A Note on Wallis' Formula

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Abstract We present some elementary proof methods for Wallis product formula by the use of integration equation, Wallis sine formula and gamma function.

Keywords: Wallis product formula, Wallis sine formula, gamma function.

1 Introduction

John Wallis (1655) gave a representation of π by the use of infinite product [1][2], that is the famous Wallis product formula

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\,\frac{8\cdot 8}{7\cdot 9}\,\cdots = \prod_{n=1}^{\infty}\frac{2n\cdot 2n}{(2n-1)(2n+1)}. \tag{1}$$

Equation (1) is equivalent to the following limit expression

$$\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2\cdot\frac{1}{2n+1}. \tag{2}$$

The Wallis product formula is closely related to the Riemann Zeta function and the Stirling formula [3][4]. The applications and proof methods of Wallis-type equations and inequalities have attracted much attention of mathematicians [5][6][7]. It is well-known that the following Wallis sine and cosine formula can be proved immediately with the help of integration by parts.

$$\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x = \int_0^{\frac{\pi}{2}}\cos^n x\mathrm{d}x = \begin{cases}\frac{(n-1)!!}{n!!}, & n \text{ is an odd}\\ \frac{(n-1)!!}{n!!}\frac{\pi}{2}, & n \text{ is an even}\end{cases}. \tag{3}$$

In this paper, three elementary proof methods for Wallis product formula (1) are given based on an integral equation (4), Wallis sine and cosine formula (3) and an asymptotic equation of gamma function (5).

2 Main Results

2.1 Proof of the Wallis Product Formula with an Integral Equation

Proof. It is fairly straightforward that

$$\int_0^{+\infty}\frac{\mathrm{d}x}{x^2+s} = \frac{\pi}{2\sqrt{s}}, \quad (s>0)$$

take the derivative of both sides n times with respect to s to get that

$$\int_0^{+\infty}\frac{\mathrm{d}x}{(x^2+s)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{\pi}{2}\cdot\frac{1}{s^{n+\frac{1}{2}}}, \quad s>0. \tag{4}$$

Set $x=\frac{y}{\sqrt{n}}, s=1$, Eq. (4) therefore becomes

$$\int_0^{+\infty}\frac{1}{(1+\frac{y^2}{n})^{n+1}}\mathrm{d}y = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}\cdot\sqrt{n},$$

take the limit with respect to n on both sides

$$\begin{aligned}\lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}\cdot\sqrt{n}&=\lim_{n\to\infty}\int_0^{+\infty}\frac{1}{(1+\frac{y^2}{n})^{n+1}}\mathrm{d}y\\&=\int_0^{+\infty}e^{-y^2}\mathrm{d}y\\&=\frac{\sqrt{\pi}}{2},\end{aligned}$$

rearrange it

$$\lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!}\ \frac{1}{\sqrt{n}}=\sqrt{\pi}.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{(2n)!!}{(2n-1)!!\cdot\sqrt{n}}\right]^2\cdot\frac{1}{2}&=\lim_{n\to\infty}\frac{[(2n)!]^2}{(2n-1)!(2n+1)!}\cdot\frac{2n+1}{2n}\\&=\prod_{n=1}^{\infty}\frac{2n\cdot 2n}{(2n-1)(2n+1)}=\frac{\pi}{2}.\end{aligned}$$

□

2.2 Proof of the Wallis Product Formula with Wallis Sine Formula

Proof. Let $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x=\int_0^{\frac{\pi}{2}}\cos^n x\mathrm{d}x, n=0,1,\ldots$, particularly, $I_0=\pi/2, I_1=1$. On the one hand, the recurrence relation $I_n=\frac{n-1}{n}I_{n-2}$ can be derived by using integration by parts on I_n, for $n\geq 2$. Consequently

$$\begin{aligned}I_{2n+1}&=\frac{2n}{2n+1}I_{2n-1}=\frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}I_1=\prod_{k=1}^{n}\frac{2k}{2k+1},\\I_{2n}&=\frac{2n-1}{2n}I_{2n-2}=\frac{2n-1}{2n}\frac{2n-3}{2n-2}\cdots\frac{1}{2}I_0=\frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}.\end{aligned}$$

On the other hand, in virtue of $\sin^{n+1}x\leq\sin^n x, 0\leq x\leq\pi/2$, we then have $0<\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\mathrm{d}x<\int_0^{\frac{\pi}{2}}\sin^{2n}x\mathrm{d}x<\int_0^{\frac{\pi}{2}}\sin^{2n-1}x\mathrm{d}x$, that is to say $0<I_{2n+1}<I_{2n}<I_{2n-1}$, accordingly

$$1\leq\frac{I_{2n}}{I_{2n+1}}\leq\frac{I_{2n-1}}{I_{2n+1}}=\frac{2n+1}{2n},$$

by the use of the squeeze rule

$$\lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}}=\lim_{n\to\infty}\frac{\pi}{2}\prod_{k=1}^{n}\left(\frac{2k-1}{2k}\cdot\frac{2k+1}{2k}\right)=1,$$

it follows that

$$\frac{\pi}{2}=\prod_{k=1}^{\infty}\frac{2k\cdot 2k}{(2k-1)(2k+1)}.$$

□

2.3 Proof of the Wallis Product Formula with Gamma Function

Proof. The Euler gamma function [8] is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} \mathrm{e}^{-x} \mathrm{d}x,$$

and the relations below are well-known for $n = 1, 2, \ldots, s \geq 0$,

$$\Gamma(n) = (n-1)!, \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!}{2^n}\sqrt{n}, \quad \lim_{n\to\infty} \frac{\Gamma(n+s)}{n^s \Gamma(n)} = 1. \tag{5}$$

Specifically, let $s = 1/2$, then

$$\begin{aligned} \lim_{n\to\infty} \left[\frac{\Gamma(n+\frac{1}{2})}{\sqrt{n}\Gamma(n)}\right]^2 &= \lim_{n\to\infty} \left[\frac{(2n-1)!!\sqrt{\pi}}{2^n(n-1)!\sqrt{n}}\right]^2 \\ &= \lim_{n\to\infty} \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \cdot (2n+1) \cdot \frac{n}{2n+1}\pi = 1. \end{aligned}$$

Therefore, Wallis product formula (2) can be obtained immediately from the above limit expression. □

2.4 The Applications of Wallis Product Formula

Here are two classical examples, we will solve them by using Wallis product formula typically.

Example 2.1. Determine the Poisson integration $I = \int_0^{+\infty} e^{-x^2} \mathrm{d}x$.

For any subinterval $[a, b]$ on $[0, +\infty)$, when $n \to \infty$, the sequence of continuous function $\left\{\left(1 + \frac{x^2}{n}\right)^{-n}\right\}$ converge to e^{-x^2} uniformly on $[a, b]$, hence

$$\begin{aligned} I = \int_0^{+\infty} e^{-x^2} \mathrm{d}x &= \int_0^{+\infty} \lim_{n\to+\infty} \left(1 + \frac{x^2}{n}\right)^{-n} \mathrm{d}x \\ &= \lim_{n\to+\infty} \int_0^{+\infty} \left(1 + \frac{x^2}{n}\right)^{-n} \mathrm{d}x. \end{aligned}$$

Let $x = \sqrt{n}\cot t$, and by the Wallis sine formula (3), we have

$$\begin{aligned} I &= \lim_{n\to+\infty} \int_0^{\frac{\pi}{2}} \sqrt{n} \sin^{2n-2} t \mathrm{d}t = \lim_{n\to+\infty} \sqrt{n} \cdot \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{\pi}{2} \\ &= \lim_{n\to+\infty} \frac{\sqrt{n}}{\sqrt{2n-1}} \cdot \frac{1}{\left\{\frac{1}{2n-1}\left[\frac{(2n-2)!!}{(2n-3)!!}\right]^2\right\}^{\frac{1}{2}}} \cdot \frac{\pi}{2} \\ &= \frac{\sqrt{\pi}}{2}. \end{aligned}$$

Example 2.2. Discuss the convergence of series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{(2n-1)!!}{(2n)!!}\right]^p, \quad p > 0.$$

Let $a_n = \frac{(2n-1)!!}{(2n)!!}, b_n = a_n^p$, in view of the Wallis product formula (2), it becomes that

$$a_n \sim \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2n+1}}, \quad (n \to \infty)$$

so $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{p}{2}}}$ have the same convergerce property. According to the p convergence criterion and Leibniz discriminance, the original series is conditionally convergent for $0 < p \leq 2$ and absolutely convergent for $p > 2$.

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References

1. C. Chen and F. Qi, "The best bounds in wallis inequality," *Proceedings of the American Mathematical Society*, vol. 133, no. 2, pp. 397–401, 2005.
2. J.-G. X. Senlin Guo and F. Qi, "Some exact constants for the approximation of the quantity in the wallis' formula," *Journal of Inequalities and Applications*, pp. 1–7, 2013. [Online]. Available: http://www.journalofinequalitiesandapplications.com/content/2013/1/6
3. S. J. Miller, "A probabilistic proof of wallis formula for π," *American Mathematical Monthly*, vol. 115, no. 8, pp. 740–745, 2008.
4. D. Bhattacharjee and N. Mukhopadhyay, "Stirling's formula and its extensions: Heuristic approaches," *Communications in Statistics-Theory and Methods*, vol. 39, no. 3, pp. 1046–1053, 2010.
5. F. Qi and C. Mortici, "Some best approximation formulas and inequalities for the wallis ratio," *Applied Mathematics and Computation*, vol. 253, pp. 363–368, 2015.
6. E. C. Popa and N.-A. Secelean, "Estimates for the constants of landau and lebesgue via some inequalities for the wallis ratio," *Journal of Computational and Applied Mathematics*, vol. 269, pp. 68–74, 2014.
7. C. Mortici, "Completely monotone functions and the wallis ratio," *Applied Mathematics Letters*, vol. 25, no. 4, pp. 717–722, 2012.
8. G. E. A. Ranjan Roy and R. A. Askey, *Special Functions, Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1999.

