s-a)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1)}{(s-a)^q+(b-s)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1)-\frac{\beta_1}{\alpha_1}\Gamma(q+1)}\right],$$

and

$$A=\frac{1}{\Gamma(q+1)}\left[\frac{(b-s)^q-\frac{\beta_1}{\alpha_1}\Gamma(q+1)}{(s-a)^q+(b-s)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1)-\frac{\beta_1}{\alpha_1}\Gamma(q+1)}\right].$$

Hence, we obtain the Green's function from (3.13) of the form

$$G(x,s)=\begin{cases}\frac{1}{\Gamma(q+1)}\left[\frac{((x-a)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1))((b-s)^q-\frac{\beta_1}{\alpha_1}\Gamma(q+1))}{((s-a)^q+(b-s)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1)-\frac{\beta_1}{\alpha_1}\Gamma(q+1))}\right]\ x<s\\ \frac{1}{\Gamma(q+1)}\left[\frac{((s-a)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1))((b-x)^q+\frac{\beta_1}{\alpha_1}\Gamma(q+1))}{((s-a)^q+(b-s)^q+\frac{\beta_0}{\alpha_0}\Gamma(q+1)-\frac{\beta_1}{\alpha_1}\Gamma(q+1))}\right]\ x>s.\end{cases}$$

Thus, (3.9) can be written as

$$\begin{aligned}u(x)=&\frac{b_1(\alpha_0(b-a)^q-\beta_0\Gamma(q+1))-b_0\beta_1\Gamma(q+1)}{\alpha_0\alpha_1(b-a)^{2q}+2\beta_0\beta_1(\Gamma(q+1))^2-(b-a)^q\Gamma(q+1)(\alpha_0\beta_1+\alpha_1\beta_0)}(x-a)^q\\ &+\frac{b_0(\alpha_1(b-a)^q-\beta_1\Gamma(q+1))+b_1\beta_0\Gamma(q+1)}{\alpha_0\alpha_1(b-a)^{2q}+2\beta_0\beta_1(\Gamma(q+1))^2-(b-a)^q\Gamma(q+1)(\alpha_0\beta_1+\alpha_1\beta_0)}(b-x)^q\\ &+\int_a^b F(s)G(x,s)\mathrm{d}s.\end{aligned}\tag{3.16}$$

 

If $b_1 = 0$, $b_2 = 0$, in (3.8) then the unique solution of the boundary value problem is given by

$$u(x) = \int_a^b G(x,s)F(s)\mathrm{d}s.$$

Remark: When $q = 1$ in (3.16), we obtain the integer result as a special case. Hence, all our results throughout this paper yields integer result as a special case.

From now on, let us consider the Caputo fractional boundary value problem with mixed boundary condition in the following form

$$\begin{aligned} -{}^cD^{2q}u_n(x) &= F_n(x, u_{n-1}) \\ \alpha_0 u_n(a) - \beta_0 {}^cD^q u_n(a) &= b_0 \\ \alpha_1 u_n(b) + \beta_1 {}^cD^q u_n(b) &= b_1, \end{aligned}$$

on $J = [a,b]$, $F_n \in C[J \times \mathbb{R}, \mathbb{R}]$ where $F_n = f_n + g_n$.

Now we prove that the sequence $\{u_n(x)\}$ is equicontinuous which is useful in our main result.

**Lemma 3.5.** *Let $\{u_n(x)\}$ be a family of continuous function on $[a,b]$ for each $n > 0$,*

$$\begin{aligned} -{}^cD^{2q}u_n(x) &= F_n(x, u_{n-1}(x)) \\ \alpha_0 u_n(a) - \beta_0 {}^cD^q u_n(a) &= b_0 \\ \alpha_1 u_n(b) + \beta_1 {}^cD^q u_n(b) &= b_1, \end{aligned}$$

*where $F_n(x, u_{n-1}(x))$ is uniformly bounded on $[a,b]$. Then $\{u_n(x)\}$ is equicontinuous on $[a,b]$.*

*Proof.* Let

$$v_n(x) = A(x-a)^q + B(b-x)^q + \int_a^b G(x,s)F(s, v_{n-1}(s), w_{n-1}(s))\mathrm{d}s,$$

and

$$w_n(x) = A(x-a)^q + B(b-x)^q + \int_a^b G(x,s)F(s, v_{n-1}(s), w_{n-1}(s))\mathrm{d}s.$$

Since $F(s)$ is continuous on a closed bounded set and $|v_{n-1}(x)| \le M_1$ and $|w_{n-1}(x)| \le M_2$ are uniformly bounded, we have $F(s, v_{n-1}(s), w_{n-1}(s))$ is uniformly bounded. Without loss of generality, we assume that $x_2 > x_1$,

$$\begin{aligned} v_n(x_2) - v_n(x_1) &= A((x_2-a)^q - (x_1-a)^q) + B((b-x_2)^q - (b-x_1)^q) \\ &\quad + \int_a^b (G(x_2,s) - G(x_1,s))F(s, v_{n-1}(s), w_{n-1}(s))\mathrm{d}s. \end{aligned}$$

We first need to prove that

$$|(x_2-a)^q - (x_1-a)^q| < K_1|x_2-x_1|^q, \tag{3.17}$$

for some fixed $K_1$ and $x_1, x_2 \in [a,b]$.

Since $x_2 > x_1$, $x_2 - a > x_1 - a$,

$$\begin{aligned} \frac{1}{x_2-a} &< \frac{1}{x_1-a}, \\ \frac{1}{(x_2-a)^p} &< \frac{1}{(x_1-a)^p}, \\ -\frac{1}{(x_2-a)^p} &> -\frac{1}{(x_1-a)^p}. \end{aligned} \tag{3.18}$$

By multiplying and dividing the expression $(x_2-a)^q - (x_1-a)^q$ by $(x_2-a)^p$ and $(x_1-a)^p$, we get

$$(x_2-a)^q - (x_1-a)^q = (x_2-a)^q\frac{(x_2-a)^p}{(x_2-a)^p} - (x_1-a)^q\frac{(x_1-a)^p}{(x_1-a)^p}.$$

Since $p+q=1$, the above expression is reduced to

$$(x_2-a)^q-(x_1-a)^q=\frac{(x_2-a)}{(x_2-a)^p}-\frac{(x_1-a)}{(x_1-a)^p}.$$

By the equation (3.18), the above expression is reduced to

$$\frac{(x_2-a)}{(x_2-a)^p}-\frac{(x_1-a)}{(x_1-a)^p}<\frac{(x_2-a)}{(x_2-a)^p}-\frac{(x_1-a)}{(x_2-a)^p}=\frac{(x_2-x_1)}{(x_2-a)^p}. \tag{3.19}$$

Let $a<x_1$, $(x_2-a)>(x_2-x_1)$, $\dfrac{1}{x_2-a}<\dfrac{1}{x_2-x_1}$

$$\frac{1}{(x_2-a)^p}<\frac{1}{(x_2-x_1)^p}. \tag{3.20}$$

Hence, equation (3.19) becomes

$$\frac{(x_2-x_1)}{(x_2-a)^p}<\frac{(x_2-x_1)}{(x_2-x_1)^p}=(x_2-x_1)^q.$$

Therefore, $|(x_2-a)^q-(x_1-a)^q|<|x_2-x_1|^q<\frac{\epsilon}{K_1}$.
Next, we need to prove that

$$|(b-x_2)^q-(b-x_1)^q|<K_2|x_2-x_1|^q, \tag{3.21}$$

for some fixed $K_2$. Since $x_2>x_1$,

$$b-x_2<b-x_1,\ \frac{1}{(b-x_2)}>\frac{1}{(b-x_1)},\ \frac{1}{(b-x_2)^p}>\frac{1}{(b-x_1)^p},$$

$$-\frac{1}{(b-x_2)^p}<-\frac{1}{(b-x_1)^p}, \tag{3.22}$$

$$(b-x_2)^q-(b-x_1)^q=(b-x_2)^q\frac{(b-x_2)^p}{(b-x_2)^p}-(b-x_1)^q\frac{(b-x_1)^p}{(b-x_1)^p},$$

$$\frac{b-x_2}{(b-x_2)^p}-\frac{b-x_1}{(b-x_1)^p}.$$

By equation (3.22) we get

$$\frac{b-x_2}{(b-x_2)^p}-\frac{b-x_1}{(b-x_1)^p}<-\frac{(b-x_2)}{(b-x_1)^p}+\frac{(b-x_1)}{(b-x_1)^p}=\frac{x_2-x_1}{(b-x_1)^p}. \tag{3.23}$$

Let $b>x_1$, $\quad b-x_1>x_2-x_1$, $\dfrac{1}{(b-x_1)^p}<\dfrac{1}{(x_2-x_1)^p}$. Hence (3.23) becomes

$$\frac{x_2-x_1}{(b-x_1)^p}<\frac{x_2-x_1}{(x_2-x_1)^p}=(x_2-x_1)^q.$$

Thus, $|(b-x_2)^q-(b-x_1)^q|<|x_2-x_1|^q<\frac{\epsilon}{K_2}$.

Now we need to prove that

$$\left|\int_a^b (G(x_2,s)-G(x_1,s))F(s,v_{n-1}(s),w_{n-1}(s)\mathrm{d}s\right|<K_3|x_2-x_1|^q.$$

Let

$$G(x_2,s)-G(x_1,s)=\begin{cases} c_1((x_2-a)^q-(x_1-a)^q),\ x<s \\ c_2((b-x_2)^q-(b-x_1)^q),\ x>s, \end{cases}$$

 

where $c_1$ and $c_2$ are constants. From (3.17) and(3.21), we have

$$G(x_2,s)-G(x_1,s)=\begin{cases}K(x_2-x_1)^q, \ x<s\\ K(x_2-x_1)^q, \ x>s,\end{cases}$$

where $K = max\ \{K_1, K_2\}$. Now

$$\begin{aligned}&\left|\int_a^b (G(x_2,s)-G(x_1,s))F(s,v_{n-1}(s),w_{n-1}(s))\mathrm{d}s\right|\\ &\leq \int_a^b |(G(x_2,s)-G(x_1,s))||F(s,v_{n-1}(s),w_{n-1}(s))|\mathrm{d}s,\\ &\leq max_{s\in[a,b]}F_n(s)\,(b-a)\,max_{s\in[a,b]}\,|G(x_2,s)-G(x_1,s)| \leq K_3|x_2-x_1|^q.\end{aligned}$$

Hence,

$$\left|\int_a^b (G(x_2,s)-G(x_1,s))F(s,v_{n-1}(s),w_{n-1}(s))\mathrm{d}s\right| < K_3|x_2-x_1|^q < \frac{\epsilon}{K_3}.$$

Therefore $\epsilon > 0$, for any $\delta > 0$, $|v_n(x_2) - v_n(x_1)| < \epsilon$ (independent of $n$) and $\{v_n(x)\}$ is equicontinuous. By a similar argument, we can prove $\{w_n(x)\}$ is equicontinuous.

□

In the next section, we state the theorem related to coupled lower and upper solutions and develop a generalized monotone method.

## 4 Main Results

In this section, we develop the generalized monotone method with coupled lower and upper solutions of the Caputo fractional boundary value problem with mixed boundary conditions. We also obtain the existence and uniqueness of the solution of the boundary value problem. We use the Green's function representation to construct the solution for the Caputo fractional boundary value problem with mixed boundary conditions.

**Theorem 4.1.** *Assume that*
*(i)* $v_0(x), w_0(x) \in C^2[J,\mathbb{R}]$ *are the coupled lower and upper solutions of* (3.7) *with* $v_0(x) < u(x) < w_0(x)$ *on* $J$.
*(ii)* $f, g \in C[J\times\mathbb{R},\mathbb{R}]$ *and* $f(x,u)$ *is increasing in* $u$ *and* $g(x,u)$ *is decreasing in* $u$ *on* $J$.
*Then there exists a sequence defined by*

$$\begin{aligned}-{}^cD^{2q}v_{n+1} &= f(x,v_n)+g(x,w_n)\\ \alpha_0 v_{n+1}(a)-\beta_0{}^cD^q v_{n+1}(a) &= b_0\\ \alpha_1 v_{n+1}(b)+\beta_1{}^cD^q v_{n+1}(b) &= b_1,\end{aligned}$$

*and*

$$\begin{aligned}-{}^cD^{2q}w_{n+1} &= f(x,w_n)+g(x,v_n)\\ \alpha_0 w_{n+1}(a)-\beta_0{}^cD^q w_{n+1}(a) &= b_0\\ \alpha_1 w_{n+1}(b)+\beta_1{}^cD^q w_{n+1}(b) &= b_1,\end{aligned}$$

*such that* $v_n(x) \to \rho(x)$ *and* $w_n(x) \to r(x)$ *uniformly and monotonically and such that* $(\rho, r)$ *are coupled minimal and maximal solution respectively to the solution of* (3.7). *That is,* $(\rho, r)$ *satisfies*

$$\begin{aligned}-{}^cD^{2q}\rho &= f(x,\rho)+g(x,r)\\ \alpha_0\rho(a)-\beta_0{}^cD^q\rho(a) &= b_0\\ \alpha_1\rho(b)+\beta_1{}^cD^q\rho(b) &= b_1,\end{aligned}$$

*and*

$$-{}^cD^{2q}r = f(x,r) + g(x,\rho)$$
$$\alpha_0 r(a) - \beta_0 {}^cD^q r(a) = b_0$$
$$\alpha_1 r(b) + \beta_1 {}^cD^q r(b) = b_1,$$

*such that* $\rho \le r$.

*Proof.* Consider the Caputo fractional boundary value problem of order $q$, $0 < q < 1$,

$$\begin{aligned} -{}^cD^{2q}u &= f(x,u) + g(x,u), \\ \alpha_0 u(a) - \beta_0 {}^cD^q u(a) &= b_0, \\ \alpha_1 u(b) + \beta_1 {}^cD^q u(b) &= b_1, \end{aligned} \tag{4.1}$$

on $J = [a,b]$, $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$, $u \in C^2[J, \mathbb{R}]$. The representation formula for (4.1) is given by

$$u(x) = A(x-a)^q + B(b-x)^q + \int_a^b G(x,s)F(s)\mathrm{d}s,$$

where $F(s) = f(s,u(s)) + g(s,u(s))$ and $G(x,s)$ is the Green's function. Firstly, we prove the uniqueness of the Caputo fractional boundary value problem (4.1). For this, let $u_1(x), u_2(x)$ be two solutions of (4.1). Then, we have

$$-{}^cD^{2q}u_1 = F(x)$$
$$\alpha_0 u_1(a) - \beta_0 {}^cD^q u_1(a) = b_0$$
$$\alpha_1 u_1(b) + \beta_1 {}^cD^q u_1(b) = b_1,$$

and

$$-{}^cD^{2q}u_2 = F(x)$$
$$\alpha_0 u_2(a) - \beta_0 {}^cD^q u_2(a) = b_0$$
$$\alpha_1 u_2(b) + \beta_1 {}^cD^q u_2(b) = b_1,$$

where $F(x) = f(x) + g(x) \in C[J \times \mathbb{R}, \mathbb{R}]$.

Let $p(x) = u_1 - u_2$, then

$$-{}^cD^{2q}p = -{}^cD^{2q}u_1 + {}^cD^{2q}u_2,$$

and

$$\alpha_0 p(a) - \beta_0 {}^cD^q p(a) = 0$$
$$\alpha_1 p(b) + \beta_1 {}^cD^q p(b) = 0,$$

which implies

$${}^cD^{2q}p \le 0.$$

By the lemma (3.4), we get $p \le 0$, which implies that $u_1 \le u_2$. By the similar argument, we can prove $u_2 \le u_1$. Hence, $u_1 \equiv u_2$.

Now, we define the sequences $\{v_{n+1}(x)\}$ and $\{w_{n+1}(x)\}$ by

$$\begin{aligned} -{}^cD^{2q}v_{n+1} &= f(x,v_n) + g(x,w_n) \\ \alpha_0 v_{n+1}(a) - \beta_0 {}^cD^q v_{n+1}(a) &= b_0 \\ \alpha_1 v_{n+1}(b) + \beta_1 {}^cD^q v_{n+1}(b) &= b_1, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} -{}^cD^{2q}w_{n+1} &= f(x,w_n) + g(x,v_n) \\ \alpha_0 w_{n+1}(a) - \beta_0 {}^cD^q w_{n+1}(a) &= b_0 \\ \alpha_1 w_{n+1}(b) + \beta_1 {}^cD^q w_{n+1}(b) &= b_1. \end{aligned} \tag{4.3}$$

The representation formula for $v_{n+1}(x)$ and $w_{n+1}(x)$ is given by

$$v_{n+1}(x) = A(x-a)^q + B(b-x)^q + \int_a^b G(x,s)F(v_n(s), w_n(s))\mathrm{d}s,$$

and

$$w_{n+1}(x) = A(x-a)^q + B(b-x)^q + \int_a^b G(x,s)F(v_n(s), w_n(s))\mathrm{d}s,$$

 

where $G(x,s)$ is the Green's function.

By the hypotheses, we have $v_0(x) \leq u(x) \leq w_0(x)$. We show that $v_0(x) \leq v_1(x) \leq u(x) \leq w_1(x) \leq w_0(x)$. Since $v_0$ and $w_0$ are the coupled lower and upper solution of (4.1),

$$\begin{aligned} -{}^cD^{2q}v_0 \leq f(x,v_0) + g(x,w_0) \\ \alpha_0 v(a) - \beta_0{}^cD^q v(a) \leq b_0 \\ \alpha_1 v(b) + \beta_1{}^cD^q v(b) \leq b_1, \end{aligned}$$

and

$$\begin{aligned} -{}^cD^{2q}w_0 \geq f(x,w_0) + g(x,v_0) \\ \alpha_0 w(a) - \beta_0{}^cD^q w(a) \geq b_0 \\ \alpha_1 w(b) + \beta_1{}^cD^q w(b) \geq b_1. \end{aligned}$$

When $n = 0$ in (4.2) we have

$$\begin{aligned} -{}^cD^{2q}v_1 = f(x,v_0) + g(x,w_0) \\ \alpha_0 v_1(a) - \beta_0{}^cD^q v_1(a) = b_0 \\ \alpha_1 v_1(b) + \beta_1{}^cD^q v_1(b) = b_1. \end{aligned}$$

Let $p(x) = v_0 - v_1$, then (4.1) is reduced to

$$-{}^cD^{2q}p = -{}^cD^{2q}v_0 + {}^cD^{2q}v_1,$$

with

$$\begin{aligned} \alpha_0 p(a) - \beta_0{}^cD^q p(a) = 0 \\ \alpha_1 p(b) + \beta_1{}^cD^q p(b) = 0, \end{aligned}$$

which implies $(f(x,v_0) + g(x,w_0)) - (f(x,v_0) + g(x,w_0)) \leq 0$. By the lemma (3.4), we have

$$p(x) \leq 0,$$

which implies that $v_0 \leq v_1$. Similarly we can prove $w_1 \leq w_0$. Next we prove $v_1 \leq v_2$.
Let $p(x) = v_1 - v_2$, then (4.1) is reduced to

$$-{}^cD^{2q}p = -{}^cD^{2q}v_1 + {}^cD^{2q}v_2,$$

with

$$\begin{aligned} \alpha_0 p(a) - \beta_0{}^cD^q p(a) = 0 \\ \alpha_1 p(b) + \beta_1{}^cD^q p(b) = 0, \end{aligned}$$

which implies that $(f(x,v_0) + g(x,w_0)) - (f(x,v_1) + g(x,w_1)) \leq 0$. Since $f(x)$ is increasing in $u$ and $g(x)$ is decreasing in $u$. By the lemma (3.4) we have

$$p(x) \leq 0,$$

which implies that $v_1 \leq v_2$. Similarly we can prove $w_2 \leq w_1$. Hence, $v_0 \leq v_1 \leq v_2$ and $w_2 \leq w_1 \leq w_0$.

Next we prove that $v_1 \leq u$, if $u$ is any solution such that $v_0 \leq u \leq w_0$.

$$\begin{aligned} -{}^cD^{2q}v_1 = f(x,v_0) + g(x,w_0) \\ \alpha_0 v_1(a) - \beta_0{}^cD^q v_1(a) = b_0 \\ \alpha_1 v_1(b) + \beta_1{}^cD^q v_1(b) = b_1, \end{aligned}$$

and

$$\begin{aligned} -{}^cD^{2q}u = f(x,u) + g(x,u) \\ \alpha_0 u(a) - \beta_0{}^cD^q u(a) = b_0 \\ \alpha_1 u(b) + \beta_1{}^cD^q u(b) = b_1. \end{aligned}$$

Let $p(x) = v_1 - u$, we have

$$-{}^cD^{2q}p = -{}^cD^{2q}v_1 + {}^cD^{2q}u,$$

 

with

$$\alpha_0 p(a) - \beta_0 {}^c D^q p(a) = 0$$
$$\alpha_1 p(b) + \beta_1 {}^c D^q p(b) = 0,$$

which implies that $(f(x, v_0) + g(x, w_0)) - (f(x, u) + g(x, u)) \leq 0$. Since $f(x)$ is increasing in $u$ and $g(x)$ is decreasing in $u$. By the lemma (3.4), we have $p \leq 0$, which implies that $v_1 \leq u$. Similarly we can prove that $u \leq w_1$. Hence, $v_0 \leq v_1 \leq u \leq w_1 \leq w_0$.

Next we need to prove that $v_n \leq u \leq w_n$, where $u$ is any solution of (4.1) proved by induction. It is certainly true that $v_0 \leq u \leq w_0$ by existence theorem.

Let us assume that $v_k \leq u \leq w_k$, we will prove $v_{k+1} \leq u \leq w_{k+1}$ for $k \geq 1$. From (4.2) and (4.3), we get

$$-{}^c D^{2q} v_{k+1} = f(x, v_k) + g(x, w_k)$$
$$\alpha_0 v_{k+1}(a) - \beta_0 {}^c D^q v_{k+1}(a) = b_0$$
$$\alpha_1 v_{k+1}(b) + \beta_1 {}^c D^q v_{k+1}(b) = b_1,$$

and

$$-{}^c D^{2q} w_{k+1} = f(x, w_k) + g(x, v_k)$$
$$\alpha_0 w_{k+1}(a) - \beta_0 {}^c D^q w_{k+1}(a) = b_0$$
$$\alpha_1 w_{k+1}(b) + \beta_1 {}^c D^q w_{k+1}(b) = b_1.$$

Let us start with $v_{k+1} \leq u$, and let $p(x) = v_{k+1} - u$. Then

$$-{}^c D^{2q} p = -{}^c D^{2q} v_{k+1} + {}^c D^{2q} u,$$

with

$$\alpha_0 p(a) - \beta_0 {}^c D^q p(a) = 0$$
$$\alpha_1 p(b) + \beta_1 {}^c D^q p(b) = 0,$$

which implies that $(f(x, v_k) + g(x, w_k)) - (f(x, u) + g(x, u)) \leq 0$. Since $f(x)$ is increasing in $u$ and $g(x)$ is decreasing in $u$. By the lemma (3.4), we have $p \leq 0$, which implies that $v_{k+1} \leq u$. Similarly, we can prove that $u \leq w_{k+1}$. Hence $v_n \leq u \leq w_n$.

Therefore for $n \geq 1$,

$$v_0 \leq v_1 \leq v_2 \leq ... \leq v_n \leq u \leq w_n \leq ... \leq w_2 \leq w_1 \leq w_0.$$

Now we need to prove that the sequence converges uniformly. Using Arzelá-Ascoli theorem, we will prove that the sequences are uniformly bounded and equicontinuous.

First we will prove uniform boundedness. Since $v_0$ and $w_0$ are bounded on [a,b], there exists $M > 0$ such that for any $x \in [a, b]$, $|v_0(x)| \leq M$, and $|w_0(x)| \leq M$.
Since $v_0(x) \leq v_n(x) \leq w_0(x)$ for each $n > 0$, it follows that

$$0 \leq v_n(x) - v_0(x) \leq w_0(x) - v_0(x),$$

since $|w_0(x) - v_0(x)|$ are continuous and bounded and hence uniformly bounded.
Let

$$\begin{aligned} |v_n(x)| &= |v_n(x) - v_0(x) + v_0(x)| \\ &\leq |v_n(x) - v_0(x)| + |v_0(x)| \\ &\leq M_1 + M_2 \leq M. \end{aligned}$$

Hence $\{v_n(x)\}$ is uniformly bounded. Similarly, we can prove $\{w_n(x)\}$ is also uniformly bounded.

Next we need to prove that $\{v_n(x)\}$ and $\{w_n(x)\}$ are equicontinuous on $[a, b]$. By recalling the lemma (3.5), we can prove that $\{v_n(x)\}$ and $\{w_n(x)\}$ are equicontinuous on $[a, b]$. Hence we proved that $\{v_n(x)\}$ and $\{w_n(x)\}$ are uniformly bounded and equicontinuous. Therefore by the Arzelá-Ascoli theorem there exist subsequences $\{v_{n_k}(x)\}$ and $\{w_{n_k}(x)\}$ which converge uniformly to $\rho(x)$ and $r(x)$ respectively on $J$. Since the sequences are monotone, the entire sequence converges uniformly. By the Lebesgue dominated convergence theorem, we can prove that the sequences $\{v_n(x)\}$ and $\{w_n(x)\}$ converges to a coupled minimal and maximal solution of (4.1). Since $v_n \leq u \leq w_n$, $\forall n$, we get $\rho(x) \leq u(x) \leq r(x)$ on $[a, b]$, which implies that $\rho, r$ are coupled minimal and maximal solutions of (4.1) respectively. This completes the proof.

Note that if further $f$ and $g$ satisfy the one-sided Lipschitz condition of the following form:

$$f(x,u_1) - f(x,u_2) \le L(u_1 - u_2), \text{whenever } u_1 \ge u_2,\ L > 0,\ \text{for } x \in [a,b]$$

and

$$g(x,u_1) - g(x,u_2) \ge M(u_1 - u_2), \text{whenever } u_1 \ge u_2, M > 0,\ \text{for } x \in [a,b],$$

then we can prove $r \le \rho$ on $[a,b]$ in the above result. This can be achieved by setting $P(x) = r - \rho$, and using the linear comparison result for sequential Caputo fractional differential inequalities, with linear boundary conditions.

□

## 5 Numerical Examples

In this section, we present the numerical examples for Caputo fractional boundary value problem with mixed boundary conditions using the Green's functions representation.

**Example 1:** Consider the Caputo fractional boundary value problem with boundary conditions,

$$\begin{aligned} -{}^cD^{2q}u &= x^{2q}, \\ u(0) = 0,\ {}^cD^q u(1) &= 0. \end{aligned} \tag{5.1}$$

The Green's function for (5.1) satisfies

$$\begin{aligned} -{}^cD^{2q}G(x,s) &= \delta(x-s), \\ G(0,s) = 0,\ {}^cD^qG(1,s) &= 0, \end{aligned}$$

and the solution to the (5.1) is given by

$$u(x) = \int_0^1 G(x,s)(-s^{2q})\mathrm{d}s. \tag{5.2}$$

If $x \neq s$, ${}^cD^{2q}G(x,s) = 0$, and $G(x,s)$ is the Green's function given by

$$G(x,s) = \begin{cases} Ax^q + B,\ x < s, \\ C(1-x)^q + D,\ x > s, \end{cases}$$

where $A, B, C, D$ are constants and $x \in (0,1)$. Starting with the Green's function boundary conditions, we obtain

$$G(0,s) = 0 \Rightarrow B = 0.$$

$${}^cD^qG(1,s) = 0 \Rightarrow \Gamma(q+1)C = 0 \Rightarrow C = 0.$$

Hence, the Green's function $G(x,s)$ is reduced to

$$G(x,s) = \begin{cases} A(x^q),\ x < s \\ D,\ x > s. \end{cases}$$

From the continuity we have that, at $x = s$, $As^q = D$ and from the jump condition we have

$${}^cD^qG(x,s)|_{b^-} - {}^cD^qG(x,s)|_{a^+} = 1.$$

Solving the above two equations, we obtain $A = \frac{-1}{\Gamma(q+1)}$ and $D = \frac{-s^q}{\Gamma(q+1)}$.
Hence,

$$G(x,s) = \begin{cases} \frac{-1}{\Gamma(q+1)}x^q,\ x < s \\ \frac{-s^q}{\Gamma(q+1)},\ x > s, \end{cases}$$

 

Hence, the solution (5.2) becomes,

$$u(x) = \int_0^x \frac{-s^q}{\Gamma(q+1)}(-s^{2q})\mathrm{d}s + \int_x^1 \frac{-x^q}{\Gamma(q+1)}(-s^{2q})\mathrm{d}s.$$

By integrating the above expression, we obtain

$$u(x) = \frac{1}{\Gamma(q+1)}\left[\frac{x^q}{(2q+1)} + \frac{x^{3q+1}}{(3q+1)} - \frac{x^{2q+1}}{(2q+1)}\right]. \tag{5.3}$$

Below, we will find the numerical result for fractional boundary value problem with simple boundary conditions when $0 < q \le 1$ and for integer case.

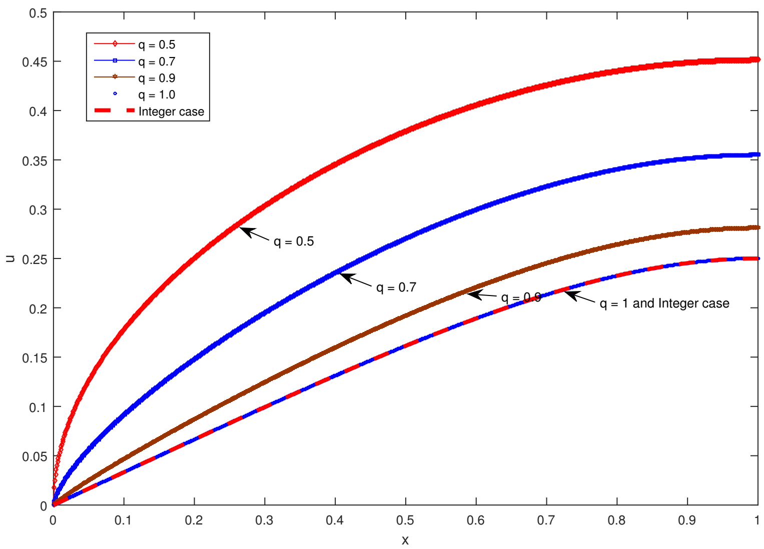


**Figure 1.** $-{}^cD^{2q}u = x^{2q}$ when $q = 0.5, 0.7, 0.9, 1.0$

The dotted lines in the Figure 1 represents the numerical result for integer derivative. Hence, we observe that when $q = 1$, our numerical result for Caputo fractional boundary value problem yields integer result as a special case.

**Example 2:** Consider the Caputo fractional boundary value problem with mixed boundary conditions

$$\begin{gathered} -{}^cD^{2q}u = x^q \\ u(0) - {}^cD^q u(0) = 0, \\ u(1) + {}^cD^q u(1) = 0. \end{gathered} \tag{5.4}$$

The Green's function for (5.4) satisfies

$$\begin{gathered} -{}^cD^{2q}G(x,s) = \delta(x-s) \\ G(0,s) - {}^cD^q G(0,s) = 0 \\ G(1,s) + {}^cD^q G(1,s) = 0. \end{gathered}$$

The solution of (5.4) is given by

$$u(x) = \int_0^1 G(x,s)(-s^q)\mathrm{d}s, \tag{5.5}$$

 

where $G(x, s)$ is the Green's function which is given by,

$$G(x,s) = \begin{cases} c_1x^q + c_2,\ x < s \\ c_3(1-x)^q + c_4,\ x > s, \end{cases}$$

where $c_1, c_2, c_3, c_4$ are constants and $x \in (0, 1)$. By applying the Green's function boundary conditions we get

$$G(x,s) = \begin{cases} c_1x^q + c_1(\Gamma(q+1)),\ x < s, \\ c_3(1-x)^q + c_3(\Gamma(q+1)),\ x > s. \end{cases}$$

From the continuity and the jump condition, we obtain $G(x, s)$ as follows

$$G(x,s) = \begin{cases} -\frac{(1-s)^q+\Gamma(q+1)}{s^q+(1-s)^q+2\Gamma(q+1)}\frac{x^q+\Gamma(q+1)}{\Gamma(q+1)},\ x < s \\ -\frac{(s)^q+\Gamma(q+1)}{s^q+(1-s)^q+2\Gamma(q+1)}\frac{(1-x)^q+\Gamma(q+1)}{\Gamma(q+1)},\ x > s. \end{cases}$$

Hence, the solution (5.5) becomes

$$\begin{aligned} u(x) = & \int_0^x \left[-\frac{(s)^q+\Gamma(q+1)}{s^q+(1-s)^q+2\Gamma(q+1)}\right]\left[\frac{(1-x)^q+\Gamma(q+1)}{\Gamma(q+1)}\right](-s^q)\mathrm{d}s \\ & + \int_x^1 \left[-\frac{(1-s)^q+\Gamma(q+1)}{s^q+(1-s)^q+2\Gamma(q+1)}\right]\left[\frac{x^q+\Gamma(q+1)}{\Gamma(q+1)}\right](-s^q)\mathrm{d}s, \end{aligned}$$

which simplifies to

$$\begin{aligned} u(x) = & \left[\frac{(1-x)^q+\Gamma(q+1)}{\Gamma(q+1)}\right]\int_0^x \left[\frac{(s)^q+\Gamma(q+1)}{s^q+(1-s)^q+2\Gamma(q+1)}\right]s^q\mathrm{d}s \\ & + \left[\frac{x^q+\Gamma(q+1)}{\Gamma(q+1)}\right]\int_x^1 \left[\frac{(1-s)^q+\Gamma(q+1)}{s^q+(1-s)^q+2\Gamma(q+1)}\right]s^q\mathrm{d}s. \end{aligned}$$

Below we find the numerical result for fractional boundary value problem with mixed boundary conditions when $0 < q \leq 1$ and for the integer derivative.

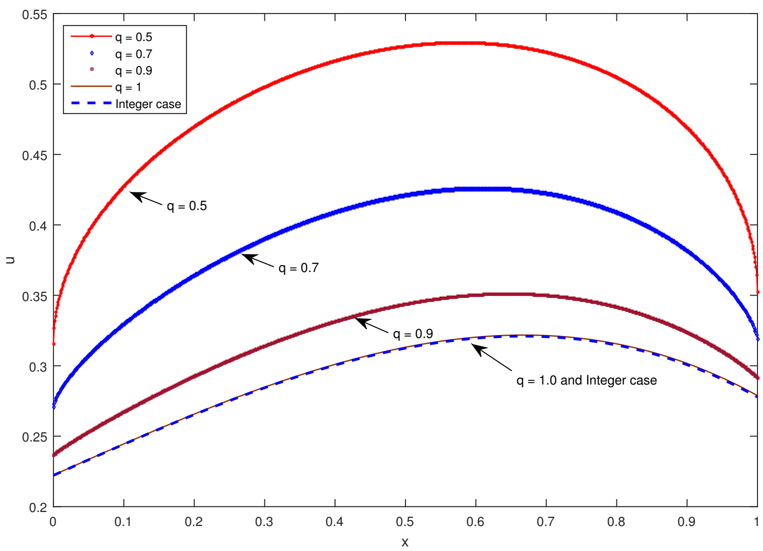


**Figure 2.** $-{}^cD^{2q}u = x^q$ when $q = 0.5, 0.7, 0.9, 1.0$

The dotted lines in the Figure 2 represents the integer derivative result. Hence we observe that when $q = 1$, our numerical result for Caputo fractional boundary value problem with mixed boundary conditions yields integer result as a special case.

 

## 6 Conclusion

Generalized monotone method is a useful tool for any nonlinear dynamic systems when the forcing function is the sum of an increasing and decreasing functions in the the nonlinear term. Further, generalized monotone method is both a theoretical and a constructive method for computing the coupled minimal and maximal solution of the nonlinear Problem. If further uniqueness condition is satisfied, then the monotone sequences converge to the unique solution of the nonlinear problem. In this work, we have developed generalized monotone iterative technique together with coupled lower and upper solutions for the nonlinear Caputo fractional boundary value problem with mixed boundary conditions. Using the Green's function as a tool, a representation form for the solution of the nonhomogeneous linear Caputo fractional boundary value problem has been developed. We have developed a linear comparison result which has been beneficial in proving the monotonicity and also the uniqueness of the solution of the linear Caputo fractional boundary value problem. Under uniqueness condition we could prove that the coupled maximal and minimal boundary value problem converges to the unique solution of the nonlinear Caputo fractional boundary value problem. Numerical results are presented for the linear sequential Caputo boundary value problem with mixed boundary conditions. All our results yield the integer results as a special cases. In future, we plan to develop a code to solve linear sequential Caputo fractional boundary value problem for a general nonhomogeneous term. This will be a useful tool to solve the nonlinear sequential Caputo fractional boundary value problem numerically with greater accuracy.

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