Sharp Inequalities Involving Neuman Means of the Second Kind with Applications

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Abstract In this paper, we give the explicit formulas for Neuman means of the second kind $N_{GQ}(a,b)$ and $N_{QG}(a,b)$, and find the best possible parameters $\alpha_i, \beta_i \in (0,1) (i = 1,2,3,\cdots,6)$ such that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)G(a,b) < N_{QG}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)G(a,b),$$

$$\frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{Q(a,b)} < \frac{1}{N_{QG}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{Q(a,b)},$$

$$\alpha_3 Q(a,b) + (1-\alpha_3)G(a,b) < N_{GQ}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)G(a,b),$$

$$\frac{\alpha_4}{G(a,b)} + \frac{1-\alpha_4}{Q(a,b)} < \frac{1}{N_{GQ}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1-\beta_4}{Q(a,b)},$$

$$\alpha_5 Q(a,b) + (1-\alpha_5)V(a,b) < N_{QG}(a,b) < \beta_5 Q(a,b) + (1-\beta_5)V(a,b),$$

$$\alpha_6 Q(a,b) + (1-\alpha_6)U(a,b) < N_{GQ}(a,b) < \beta_6 Q(a,b) + (1-\beta_6)U(a,b),$$

holds for all $a,b > 0$ with $a \neq b$, where $G(a,b)$ and $Q(a,b)$ are the classical geometric and quadratic means, $V(a,b)$, $U(a,b)$, $N_{QG}(a,b)$ and $N_{GQ}(a,b)$ are Yang and Neuman mean of the second kind.

Keywords: geometric mean, quadratic mean, Neuman means of the second kind, Yang means, inequalities.

1 Introduction

For $a,b > 0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a,b)$[1,2] is defined by

$$SB(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & if\ a < b\ , \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & if\ a > b\ . \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = log(x + \sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well-known that $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean, for example, the first and second Seiffert means, Neuman-Sándor mean, logarithmic mean and two Yang means [3] are respectively defined by

$$P = P(a,b) = \frac{a-b}{2\sin^{-1}\left[(a-b)/(a+b)\right]} = SB(G,A)\ ,$$

$$T = T(a,b) = \frac{a-b}{2\tan^{-1}\left[(a-b)/(a+b)\right]} = SB(A,Q)\ ,$$

$$M=M(a,b)=\frac{a-b}{2\sinh^{-1}\left[(a-b)/(a+b)\right]}=SB(Q,A)\ ,$$

$$L=L(a,b)=\frac{a-b}{2\tanh^{-1}\left[(a-b)/(a+b)\right]}=SB(A,G)\ ,$$

$$U=U(a,b)=\frac{a-b}{\sqrt{2}\tan^{-1}\left[(a-b)/\sqrt{2ab}\right]}=SB(G,Q)\ , \tag{1}$$

and

$$V=V(a,b)=\frac{a-b}{\sqrt{2}\sinh^{-1}\left[(a-b)/\sqrt{2ab}\right]}=SB(Q,G)\ . \tag{2}$$

where $G=G(a,b)=\sqrt{ab}$, $A=A(a,b)=(a+b)/2$ and $Q=Q(a,b)=\sqrt{(a^2+b^2)/2}$ are the classical geometric, arithmetic and quadratic means of a and b.

Let $X=X(a,b)$ and $Y=Y(a,b)$ be the symmetric bivariate means of a and b. Then Neuman mean of the second kind $N_{XY}(a,b)$[4] is defined by

$$N_{XY}(a,b)=\frac{1}{2}\left[X+\frac{Y^2}{SB(X,Y)}\right]\ . \tag{3}$$

Moreover, without loss of generality, let $a>b$, $v=(a-b)/(a+b)\in(0,1)$, then Neuman [4] gave explicit formulas

$$N_{AG}(a,b)=\frac{1}{2}A\Big[1+(1-v^2)\frac{tanh^{-1}(v)}{v}\Big],N_{GA}(a,b)=\frac{1}{2}A\Big[\sqrt{1-v^2}+\frac{sin^{-1}(v)}{v}\Big]$$

$$N_{AQ}(a,b)=\frac{1}{2}A\Big[1+(1+v^2)\frac{tan^{-1}(v)}{v}\Big],N_{QA}(a,b)=\frac{1}{2}A\Big[\sqrt{1+v^2}+\frac{sinh^{-1}(v)}{v}\Big]$$

and inequalities

$$G(a,b)<L(a,b)<N_{AG}(a,b)<P(a,b)<N_{GA}(a,b)<A(a,b)$$

$$<M(a,b)<N_{QA}(a,b)<T(a,b)<N_{AQ}(a,b)<Q(a,b)\ .$$

for all $a,b>0$ with $a\neq b$.

In the recent past, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for Schwab-Borchardt mean and its generated means can be found in the literature [4-14].

In [4], Neuman found the best possible constants $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and $\beta_1,\beta_2,\beta_3,\beta_4$ such that the double inequalities

$$\begin{aligned}
\alpha_1A(a,b)+(1-\alpha_1)G(a,b)<N_{GA}(a,b)<\beta_1A(a,b)+(1-\beta_1)G(a,b)\\
\alpha_2Q(a,b)+(1-\alpha_2)A(a,b)<N_{AQ}(a,b)<\beta_2Q(a,b)+(1-\beta_2)A(a,b)\\
\alpha_3A(a,b)+(1-\alpha_3)G(a,b)<N_{AG}(a,b)<\beta_3A(a,b)+(1-\beta_3)G(a,b)\\
\alpha_4Q(a,b)+(1-\alpha_4)A(a,b)<N_{QA}(a,b)<\beta_4Q(a,b)+(1-\beta_4)A(a,b)
\end{aligned}$$

hold for $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 2/3$, $\beta_1\geq\pi/4$, $\alpha_2\leq 2/3$, $\beta_2\geq(\pi-2)/\big[4(\sqrt{2}-1)\big]=0.689\cdots$, $\alpha_3\leq 1/3$, $\beta_3\geq 1/2$ and $\alpha_4\leq 1/3$, $\beta_4\geq\big[\log(1+\sqrt{2})+\sqrt{2}-2\big]/\big[2(\sqrt{2}-1)\big]=0.356\cdots$

Zhang et al. [11] presented the best possible parameters $\alpha_1,\alpha_2,\beta_1,\beta_2\in[0,1/2]$ and $\alpha_3,\alpha_4,\beta_3,\beta_4\in[1/2,1]$ such that the double inequalities

$$\begin{aligned}
G\big(\alpha_1a+(1-\alpha_1)b,\alpha_1b+(1-\alpha_1)a\big)<N_{AG}(a,b)<G\big(\beta_1a+(1-\beta_1)b,\beta_1b+(1-\beta_1)a\big)\\
G\big(\alpha_2a+(1-\alpha_2)b,\alpha_2b+(1-\alpha_2)a\big)<N_{GA}(a,b)<G\big(\beta_2a+(1-\beta_2)b,\beta_2b+(1-\beta_2)a\big)\\
Q\big(\alpha_3a+(1-\alpha_3)b,\alpha_3b+(1-\alpha_3)a\big)<N_{QA}(a,b)<Q\big(\beta_3a+(1-\beta_3)b,\beta_3b+(1-\beta_3)a\big)\\
Q\big(\alpha_4a+(1-\alpha_4)b,\alpha_4b+(1-\alpha_4)a\big)<N_{AQ}(a,b)<Q\big(\beta_4a+(1-\beta_4)b,\beta_4b+(1-\beta_4)a\big)\ .
\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$.

Guo et.al. [12] proved that the double inequalities

$$\begin{aligned}
A^{p_1}(a,b)G^{1-p_1}(a,b) &< N_{GA}(a,b) < A^{q_1}(a,b)G^{1-q_1}(a,b)\ ,\\
\frac{p_2}{G(a,b)}+\frac{1-p_2}{A(a,b)} &< N_{GA}(a,b) < \frac{q_2}{G(a,b)}+\frac{1-q_2}{A(a,b)}\ ,\\
A^{p_3}(a,b)G^{1-p_3}(a,b) &< N_{AG}(a,b) < A^{q_3}(a,b)G^{1-q_3}(a,b)\ ,\\
\frac{p_4}{G(a,b)}+\frac{1-p_4}{A(a,b)} &< N_{AG}(a,b) < \frac{q_4}{G(a,b)}+\frac{1-q_4}{A(a,b)}\ ,\\
Q^{p_5}(a,b)A^{1-p_5}(a,b) &< N_{AQ}(a,b) < Q^{q_5}(a,b)A^{1-q_5}(a,b)\ ,\\
\frac{p_6}{A(a,b)}+\frac{1-p_6}{Q(a,b)} &< N_{AQ}(a,b) < \frac{q_6}{A(a,b)}+\frac{1-q_6}{Q(a,b)}\ ,\\
Q^{p_7}(a,b)A^{1-p_7}(a,b) &< N_{QA}(a,b) < Q^{q_7}(a,b)A^{1-q_7}(a,b)\ ,\\
\frac{p_8}{A(a,b)}+\frac{1-p_8}{Q(a,b)} &< N_{QA}(a,b) < \frac{q_8}{A(a,b)}+\frac{1-q_8}{Q(a,b)}\ .
\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $p_1\leq 2/3$, $q_1\geq 1$, $p_2\leq 0$, $q_2\geq 1/3$, $p_3\leq 1/3$, $q_3\geq 1$, $p_4\leq 0$, $q_4\geq 2/3$, $p_5\leq 2/3$, $q_5\geq 2\log(\pi+2)/\log 2-4=0.7244\cdots$, $p_6\leq\left[6+2\sqrt{2}-(1+\sqrt{2})\pi\right]/(\pi+2)=0.2419\cdots$, $q_6\geq 1/3$, $p_7\leq 1/3$, $q_7\geq 2\log\left[\sqrt{2}+\log(1+\sqrt{2})\right]/\log 2-2=0.3977\cdots$ and $p_8\leq\left[2+\sqrt{2}-(1+\sqrt{2})\log(1+\sqrt{2})\right]/\left[\sqrt{2}+\log(1+\sqrt{2})\right]=0.5603\cdots$, $q_8\geq 2/3$.

Let $a>b>0$, $u=(a-b)/\sqrt{2ab}\in(0,+\infty)$. Then from (1)-(3) we gave the explicit formulas

$$N_{QG}(a,b)=\frac{1}{2}G(a,b)\left[\sqrt{1+u^2}+\frac{\sinh^{-1}(u)}{u}\right]. \tag{4}$$

$$N_{GQ}(a,b)=\frac{1}{2}G(a,b)\left[1+(1+u^2)\frac{\tan^{-1}(u)}{u}\right]. \tag{5}$$

The main purpose of this paper is to find the best possible parameters $\alpha_i,\beta_i\in(0,1)(i=1,2,3,\cdots,6)$ such that the double inequalities

$$\begin{aligned}
\alpha_1 Q(a,b)+(1-\alpha_1)G(a,b) &< N_{QG}(a,b) < \beta_1 Q(a,b)+(1-\beta_1)G(a,b)\ ,\\
\frac{\alpha_2}{G(a,b)}+\frac{1-\alpha_2}{Q(a,b)} &< \frac{1}{N_{QG}(a,b)} < \frac{\beta_2}{G(a,b)}+\frac{1-\beta_2}{Q(a,b)}\ ,\\
\alpha_3 Q(a,b)+(1-\alpha_3)G(a,b) &< N_{GQ}(a,b) < \beta_3 Q(a,b)+(1-\beta_3)G(a,b)\ ,\\
\frac{\alpha_4}{G(a,b)}+\frac{1-\alpha_4}{Q(a,b)} &< \frac{1}{N_{GQ}(a,b)} < \frac{\beta_4}{G(a,b)}+\frac{1-\beta_4}{Q(a,b)}\ ,\\
\alpha_5 Q(a,b)+(1-\alpha_5)V(a,b) &< N_{QG}(a,b) < \beta_5 Q(a,b)+(1-\beta_5)V(a,b)\ ,\\
\alpha_6 Q(a,b)+(1-\alpha_6)U(a,b) &< N_{GQ}(a,b) < \beta_6 Q(a,b)+(1-\beta_6)U(a,b)\ .
\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$.

2 Lemma

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 (see[15]) For $-\infty<a<b<+\infty$, let $f,g:[a,b]\to R$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are

$$\frac{f(x)-f(a)}{g(x)-g(a)},\ \ \frac{f(x)-f(b)}{g(x)-g(b)}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (see [16]). Suppose that the power series $f(x)=\sum\limits_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum\limits_{n=0}^{\infty}b_nx^n$ have the radius of convergence $r>0$ with $a_n,b_n>0$ for all $n=0,1,2,\cdots$. Let $h(x)=f(x)/g(x)$, if the sequence series $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0,r)$.

Lemma 2.3 (1) (See [17], Lemma 2.4) The function

$$\varphi_1(x)=\frac{2x+\sinh(2x)-4\sinh(x)}{\sinh(2x)-2\sinh(x)}$$

is strictly increasing from $(0,+\infty)$ onto$(2/3,1)$.

(2)(See [17], Lemma 2.6) The function

$$\varphi_2(x)=\frac{\sinh(x)\cosh(x)-x}{\big[\cosh(x)-1\big]\big[x+\sinh(x)\cosh(x)\big]}$$

is strictly decreasing from $(0,+\infty)$ onto $(0,2/3)$.

(3)(See [17], Lemma 2.5) The function

$$\varphi_3(x)=\frac{2x-\sin(2x)}{\sin(x)\big[1-\cos(x)\big]}$$

is strictly increasing from $(0,\pi/2)$ onto$(8/3,\pi)$.

(4)(See [17], Lemma 2.8) The function

$$\varphi_4(x)=\frac{\sin(x)\cos(x)-x}{\big[1-\cos(x)\big]\big[x+\sin(x)\cos(x)\big]}$$

is strictly decreasing from $(0,\pi/2)$ onto$(-1,-2/3)$.

Lemma 2.4 The function

$$\varphi_5(x)=\frac{x\sinh(2x)-2x^2}{x\sinh(2x)-\cosh(2x)+1}$$

is strictly decreasing from $(0,+\infty)$ onto$(1,2)$.

Proof. Let $f_1(x)=x\sinh(2x)-2x^2$, $g_1(x)=x\sinh(2x)-\cosh(2x)+1$. Then simple computations lead to

$$\varphi_5(x)=\frac{f_1(x)}{g_1(x)}=\frac{f_1(x)-f_1(0^+)}{g_1(x)-g_1(0^+)}\ . \tag{6}$$

$$\begin{aligned}\frac{f_1'(x)}{g_1'(x)}&=\frac{\sinh(2x)+2x\cosh(2x)-4x}{2x\cosh(2x)-\sinh(2x)}\\&=\frac{2x\sum\limits_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}x^{2n}+\sum\limits_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}x^{2n+1}-4x}{2x\sum\limits_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}x^{2n}-\sum\limits_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}x^{2n+1}}\\&=\frac{\sum\limits_{n=1}^{\infty}\frac{(n+1)\times 2^{2n+2}}{(2n+1)!}x^{2n+1}}{\sum\limits_{n=1}^{\infty}\frac{n\times 2^{2n+2}}{(2n+1)!}x^{2n+1}}=\frac{\sum\limits_{n=0}^{\infty}\frac{(n+2)\times 2^{2n+4}}{(2n+3)!}x^{2n}}{\sum\limits_{n=0}^{\infty}\frac{(n+1)\times 2^{2n+4}}{(2n+3)!}x^{2n}}\ .\end{aligned} \tag{7}$$

Let

$$a_n=\frac{(n+2)\times 2^{2n+4}}{(2n+3)!}>0, b_n=\frac{(n+1)\times 2^{2n+4}}{(2n+3)!}>0\ . \tag{8}$$

and

$$\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=-\frac{1}{(n+1)(n+2)}<0\ . \tag{9}$$

for all $n \geq 0$.

It follows from Lemma 2.2 and (7)-(9) that $f_1'(x)/g_1'(x)$ is strictly decreasing on $(0, +\infty)$. Note that

$$\varphi_5(0^+) = \frac{a_0}{b_0} = 2, \varphi_5(+\infty) = 1 \ . \tag{10}$$

Therefore, Lemma 2.4 follows easily from Lemma 2.1 and (6), (10) together with the monotonicity of $f_1'(x)/g_1'(x)$.

Lemma 2.5 The function

$$\varphi_6(x) = \frac{x^2 + x\sin(x)\cos(x) - 2\sin^2(x)}{\sin(x)\big[x - \sin(x)\big]}$$

is strictly increasing from $(0, \pi/2)$ onto $(0, (\pi^2 - 8)/[2(\pi - 2)])$.

Proof. The function $\varphi_6(x)$ can be rewritten as

$$\varphi_6(x) = \frac{x}{\sin(x)} + \frac{x + x\cos(x) - 2\sin(x)}{x - \sin(x)} = \varphi_7(x) + \varphi_8(x) \ . \tag{11}$$

where $\varphi_7(x) = x/\sin(x)$ and $\varphi_8(x) = [x + x\cos(x) - 2\sin(x)]/[x - \sin(x)]$.

Let $f_2(x) = x + x\cos(x) - 2\sin(x)$, $g_2(x) = x - \sin(x)$, $f_3(x) = 1 - \cos(x) - x\sin(x)$ and $g_3(x) = 1 - \cos(x)$. Then simple computations lead to

$$\varphi_8(x) = \frac{f_2(x)}{g_2(x)} = \frac{f_2(x) - f_2(0^+)}{g_2(x) - g_2(0^+)} \ . \tag{12}$$

$$\frac{f_2'(x)}{g_2'(x)} = \frac{f_3(x)}{g_3(x)} = \frac{f_3(x) - f_3(0^+)}{g_3(x) - g_3(0^+)} \ . \tag{13}$$

and

$$\frac{f_3'(x)}{g_3'(x)} = -\frac{x}{\tan(x)} \ . \tag{14}$$

Since the function $x \to x/\tan(x)$ is strictly decreasing on $(0, \pi/2)$, hence Lemma 2.1 and (12)-(14) lead to that $\varphi_8(x)$ is strictly increasing on $(0, \pi/2)$. From (11) and the fact that the function $\varphi_7(x) = x/\sin(x)$ is strictly increasing on $(0, \pi/2)$ together with the monotonicity of $\varphi_8(x)$ we can reach the conclusion that $\varphi_6(x)$ is strictly increasing on $(0, \pi/2)$.

Note that

$$\varphi_6(0^+) = 0, \varphi_6(\frac{\pi}{2}) = \frac{\pi^2 - 8}{2(\pi - 2)} \ . \tag{15}$$

Therefore, Lemma 2.5 follows easily from (15) and the monotonicity of $\varphi_6(x)$.

3 Main Results

Theorem 3.1 The double inequalities

$$\alpha_1 Q(a,b) + (1 - \alpha_1)G(a,b) < N_{QG}(a,b) < \beta_1 Q(a,b) + (1 - \beta_1)G(a,b) \ . \tag{16}$$

$$\frac{\alpha_2}{G(a,b)} + \frac{1 - \alpha_2}{Q(a,b)} < \frac{1}{N_{QG}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1 - \beta_2}{Q(a,b)} \ . \tag{17}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$, $\beta_1 \geq 1/2$, $\alpha_2 \leq 0$ and $\beta_2 \geq 2/3$.

Proof. We clearly see that inequalities (16) and (17) can be rewritten as

$$\alpha_1 < \frac{N_{QG}(a,b) - G(a,b)}{Q(a,b) - G(a,b)} < \beta_1 \ . \tag{18}$$

and

$$\alpha_2 < \frac{1/N_{QG}(a,b) - 1/Q(a,b)}{1/G(a,b) - 1/Q(a,b)} < \beta_2 \ . \tag{19}$$

respectively.

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$. Let $u=(a-b)/\sqrt{2ab}\in(0,+\infty)$. Then from (4) and (18)-(19) together with $Q(a,b)=G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_1 < \frac{\frac{1}{2}\left[\sqrt{1+u^2}+\frac{\sinh^{-1}(u)}{u}\right]-1}{\sqrt{1+u^2}-1} < \beta_1 \ . \tag{20}$$

and

$$\alpha_2 < \frac{u\sqrt{1+u^2}-\sinh^{-1}(u)}{(\sqrt{1+u^2}-1)\left[u\sqrt{1+u^2}+\sinh^{-1}(u)\right]} < \beta_2 \ . \tag{21}$$

respectively.

Let $x=\sinh^{-1}(u)$. Then $x\in(0,+\infty)$,

$$\begin{aligned}&\frac{\frac{1}{2}\left[\sqrt{1+u^2}+\frac{\sinh^{-1}(u)}{u}\right]-1}{\sqrt{1+u^2}-1}\\ &=\frac{1}{2}\frac{2x+\sinh(2x)-4\sinh(x)}{\sinh(2x)-2\sinh(x)}=\frac{1}{2}\varphi_1(x)\ .\end{aligned} \tag{22}$$

$$\begin{aligned}&\frac{u\sqrt{1+u^2}-\sinh^{-1}(u)}{(\sqrt{1+u^2}-1)\left[u\sqrt{1+u^2}+\sinh^{-1}(u)\right]}\\ &=\frac{\sinh(x)\cosh(x)-x}{[\cosh(x)-1][x+\sinh(x)\cosh(x)]}=\varphi_2(x)\ .\end{aligned} \tag{23}$$

where the functions $\varphi_1(x)$ and $\varphi_2(x)$ are defined as in Lemma 2.3(1) and (2).

Therefore, inequality (16) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 1/3$ and $\beta_1\geq 1/2$ follows from (20) and (22) together with Lemma 2.3(1), inequality (17) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2\leq 0$ and $\beta_2\geq 2/3$ follows from (21) and (23) together with Lemma 2.3(2).

Theorem 3.2 The double inequalities

$$\alpha_3 Q(a,b)+(1-\alpha_3)G(a,b) < N_{GQ}(a,b) < \beta_3 Q(a,b)+(1-\beta_3)G(a,b)\ . \tag{24}$$

$$\frac{\alpha_4}{G(a,b)}+\frac{1-\alpha_4}{Q(a,b)} < \frac{1}{N_{GQ}(a,b)} < \frac{\beta_4}{G(a,b)}+\frac{1-\beta_4}{Q(a,b)}\ . \tag{25}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_3\leq 2/3$, $\beta_3\geq\pi/4$, $\alpha_4\leq 0$ and $\beta_4\geq 1/3$.

Proof. We clearly see that inequalities (24) and (25) can be rewritten as

$$\alpha_3 < \frac{N_{GQ}(a,b)-G(a,b)}{Q(a,b)-G(a,b)} < \beta_3\ . \tag{26}$$

and

$$\alpha_4 < \frac{1/N_{GQ}(a,b)-1/Q(a,b)}{1/G(a,b)-1/Q(a,b)} < \beta_4\ . \tag{27}$$

respectively.

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$. Let $u=(a-b)/\sqrt{2ab}\in(0,+\infty)$. Then from (5) and (26)-(27) together with $Q(a,b)=G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_3 < \frac{\frac{1}{2}\left[1+(1+u^2)\frac{\tan^{-1}(u)}{u}\right]-1}{\sqrt{1+u^2}-1} < \beta_3\ . \tag{28}$$

and

$$\alpha_4 < \frac{2u\sqrt{1+u^2} - [u + (1+u^2)\tan^{-1}(u)]}{(\sqrt{1+u^2}-1)\Big[u + (1+u^2)\tan^{-1}(u)\Big]} < \beta_4 \ . \tag{29}$$

respectively.

Let $x = \tan^{-1}(u)$. Then $x \in (0, \pi/2)$,

$$\begin{aligned} &\frac{\frac{1}{2}\Big[1 + (1+u^2)\frac{\tan^{-1}(u)}{u}\Big] - 1}{\sqrt{1+u^2}-1} \\ &= \frac{1}{4}\frac{2x - \sin(2x)}{\sin(x)[1-\cos(x)]} = \frac{1}{4}\varphi_3(x) \ . \end{aligned} \tag{30}$$

$$\begin{aligned} &\frac{2u\sqrt{1+u^2} - [u + (1+u^2)\tan^{-1}(u)]}{(\sqrt{1+u^2}-1)\Big[u + (1+u^2)\tan^{-1}(u)\Big]} \\ &= 1 + \frac{\sin(x)\cos(x) - x}{[1-\cos(x)][x + \sin(x)\cos(x)]} = 1 + \varphi_4(x) \ . \end{aligned} \tag{31}$$

where the functions $\varphi_3(x)$ and $\varphi_4(x)$ are defined as in Lemma 2.3(3) and 2.3(4).

Therefore, inequality (24) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 2/3$ and $\beta_3 \geq \pi/4$ follows from (28) and (30) together with Lemma 2.3(3), inequality (25) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$ follows from (29) and (31) together with Lemma 2.3(4).

Theorem 3.3 The double inequalities

$$\alpha_5 Q(a,b) + (1-\alpha_5)V(a,b) < N_{QG}(a,b) < \beta_5 Q(a,b) + (1-\beta_5)V(a,b) \ . \tag{32}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_5 \leq 0$ and $\beta_5 \geq 1/2$.

Proof. We clearly see that inequalities (32) can be rewritten as

$$\alpha_5 < \frac{N_{QG}(a,b) - V(a,b)}{Q(a,b) - V(a,b)} < \beta_5 \ . \tag{33}$$

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$. Then from (4) and (33) together with $Q(a,b) = G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_5 < \frac{\frac{1}{2}\Big[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\Big] - \frac{u}{\sinh^{-1}(u)}}{\sqrt{1+u^2} - \frac{u}{\sinh^{-1}(u)}} < \beta_5 \ . \tag{34}$$

Let $x = \sinh^{-1}(u)$. Then $x \in (0, +\infty)$,

$$\begin{aligned} &\frac{\frac{1}{2}\Big[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\Big] - \frac{u}{\sinh^{-1}(u)}}{\sqrt{1+u^2} - \frac{u}{\sinh^{-1}(u)}} \\ &= 1 - \frac{1}{2}\frac{x\sinh(2x) - 2x^2}{x\sinh(2x) - \cosh(2x) + 1} = 1 - \frac{1}{2}\varphi_5(x) \ . \end{aligned} \tag{35}$$

where the functions $\varphi_5(x)$ is defined as in Lemma 2.4.

Therefore, inequality (32) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_5 \leq 0$ and $\beta_5 \geq 1/2$ follows from (34) and (35) together with Lemma 2.4.

Theorem 3.4 The double inequalities

$$\alpha_6 Q(a,b)+(1-\alpha_6)U(a,b) < N_{GQ}(a,b) < \beta_6 Q(a,b)+(1-\beta_6)U(a,b)\ . \tag{36}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_6\leq 0$, $\beta_6\geq(\pi^2-8)/[4(\pi-2)]=0.4094\cdots$.

Proof. We clearly see that inequalities (36) can be rewritten as

$$\alpha_6 < \frac{N_{GQ}(a,b)-U(a,b)}{Q(a,b)-U(a,b)} < \beta_6\ . \tag{37}$$

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$. Let $u=(a-b)/\sqrt{2ab}\in(0,+\infty)$. Then from (5) and (36) together with $Q(a,b)=G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_6 < \frac{\frac{1}{2}\Big[1+(1+u^2)\frac{\tan^{-1}(u)}{u}\Big]-\frac{u}{\tan^{-1}(u)}}{\sqrt{1+u^2}-\frac{u}{\tan^{-1}(u)}} < \beta_6\ . \tag{38}$$

Let $x=\tan^{-1}(u)$. Then $x\in(0,\pi/2)$,

$$\frac{\frac{1}{2}\Big[1+(1+u^2)\frac{\tan^{-1}(u)}{u}\Big]-\frac{u}{\tan^{-1}(u)}}{\sqrt{1+u^2}-\frac{u}{\tan^{-1}(u)}}\ .=\frac{1}{2}\varphi_6(x), \tag{39}$$

where the function $\varphi_6(x)$ is defined as in Lemma 2.5.

Therefore, inequality (36) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_6\leq 0$ and $\beta_6\geq(\pi^2-8)/[4(\pi-2)]=0.4094\cdots$ follows from (37)-(39) together with Lemma 2.5.

4 Applications

In this section, we will establish several sharp inequalities involving the hyperbolic, inverse hyperbolic, trigonometric and inverse trigonometric functions by use of Theorems 3.1-3.4.

From (3) we clearly see that

$$N_{QG}(a,b)=\frac{1}{2}\Big[Q(a,b)+\frac{G^2(a,b)}{V(a,b)}\Big],\ N_{GQ}(a,b)=\frac{1}{2}\Big[G(a,b)+\frac{Q^2(a,b)}{U(a,b)}\Big]. \tag{40}$$

Let $a>b$ and $x=\sinh^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)\in(0,\infty)$. Then simple computations lead to

$$\frac{Q(a,b)}{G(a,b)}=\cosh(x),\frac{V(a,b)}{G(a,b)}=\frac{\sinh(x)}{x},\frac{U(a,b)}{G(a,b)}=\frac{\sinh(x)}{\tan^{-1}\big[\sinh(x)\big]}. \tag{41}$$

Theorems 3.1-3.4 and (40)-(41) lead to Theorem 4.1.

Theorem 4.1 The double inequalities

$$2\alpha_1\cosh(x)+2(1-\alpha_1)<\cosh(x)+\frac{x}{\sinh(x)}<2\beta_1\cosh(x)+2(1-\beta_1),$$

$$\frac{1}{2}\big[\alpha_2\cosh(x)+(1-\alpha_2)\big]<1-\frac{2x}{\sinh(2x)+2x}<\frac{1}{2}\big[\beta_2\cosh(x)+(1-\beta_2)\big],$$

$$2\alpha_3\cosh(x)+(1-2\alpha_3)<\cosh(x)\coth(x)\tan^{-1}\big[\sinh(x)\big]<2\beta_3\cosh(x)+(1-2\beta_3),$$

$$\frac{\alpha_4\cosh(x)+(1-\alpha_4)}{2\cosh(x)}<\frac{1}{1+\cosh(x)\coth(x)\tan^{-1}\big[\sinh(x)\big]}<\frac{\beta_4\cosh(x)+(1-\beta_4)}{2\cosh(x)},$$

$$2\alpha_5\cosh(x)+2(1-\alpha_5)\frac{\sinh(x)}{x}<\cosh(x)+\frac{x}{\sinh(x)}<2\beta_5\cosh(x)+2(1-\beta_5)\frac{\sinh(x)}{x},$$

$$2\alpha_6\cosh(x)+2(1-\alpha_6)\frac{\sinh(x)}{\tan^{-1}\big[\sinh(x)\big]}<1+\cosh(x)\coth(x)\tan^{-1}\big[\sinh(x)\big]$$

$$<2\beta_6\cosh(x)+2(1-\beta_6)\frac{\sinh(x)}{\tan^{-1}\big[\sinh(x)\big]}.$$

hold for all $x>0$ if and only if $\alpha_1\le 1/3$, $\beta_1\ge 1/2$, $\alpha_2\le 0$, $\beta_2\ge 2/3$, $\alpha_3\le 2/3$, $\beta_3\ge \pi/4$, $\alpha_4\le 0$, $\beta_4\ge 1/3$, $\alpha_5\le 0$, $\beta_5\ge 1/2$, $\alpha_6\le 0$ and $\beta_6\ge(\pi^2-8)/\big[4(\pi-2)\big]$.

Let $a>b$ and $x=\tan^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)\in(0,\pi/2)$. Then it is not difficult to verify that

$$\frac{Q(a,b)}{G(a,b)}=\sec(x),\ \frac{V(a,b)}{G(a,b)}=\frac{\tan(x)}{\sinh^{-1}\big[\tan(x)\big]},\ \frac{U(a,b)}{G(a,b)}=\frac{\tan(x)}{x}. \tag{42}$$

From Theorems 3.1-3.4 and (40), (42) we get Theorem 4.2 immediately.

Theorem 4.2 The double inequalities

$$2\alpha_1\sec(x)+2(1-\alpha_1)<\sec(x)+\frac{\sinh^{-1}\big[\tan(x)\big]}{\tan(x)}<2\beta_1\sec(x)+2(1-\beta_1),$$

$$\frac{1}{2}\big[\alpha_2+(1-\alpha_2)\cos(x)\big]<\frac{\tan(x)}{\sec(x)\tan(x)+\sinh^{-1}\big[\tan(x)\big]}<\frac{1}{2}\big[\beta_2+(1-\beta_2)\cos(x)\big],$$

$$2\alpha_3\sec(x)+2(1-\alpha_3)<1+\frac{2x}{\sin(2x)}<2\beta_3\sec(x)+2(1-\beta_3),$$

$$\frac{1}{2}\big[\alpha_4+(1-\alpha_4)\cos(x)\big]<1-\frac{2x}{\sin(2x)+2x}<\frac{1}{2}\big[\beta_4+(1-\beta_4)\cos(x)\big],$$

$$2\alpha_5\sec(x)+2(1-\alpha_5)\frac{\tan(x)}{\sinh^{-1}\big[\tan(x)\big]}<\sec(x)+\frac{\sinh^{-1}\big[\tan(x)\big]}{\tan(x)}<2\beta_5\sec(x)+2(1-\beta_5)\frac{\tan(x)}{\sinh^{-1}\big[\tan(x)\big]},$$

$$2\alpha_6\sec(x)+2(1-\alpha_6)\frac{\tan(x)}{x}<1+\frac{2x}{\sin(2x)}<2\beta_6\sec(x)+2(1-\beta_6)\frac{\tan(x)}{x}.$$

hold for all $x\in(0,\pi/2)$ if and only if $\alpha_1\le 1/3$, $\beta_1\ge 1/2$, $\alpha_2\le 0$, $\beta_2\ge 2/3$, $\alpha_3\le 2/3$, $\beta_3\ge \pi/4$, $\alpha_4\le 0$, $\beta_4\ge 1/3$, $\alpha_5\le 0$, $\beta_5\ge 1/2$, $\alpha_6\le 0$ and $\beta_6\ge(\pi^2-8)/\big[4(\pi-2)\big]$.

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