Estimation of the Stress-Strength Reliability for the Dagum Distribution

Bander Al-Zahrani and Samerah Basloom

Department of Statistics, King Abdulaziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia
Email: bmalzahrani@kau.edu.sa

Abstract The paper deals with the problem of estimating the reliability of a single and multicomponent stress-strength based on a three-parameter Dagum distribution. The maximum likelihood estimators and the Bayes estimators of $R = P(Y < X)$ when X and Y are two independent random variables follow Dagum distribution and their asymptotic distributions are obtained. Also, the reliability of a multicomponent stress-strength model is estimated using the maximum likelihood. Consequently, the asymptotic confidence intervals of the reliability of a single and multicomponent stress-strength model are constructed. Gibbs and Metropolis sampling were used to provide sample-based estimates of the reliability and its associated credible intervals. Finally, Monte Carlo simulations are carried out for illustrative purposes.

Keywords: Dagum distribution, maximum likelihood estimator, Bayes estimator, stress-strength models.

1 Introduction

Dagum distribution was introduced by Dagum [1] for modeling personal income data as an alternative to the Pareto and log-normal models. This distribution has been extensively used in various fields such as, income and wealth data, meterological data, reliability and survival analysis. The Dagum distribution is also known as the inverse Burr XII distribution, especially in the actuarial literature. An important characteristic of Dagum distribution is that its hazard function can be monotonically decreasing, an upside-down bathtub, or bathtub and then upside-down bathtub shaped, for details see Domma [2]. This behavior of the distribution has led several authors to study the model in different fields. In fact, recently, the Dagum distribution has been studied from a reliability point of view and used to analyze survival data (see Domma et al., [3], and Domma and Condino [4]). Kleiber and Kotz [5] and Kleiber [6] provided an exhaustive review on the origin of the Dagum model and its applications. Domma et al. [3] estimated the parameters of Dagum distribution with censored samples. Shahzad and Asghar [7] used TL-moments to estimate the parameter of this distribution. Oluyede and Ye [8] presented the class of weighted Dagum and related distributions. Domma and Condino [4] proposed the five parameter beta-Dagum distribution.

A continuous random variable T is said to have a three-parameter Dagum distribution, abbreviated as $T \sim Dagum(\alpha, \beta, \delta)$, if its density probability function (pdf) is given as

$$f\left(t; \alpha, \beta, \delta\right) = \beta\alpha\delta t^{-\delta-1}\left(1 + \alpha t^{-\delta}\right)^{-\beta-1}, \quad t > 0, \tag{1.1}$$

where $\alpha > 0$ is the scale parameter and its two shape parameters β and δ are both positive. The corresponding distribution function is given by

$$F\left(t; \alpha, \beta, \delta\right) = \left(1 + \alpha t^{-\delta}\right)^{-\beta}, \quad t > 0,\ \beta, \alpha, \delta > 0. \tag{1.2}$$

In this paper we focus our attention on estimating the reliability of a single component stress-strength, $R = P(Y < X)$ where X and Y are two independent random variables following Dagum distribution with different shape parameter β and have the same scale parameter α and shape parameter δ. Here R is the probability that the strength X of a component during a given period of time exceeds the stress Y. We also focus on estimating the reliability of multicomponent stress-strength $R_{s,k}$, where s and k are two intgers and $s \leq k$. The problem of estimating R and $R_{s,k}$ has attracted the attention of many authors, a good overview on estimating R can be found in the monograph of Kotz et al. [9]. See also, Pandey and Uddin [10] and Srinivasa and Kantam [11] for details on estimation of $R_{s,k}$. It is worth mentioning that the problem here seems to be clearly an application of well known mathematical methods. However, there were some challenges in finding the estimators.

The rest of the paper is organized as follows. In Section 2, we obtain the maximum likelihood estimatior (MLE) of R and the asymptotic distribution of $\hat{R}$. The MLE and Bayes estimators of R when α is known is considered. Also, simulation studies have been presented for illustrative purposes. In section 3, the MLEs are obtained and employed to get the asymptotic distribution and confidence intervals for $R_{s,k}$. Finally, the conclusion and comments are provided in Section 4.

2 Estimation of R with Different β and Common δ and α

In this section, we investigate the properties of R when the shape parameters β are different, the scale parameter α and the shape parameter δ are constants. The other cases, i.e. the parameters α and δ are not constants and the general case where all parameters are different, can be studied in a similar way to the case presented in this paper.

2.1 Maximum Likelihood Estimator of R

Let $X \sim Dagum(\beta_1, \alpha, \delta)$ and $Y \sim Dagum(\beta_2, \alpha, \delta)$, where X and Y are two independent random variables. All three parameters, β, α and δ are unknown to us. Then it can be shown that

$$R = P(Y < X) = \iint\limits_{0<y<x} f(x,y)dxdy = \frac{\beta_1}{\beta_1 + \beta_2}. \tag{2.1}$$

To compute the MLE of R, suppose that $X_1, X_2, \cdots, X_n$ is a random sample from $Dagum(\beta_1, \alpha, \delta)$ and $Y_1, Y_2, \cdots, Y_m$ is another random sample from $Dagum(\beta_2, \alpha, \delta)$. Then the log-likelihood function of the observed sample is

$$\begin{aligned} \ln L(\beta_1, \beta_2, \alpha, \delta) = {} & n\ln(\beta_1) + (n+m)\ln(\alpha) + (n+m)\ln(\delta) \\ & -(\delta+1)\left[\sum_{i=1}^{n}\ln(x_i) + \sum_{j=1}^{m}\ln(y_j)\right] \\ & -(\beta_1+1)s_1(x,\alpha,\delta) + m\ln(\beta_2) - (\beta_2+1)s_2(y,\alpha,\delta), \end{aligned}$$

where

$$s_1(x,\alpha,\delta) = \sum_{i=1}^{n}\ln(1+\alpha x_i^{-\delta}), \tag{2.2}$$

and

$$s_2(y,\alpha,\delta) = \sum_{j=1}^{m}\ln(1+\alpha y_j^{-\delta}). \tag{2.3}$$

The MLEs of β, α and δ say $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\delta}$, respectively, can be obtained as the solutions of the following equations

$$\left.\begin{aligned} \frac{\partial L}{\partial \beta_1} &= \frac{n}{\beta_1} - s_1(x,\alpha,\delta) = 0. \\ \frac{\partial L}{\partial \beta_2} &= \frac{m}{\beta_2} - s_2(y,\alpha,\delta) = 0. \\ \frac{\partial L}{\partial \alpha} &= \frac{n+m}{\alpha} - (\beta_1+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{(1+\alpha x_i^{-\delta})} - (\beta_2+1)\sum_{j=1}^{m}\frac{y_j^{-\delta}}{(1+\alpha y_j^{-\delta})} = 0. \\ \frac{\partial L}{\partial \delta} &= \frac{n+m}{\delta} - \sum_{i=1}^{n}\ln(x_i) - \sum_{j=1}^{m}\ln(y_j) - (\beta_1+1)\sum_{i=1}^{n}\frac{\alpha x_i^{-\delta}\ln(x_i)}{(1+\alpha x_i^{-\delta})} \\ &\quad -(\beta_2+1)\sum_{j=1}^{m}\frac{y_j^{-\delta}\ln(y_j)}{(1+\alpha y_j^{-\delta})} = 0. \end{aligned}\right\} \tag{2.4}$$

We obtain

$$\hat{\beta}_1 = \frac{n}{\hat{s}_1(x,\alpha,\delta)}. \tag{2.5}$$

$$\hat{\beta}_2 = \frac{m}{\hat{s}_2(y,\alpha,\delta)}. \tag{2.6}$$

The estimators $\hat{\alpha}$ and $\hat{\delta}$ can be obtained as the solutions of the nonlinear equations given in (2.4). The "plug-in estimation" of R, say $\hat{R}$, is readily computed as

$$\hat{R} = \frac{\hat{\beta}_{1ML}}{\hat{\beta}_{1MR} + \hat{\beta}_{2ML}}. \tag{2.7}$$

2.2 Asymptotic Distribution of $\hat{R}$

In this section, the asymptotic distribution of $\hat{\theta} = (\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha}, \hat{\delta})$ and the asymptotic distribution of $\hat{R}$ are obtained. The Fisher information matrix of θ, denoted by $\mathbf{J}(\theta) = \mathbf{E}(\mathbf{I}, \theta)$, is giving below, where $\mathbf{I} = [I_{i,j}]_{i,j=1,2,3,4}$ is the observed information matrix i.e.,

$$I(\theta) = -\begin{pmatrix} \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \beta_1} & \frac{\partial^2 L}{\partial \alpha \partial \beta_2} & \frac{\partial^2 L}{\partial \alpha \partial \delta} \\ \frac{\partial^2 L}{\partial \beta_1 \partial \alpha} & \frac{\partial^2 L}{\partial \beta_1^2} & \frac{\partial^2 L}{\partial \beta_1 \partial \beta_2} & \frac{\partial^2 L}{\partial \beta_1 \partial \delta} \\ \frac{\partial^2 L}{\partial \beta_2 \partial \alpha} & \frac{\partial^2 L}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 L}{\partial \beta_2^2} & \frac{\partial^2 L}{\partial \beta_1 \partial \delta} \\ \frac{\partial^2 L}{\partial \delta \partial \alpha} & \frac{\partial^2 L}{\partial \delta \partial \beta_1} & \frac{\partial^2 L}{\partial \delta \partial \beta_2} & \frac{\partial^2 L}{\partial \delta^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix},$$

and the elements of $I(\theta)$ are as follows

$$\begin{aligned}
I_{11} &= \frac{n+m}{\alpha^2} + (\beta_1+1)\sum_{i=1}^{n} \frac{(x_i^{-\delta})^2}{(1+\alpha x_i^{-\delta})} + (\beta_2+1)\sum_{j=1}^{m} \frac{(y_j^{-\delta})^2}{(1+\alpha y_j^{-\delta})}. \\
I_{12} &= I_{21} = -\sum_{i=1}^{n} \frac{X_i^{-\delta}}{(1+\alpha x_i^{-\delta})}, \quad I_{13} = I_{31} = -\sum_{j=1}^{m} \frac{y_j^{-\delta}}{(1+\alpha y_j^{-\delta})}. \\
I_{14} &= I_{41} = (\beta_1+1)\left[\sum_{i=1}^{n} \frac{x_i^{-\delta}\ln x_i}{(1+\alpha x_i^{-\delta})} - \sum_{i=1}^{n} \frac{\left(x_i^{-\delta}\right)^2 \alpha \ln x_i}{(1+\alpha x_i^{-\delta})^2}\right] \\
&\quad + (\beta_2+1)\left[\sum_{j=1}^{m} \frac{y_j^{-\delta}\ln y_j}{(1+\alpha y_j^{-\delta})} - \sum_{j=1}^{m} \frac{\left(y_j^{-\delta}\right)^2 \alpha \ln y_j}{(1+\alpha y_j^{-\delta})^2}\right]. \\
I_{22} &= -\frac{n}{\beta_1^2}, \quad I_{23} = I_{32} = 0, \quad I_{24} = I_{42} = \sum_{i=1}^{n} \frac{\alpha x_i^{-\delta}\ln x_i}{(1+\alpha x_i^{-\delta})} \\
I_{33} &= \frac{m}{\beta_2^2}. \\
I_{34} &= I_{43} = -\frac{n+m}{\delta^2} + \sum_{j=1}^{m} \frac{\alpha y_j^{-\delta}\ln y_j}{(1+\alpha y_j^{-\delta})} - \frac{(\beta_1+1)n\alpha x^{-\delta}\ln(x)^2}{(1+\alpha x^{-\delta})} + \frac{(\beta_1+1)n\alpha^2(x^{-\delta})^2\ln(x)^2}{(1+\alpha x^{-\delta})^2} \\
&\quad - \frac{(\beta_2+1)m\alpha y^{-\delta}\ln(y)^2}{(1+\alpha y^{-\delta})} + \frac{(\beta_2+1)m\alpha^2(y^{-\delta})^2\ln(y)^2}{(1+\alpha y^{-\delta})^2}. \\
I_{44} &= -\frac{n+m}{\alpha^2} - (\beta_1+1)\left[\sum_{i=1}^{n} \frac{\alpha x_i^{-\delta}\ln(x_i)^2}{(1+\alpha x_i^{-\delta})} - \sum_{i=1}^{n} \frac{\alpha^2(x_i^{-\delta})^2\ln(x_i)^2}{(1+\alpha x_i^{-\delta})^2}\right] \\
&\quad + (\beta_2+1)\left[\sum_{j=1}^{m} \frac{\alpha y_j^{-\delta}\ln(y_j)^2}{(1+\alpha y_j^{-\delta})} - \sum_{j=1}^{m} \frac{\alpha^2(y_j^{-\delta})^2\ln(y_j)^2}{(1+\alpha y_j^{-\delta})^2}\right].
\end{aligned}$$

The following integrals can be helpful when finding the elements of the Fisher information matrix. Note that $\psi(.)$ is the digamma function, defined as $\psi(x) = d/d(x)\ln(\Gamma(x))$.

$$\int_{\infty}^{0} \ln(t)(1+t)^{-\beta-1}\, dt = -\frac{\psi(\beta)+\gamma}{\beta}, \quad \int_{\infty}^{0} t\ln(t)(1+t)^{-\beta-2}\, dt = -\frac{\Psi(\beta)+\gamma-1}{\alpha(\beta+1)},$$
$$\int_{\infty}^{0} t\ln^2(t)(1+t)^{-\beta-3}\, dt = -\frac{\psi(\beta)'\psi^2(\beta)+\gamma^2+\frac{\pi^2}{6}}{(\beta+1)(\beta+2)} + 2\frac{[\beta(\gamma-1)+1]\,\psi(\beta)-[\gamma(\beta-1)+1]}{\beta(\beta+1)(\beta+2)},$$
$$\int_{\infty}^{0} t^2\ln(t)(1+t)^{-\beta-3}\, dt = -\frac{2\psi(\beta)-2\gamma+3}{\beta(\beta^2+3\beta+2)},$$
$$\int_{\infty}^{0} t^2\ln^2(t)(1+t)^{-\beta-3}\, dt = -\frac{6\psi^2(\beta)+12\psi(\beta)\gamma+6\gamma^2+\pi^2-18\psi(\beta)-18\gamma+6\psi(1,\beta)+6}{\beta(3\beta^2+9\beta+6)}.$$

The elements of the Fisher information matrix are obtained by taking the expectations of the observed matrix. Doing so will result in the following:

$$J_{11} = \frac{n+m}{\alpha^2} - \frac{n\beta_1(\beta_1+1)}{\alpha^2}\, B(3,\beta_1) - \frac{m\beta_2(\beta_2+1)}{\alpha^2}\, B(3,\beta_2).$$
$$J_{12} = \frac{n\beta_1}{\alpha}\, B(2,\beta_1), \quad J_{13} = J_{31} = \frac{m\beta_2}{\alpha}\, B(2,\beta_2),$$

where $B(.,.)$ is the beta function.

$$\begin{aligned} J_{14} = J_{41} = &-\frac{n\beta_1(\beta_1+1)}{\delta\alpha}\Bigg[\left(B(2,\beta_1)\ln(\alpha) + \frac{\psi(\beta_1)+\gamma-1}{\alpha(\beta_1+1)}\right) \\ &- \left(B(3,\beta_1)\ln(\alpha) + \frac{2\psi(\beta_1)-2\gamma+3}{\beta_1(\beta_1^2+3\beta_1+2)}\right)\Bigg] \\ &- \frac{m\beta_2(\beta_2+1)}{\delta\alpha}\Bigg[\left(B(2,\beta_2)\ln(\alpha) + \frac{\psi(\beta_2)+\gamma-1}{\alpha(\beta_2+1)}\right) \\ &- \left(B(3,\beta_2)\ln(\alpha) + \frac{2\psi(\beta_2)-2\gamma+3}{\beta_2(\beta_2^2+3\beta_2+2)}\right)\Bigg]. \end{aligned}$$

Again, the functions $B(.,.)$ and $\psi(.)$ are the beta function and the digamma function.

$$J_{22} = \frac{n}{\beta_1^2}, \quad J_{23} = J_{32} = 0, \quad J_{24} = J_{42} = -\frac{n\beta_1}{\delta}\left[B(2,\beta_1)\ln(\alpha) + \frac{\psi(\beta_1)+\gamma-1}{\beta_1(\beta_1+1)}\right].$$
$$J_{33} = \frac{m}{\beta_2^2}, \quad J_{34} = J_{43} = -\frac{m\beta_2}{\delta}\left[B(2,\beta_2)\ln(\alpha) + \frac{\psi(\beta_2)+\gamma-1}{\beta_1(\beta_2+1)}\right].$$

and the element J_{44} is defined as follows

$$\begin{aligned} J_{44} = &\frac{n+m}{\alpha^2} + \frac{n\beta_1(\beta_1+1)}{\delta^2}\Bigg[\Bigg(B(3,\beta_1)\ln^2(\alpha) + 2\ln(\alpha)\frac{\psi(\beta_1)+\gamma-1}{\beta_1(\beta_1+1)} \\ &+ \frac{6\psi(\beta_1)^2+12\psi(\beta_1)\gamma+6\gamma^2+\pi^2-12\psi(\beta_1)-12\gamma+6\psi(1,\beta_1)}{6\beta_1(\beta_1+1)}\Bigg) \\ &- \frac{1}{\alpha}\Bigg(\ln^2(\alpha) - 2B(3,\beta_1) + 2\frac{2\psi(\beta_1)-2\gamma+3}{\beta_1(\beta_1^2+3\beta_1+2)} \\ &- \frac{6\psi^2(\beta_1)+12\psi(\beta_1)\gamma+6\gamma^2+\pi^2-18\psi(\beta_1)-18\gamma+6\psi(1,\beta_1)+6}{\beta_1(3\beta_1^2+9\beta_1+6)}\Bigg)\Bigg] \\ &- \frac{m\beta_2(\beta_2+1)}{\delta^2}\Bigg[\Bigg(B(3,\beta_2)\ln^2(\alpha) + 2\ln(\alpha)\frac{\psi(\beta_2)+\gamma-1}{\beta_2(\beta_2+1)} \\ &+ \frac{6\psi(\beta_2)^2+12\psi(\beta_2)\gamma+6\gamma^2+\pi^2-12\psi(\beta_2)-12\gamma+6\psi(1,\beta_2)}{6\;\beta_2(\beta_2+1)}\Bigg) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\alpha}\Bigg(\ln^2(\alpha) - 2B(3,\beta_2) + 2\frac{2\psi(\beta_2) - 2\gamma + 3}{\beta_2(\beta_2^2 + 3\beta_2 + 2)} \\
&- \frac{6\psi^2(\beta_2) + 12\psi(\beta_2)\gamma + 6\gamma^2 + \pi^2 - 18\psi(\beta_2) - 18\gamma + 6\psi(1,\beta_2) + 6}{\beta_2(3\beta_2^2 + 9\beta_2 + 6)}\Bigg)\Bigg].
\end{aligned}
$$

It is worth mentioning that the Dagum family of distributions satisfies all the regularity conditions, for example see Nadarajah and Kotz [12]. Now, it turns out that we can formulate the limiting joint distribution of estimators.

Theorem 1 *As $n \to \infty$ and $m \to \infty$ and $\frac{n}{m} \to p$, where p is positive real constant, then*

$$
\left[\sqrt{m}(\hat{\beta}_1 - \beta_1), \sqrt{n}(\hat{\beta}_2 - \beta_2), \sqrt{m}(\hat{\alpha} - \alpha), \sqrt{m}(\hat{\delta} - \delta)\right] \to N_4(0, A^{-1}(\beta_1, \beta_2, \alpha, \delta)),
$$

where the covariance matrix A is given as

$$
A(\beta_1, \beta_2, \alpha, \delta) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.
$$

Proof. We first give the elements of the covariance matrix A.

$$
\begin{aligned}
a_{11} &= \lim_{n,m\to\infty} \frac{J_{11}}{m} = \frac{p+1}{\alpha^2} - \frac{p\beta_1(\beta_1+1)}{\alpha^2}\ B(3,\beta_1) - \frac{\beta_2(\beta_2+1)}{\alpha^2}\ B(3,\beta_2). \\
a_{12} &= a_{21} = \lim_{n,m\to\infty} \frac{J_{12}\sqrt{p}}{n} = \frac{\sqrt{p}\beta_1}{\alpha}\ B(2,\beta_1). \\
a_{13} &= a_{31} = \lim_{n,m\to\infty} \frac{J_{13}}{m} = \frac{\beta_2}{\alpha}\ B(2,\beta_2). \\
a_{14} &= a_{41} = \lim_{n,m\to\infty} \frac{J_{14}}{m\sqrt{p}}. \\
&= -\frac{\sqrt{p}\beta_1(\beta_1+1)}{\delta\alpha}\Bigg[\left(B(2,\beta_1)\ln(\alpha) + \frac{\psi(\beta_1) + \gamma - 1}{\alpha(\beta_1+1)}\right) \\
&\quad - \left(B(3,\beta_1)\ln(\alpha) + \frac{2\psi(\beta_1) - 2\gamma + 3}{\beta_1(\beta_1^2 + 3\beta_1 + 2)}\right)\Bigg] \\
&\quad - \frac{\beta_2(\beta_2+1)}{\sqrt{p}\alpha\delta}\Bigg[\left(B(2,\beta_2)\ln(\alpha) + \frac{\psi(\beta_2) + \gamma - 1}{\alpha(\beta_2+1)}\right) \\
&\quad - \left(B(3,\beta_2)\ln(\alpha) + \frac{2\psi(\beta_2) - 2\gamma + 3}{\beta_2(\beta_2^2 + 3\beta_2 + 2)}\right)\Bigg]. \\
a_{22} &= \lim_{n,m\to\infty} \frac{J_{22}}{n} = \frac{1}{\beta_1^2}, \quad a_{23} = a_{32} = 0. \\
a_{34} &= a_{43} = \lim_{n,m\to\infty} \frac{J_{34}}{m\sqrt{p}} = -\frac{\beta_2}{\delta\sqrt{p}}\left[B(2,\beta_2)\ln(\alpha) + \frac{\psi(\beta_2) + \gamma - 1}{\beta_2(\beta_2+1)}\right]. \\
a_{24} &= a_{42} = \lim_{n,m\to\infty} \frac{J_{24}}{n} = -\frac{B_1}{\delta}\left[B(2,\beta_1)\ln(\alpha) + \frac{\psi(\beta_1) + \gamma - 1}{\beta_1(\beta_1+1)}\right].
\end{aligned}
$$

and

$$
\begin{aligned}
a_{44} &= \lim_{n,m\to\infty} \frac{J_{44}}{n}. \\
&= \frac{1}{\alpha^2 p} + \frac{\beta_1(\beta_1+1)}{\delta^2}\Bigg[\left(B(3,\beta_1)\ln^2(\alpha) + 2\ln(\alpha)\frac{\psi(\beta_1) + \gamma - 1}{\beta_1(\beta_1+1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\frac{6\psi(\beta_1)^2+12\psi(\beta_1)\gamma+6\gamma^2+\pi^2-12\psi(\beta_1)-12\gamma+6\psi(1,\beta_1)}{6\beta_1(\beta_1+1)}\\
&-\left(\frac{1}{\alpha}\right)\Bigg(\ln^2(\alpha)-2B(3,\beta_1)+2\frac{2\psi(\beta_1)-2\gamma+3}{\beta_1(\beta_1^2+3\beta_1+2)}\\
&-\frac{6\psi^2(\beta_1)+12\psi(\beta_1)\gamma+6\gamma^2+\pi^2-18\psi(\beta_1)-18\gamma+6\psi(1,\beta_1)+6}{\beta_1(3\beta_1^2+9\beta_1+6)}\Bigg)\Bigg)\Bigg]\\
&-\left(\frac{\beta_2(\beta_2+1)}{p\delta^2}\right)\Bigg[\Bigg(B(3,\beta_2)\ln^2(\alpha)+2\ln(\alpha)\frac{\psi(\beta_2)+\gamma-1}{\beta_2(\beta_2+1)}\\
&+\frac{6\psi^2(\beta_2)+12\psi(\beta_2)\gamma+6\gamma^2+\pi^2-12\psi(\beta_2)-12\gamma+6\psi(1,\beta_2)}{6\beta_2(\beta_2+1)}\Bigg)\\
&-\left(\frac{1}{\alpha}\right)\Bigg(\ln^2(\alpha)-2B(3,\beta_2)+2\left[\frac{2\psi(\beta_2)-2\gamma+3}{\beta_2(\beta_2^2+3\beta_2+2)}\right]\\
&-\frac{(6\psi^2(\beta_2)+12\psi(\beta_2)\gamma+6\gamma^2+\pi^2-18\psi(\beta_2)-18\gamma+6\psi(1,\beta_2)+6)}{(\beta_2(3\beta_2^2+9\beta_2+6))}\Bigg)\Bigg].
\end{aligned}
$$

The proof follows immediately by invoking the asymptotic properties of MLEs and the multivariate central limit theorem. □

One of the main results here is Theorem 2, concerning the asymptotic properties of the parameter R.

Theorem 2 *As $n\to\infty$, $m\to\infty$, $\frac{n}{m}\to p$, where p is a positive number, then*

$$\sqrt{n}(\hat{R}-R)\to N(0,B_A),$$

where

$$
\begin{aligned}
B_A=\frac{1}{U_A(\beta_1+\beta_2)^4}\Bigg[&\beta_2\Bigg(\beta_2\left(a_{22}a_{33}a_{44}-a_{22}a_{43}^2-a_{33}a_{42}^2\right)\\
&+\beta_1\left(a_{21}a_{33}a_{44}-a_{21}a_{43}+a_{31}a_{42}a_{43}-a_{33}a_{41}a_{42}\right)\Bigg)\\
&-\beta_1\Bigg(-\beta_2\left(a_{21}a_{33}a_{44}-a_{21}a_{43}^2+a_{31}a_{42}^2a_{43}-a_{33}a_{41}a_{42}\right)\\
&-\beta_1\left(a_{11}a_{33}a_{44}-a_{11}a_{43}^2-a_{31}^2a_{44}+2a_{31}a_{41}a_{43}-a_{33}a_{41}^2\right)\Bigg)\Bigg].
\end{aligned}
$$

and

$$
\begin{aligned}
U_A=\;&a_{11}a_{22}a_{33}a_{44}-a_{11}a_{22}a_{43}^2-a_{11}a_{33}a_{24}^2-a_{12}^2a_{33}a_{44}+a_{12}^2a_{43}^2-2a_{12}a_{13}\\
&a_{24}a_{34}2a_{12}a_{33}a_{14}a_{24}+a_{22}a_{13}^2a_{44}+2a_{22}a_{13}a_{14}a_{34}-a_{22}a_{33}a_{14}^2+a_{13}^2a_{24}^2.
\end{aligned}
$$

Proof. Applying the delta method (see Casella and Roger, [13]) and using Theorem 1, we can write the asymptotic distribution of $\hat{R}$ where $\hat{R}=g(\hat{\beta}_1,\hat{\beta}_2,\hat{\alpha},\hat{\delta})$ and $g(\beta_1,\beta_2,\alpha,\delta)=\beta_1/(\beta_1+\beta_2)$ as the following:

$$\sqrt{n}(\hat{R}-R)\to N(0,B_A),$$

where

$$B_A=b_A^tA^{-1}b_A,$$

$$b_A=\begin{pmatrix}\frac{\partial R}{\partial\beta_1}\\ \frac{\partial R}{\partial\beta_2}\\ \frac{\partial R}{\partial\alpha}\\ \frac{\partial R}{\partial\delta}\end{pmatrix}=\frac{1}{(\beta_1+\beta_2)^2}\begin{pmatrix}\beta_2\\ \beta_1\\ 0\\ 0\end{pmatrix},$$

and

$$A^{-1} = \frac{1}{U_A}\begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}.$$

The elements of the A^{-1} are obtained as follows

$$\begin{aligned}
v_{11} &= a_{22}a_{33}a_{44} - a_{22}a_{43}^2 - a_{33}a_{42}^2. \\
v_{12} &= v_{21} = -a_{21}a_{33}a_{44} - a_{21}a_{43}^2 + a_{31}a_{42}a_{43} - a_{33}a_{41}a_{12}. \\
v_{13} &= v_{31} = -a_{21}a_{42}a_{43} + a_{22}a_{31}a_{44} - a_{22}a_{41}a_{43} - a_{31}a_{42}^2. \\
v_{14} &= v_{41} = a_{21}a_{33}a_{42} + a_{22}a_{31}a_{43} - a_{22}a_{33}a_{41}. \\
v_{22} &= a_{11}a_{33}a_{44} - a_{11}a_{43}^2 - a_{31}^2a_{44} + 2a_{31}a_{41}a_{43} - a_{33}a_{41}^2. \\
v_{23} &= v_{32} = a_{11}a_{42}a_{43} + a_{21}a_{31}a_{44} - a_{21}a_{41}a_{43} - a_{31}a_{41}a_{42}. \\
v_{24} &= v_{42} = -a_{11}a_{33}a_{42} + a_{21}a_{31}a_{43} - a_{21}a_{33}a_{41} - a_{31}^2a_{42}. \\
v_{33} &= a_{11}a_{22}a_{44} - a_{11}a_{42}^2 - a_{21}^2a_{44} + 2a_{21}a_{41}a_{42} - a_{22}a_{41}^2. \\
v_{34} &= v_{43} = -(a_{11}a_{22}a_{43} - a_{21}^2a_{43} + a_{21}a_{31}a_{42} - a_{22}a_{31}a_{41}). \\
v_{44} &= a_{11}a_{22}a_{33} - a_{21}^2a_{33} - a_{22}a_{31}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
B_A = b_A^t A^{-1} b_A = \frac{1}{U_A(\beta_1+\beta_2)^4} &\Bigg[\beta_2\Big(\beta_2\Big(a_{22}a_{33}a_{44} - a_{22}a_{43}^2 - a_{33}a_{42}^2\Big) \\
&+\beta_1\Big(a_{21}a_{33}a_{44} - a_{21}a_{43}^2 + a_{31}a_{42}a_{43} - a_{33}a_{41}a_{42}\Big)\Big)\Bigg] \\
&-\beta_1\Bigg[-\beta_2\Big(a_{21}a_{33}a_{44} - a_{21}a_{43}^2 + a_{31}a_{42}a_{43} - a_{33}a_{41}a_{42}\Big) \\
&-\beta_1\Big(a_{11}a_{33}a_{44} - a_{11}a_{43}^2 - a_{31}^2a_{44} + 2a_{31}a_{41}a_{43} - a_{33}a_{41}^2\Big)\Bigg].
\end{aligned}$$

The proof is now completed. □

It should be noted that Theorem 2 can be used to construct the asymptotic confidence intervals for R. However, the variance B_A is not known and it has to be estimated. The estimator of B_A, say $\hat{B}_A$, is obtained by replacing β_1, β_2, α and δ involved in B_A by their corresponding MLEs. The $100(1-\gamma)\%$ confidence intervals for R are given by

$$\left(\hat{R} - Z_{1-\gamma/2}\,\frac{\sqrt{\hat{B}_A}}{\sqrt{n}},\ \hat{R} + Z_{1-\gamma/2}\,\frac{\sqrt{\hat{B}_A}}{\sqrt{n}}\right), \tag{2.8}$$

where Z_γ is the $\gamma\%$ percentile of $N(0,1)$.

As mentioned in Asgharzadeh et al. [14], instead of approximating $\left(\hat{R}-R\right)/\sqrt{var(\hat{R})}$ as standard normal variable, it is possible to consider some other normalizing transformation, say $g(R)$, of R and assume that

$$\left(g(\hat{R}) - g(R)\right)/\sqrt{var[g(\hat{R})]} \sim N(0,1),$$

Now we use delta method to approximate the variance of $g(R)$, see for example Held and Bove [15]. It is written as follows

$$var\left[g(\hat{R})\right] = \left[g'(R)\right]^2 var(\hat{R}) = \left[g'(R)\right]^2 B_A/n.$$

As before, the $100(1-\gamma)\%$ confidence intervals for $g(R)$ are given as

$$g(\hat{R}) \pm Z_{1-\gamma/2}\sqrt{var\left[g(\hat{R})\right]}.$$

If the function $g(R)$ is strictly increasing then approximate $100(1-\gamma)\%$ confidence intervals for R are derived to be

$$\left(g^{-1}\left(g(\hat{R}) - Z_{1-\gamma/2}\sqrt{var(\hat{R})}\right), g^{-1}\left(g(\hat{R}) + Z_{1-\gamma/2}\sqrt{var(\hat{R})}\right)\right).$$

Specifically, we follow Asgharzadeh et al. [14] and consider the following two transformations.

1. Logit transformation: $g(R) = \ln\left(\frac{R}{1-R}\right)$ with $g'(R) = \frac{1}{R(1-R)}$
2. Arcsine transformation: $g(R) = \sin^{-1}(\sqrt{R})$ with $g'(R) = \frac{1}{2\sqrt{R(1-R)}}$

More details about these transformations one can refer to Lawless [16] or Mukherjee and Maiti [17].

2.3 Bootstrap Confidence Intervals

For small sample sizes, confidence intervals carried out based on the asymptotic results are usually expected not to perform well. Therefore, we propose to use confidence intervals based on two parametric bootstrap methods. These methods are: (i) the percentile bootstrap method, shortened as Boot-p, based on the idea of Hall [18]. (ii) the bootstrap-t method, shortened as Boot-t, based on the idea of Hall [18]. The algorithms for estimating the confidence intervals of R using both methods are summarized below.

(i) Boot-p method
1. Use the samples $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ and equations (2.4), (2.5) and (2.6) to compute the estimates $\hat{\beta}_{1ML}$, $\hat{\beta}_{2ML}$, $\hat{\alpha}_{ML}$ and $\hat{\delta}_{ML}$.
2. Use the estimates $\hat{\beta}_{1ML}$, $\hat{\alpha}_{ML}$ and $\hat{\delta}_{ML}$ to generate a bootstrap sample $\{x_1{}^*, \cdots, x_n{}^*\}$ and similarly use $\hat{\beta}_{2ML}$, $\hat{\alpha}_{ML}$ and $\hat{\delta}_{ML}$ to generate a bootstrap sample $\{y_1{}^*, \cdots, y_n{}^*\}$. Based on $\{x_1{}^*, \cdots, x_n{}^*\}$ and $\{y_1{}^*, \cdots, y_n{}^*\}$ compute the bootstrap estimate of R, say R^*, using $\hat{R}^* = \frac{\hat{\beta}_{1ML}}{\hat{\beta}_{1ML}+\hat{\beta}_{2ML}}$.
3. Repeat Step 2, NBOOT times.
4. Let $g_1(x) = P(\hat{R}^* \le x)$ be the cdf of $\hat{R}^*$ and define $\hat{R}_{Bp}(x) = g_1^{-1}(x)$ for a given x. Then the approximate $100(1-\gamma)\%$ confidence intervals for R are given by

$$\left(\hat{R}_{Bp}\left(\frac{\gamma}{2}\right), \hat{R}_{Bp}\left(1-\frac{\gamma}{2}\right)\right).$$

(ii) Boot-t method
1. Use the samples $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ and equations (2.4), (2.5) and (2.6) to compute $\hat{\beta}_{1ML}$, $\hat{\beta}_{2ML}$, $\hat{\alpha}_{ML}$ and $\hat{\delta}_{ML}$.
2. Use $\hat{\beta}_{1ML}$, $\hat{\alpha}_{ML}$ and $\hat{\delta}_{ML}$ to generate a bootstrap sample $\{x_1{}^*, \cdots, x_n{}^*\}$ and similarly use $\hat{\beta}_{2ML}$, $\hat{\alpha}_{ML}$ and $\hat{\delta}_{ML}$ to generate a bootstrap sample $\{y_1{}^*, \cdots, y_n{}^*\}$. Based on $\{x_1{}^*, \cdots, x_n{}^*\}$ and $\{y_1{}^*, \cdots, y_n{}^*\}$ compute the bootstrap estimate of R, say R^*, using $\hat{R}^* = \frac{\hat{\beta}_{1ML}}{\hat{\beta}_{1ML}+\hat{\beta}_{2ML}}$ and the statistic

$$T^* = \frac{\sqrt{n}(\hat{R}^* - \hat{R})}{\sqrt{var(\hat{R}^*)}}.$$

3. Repeat Step 2, NBOOT times.
4. Let $g_2(x) = P(T^* \le x)$ be the cdf of T^* and define $\hat{R}_{Bt}(x) = \hat{R} + g_2^{-1}(x)\sqrt{\frac{var(\hat{R})}{n}}$ for a given x. Then the approximate $100(1-\gamma)\%$ confidence intervals for R are given by

$$\left(\hat{R}_{Bt}\left(\frac{\gamma}{2}\right), \hat{R}_{Bt}\left(1-\frac{\gamma}{2}\right)\right).$$

2.4 Bayes Estimation of R

In this subsection, we discuss the estimation of the parameter R using Bayes method and assuming that the shape parameters β_1, β_2, δ and the scale parameter α are random variables and unknown to us. It is assumed that β_1, β_2, δ and α have density functions Gamma(a_1, b_1), Gamma(a_2, b_2), Gamma(a_3, b_3) and Gamma(a_4, b_4) respectively. Moreover, it is assumed that β_1, β_2, δ and α are independent. Based on the above assumptions, we obtain the likelihood function of the observed data as follows

$$L(data|\beta_1, \beta_2, \alpha, \delta) = \beta_1^n \alpha^{n+m} \delta^{n+m} \sum_{i=1}^{n} x_i^{-(\delta+1)} s_1(x, \alpha, \delta) \sum_{j=1}^{m} y_j^{-(\delta+1)} s_2(y, \alpha, \delta), \tag{2.9}$$

where $s_1(x, \alpha, \delta)$ and $s_2(y, \alpha, \delta)$ are defined as before in equations (2.2) and (2.3). The joint density of data β_1, β_2, α and δ can be obtained as

$$L(data, \beta_1, \beta_2, \alpha, \delta) = L(data; \beta_1, \beta_2, \alpha, \delta)\, \pi_1(\beta_1)\, \pi_2(\beta_2)\, \pi_3(\alpha)\, \pi_4(\delta), \tag{2.10}$$

where $\pi_1(\beta_1)$, $\pi_2(\beta_2)$, $\pi_3(\alpha)$ and $\pi_4(\delta)$ are gamma prior densities for β_1, β_2, α and δ respectively. Therefore, the joint posterior density of β_1, β_2, α and δ given the data is

$$L(data|\beta_1, \beta_2, \alpha, \delta) = \frac{L(data, \beta_1, \beta_2, \alpha, \delta)}{\int_\infty^0 \int_\infty^0 \int_\infty^0 \int_\infty^0 L(data, \beta_1, \beta_2, \alpha, \delta) d\delta d\alpha d\beta_2 d\beta_1}. \tag{2.11}$$

Equation (2.11) cannot be written in closed form, thus we apply the Gibbs sampling technique to compute the Bayes estimate of R along with the corresponding credible intervals. The posterior pdfs of β_1, β_2, α and δ can be obtained readily as follows.

$$\beta_1|\beta_2, \delta, \alpha, data \sim Gamma(n + a_1, b_1 + s_1(x, \alpha, \delta)),$$
$$\beta_2|\beta_1, \delta, \alpha, data \sim Gamma(m + a_2, b_2 + s_2(y, \alpha, \delta)),$$

$$f(\alpha|\beta_1, \beta_2, \delta, data) \propto \alpha^{n+m+a_3-1} exp\Bigg[-\alpha \left(b_3 + \sum_{i=1}^{n}(x_i) + \sum_{j=1}^{n}(y_j) \right) \\ -(\beta_1 + 1)s_1(x, \alpha, \delta) - (\beta_2 + 1)s_2(y, \alpha, \delta) \Bigg],$$

and

$$f(\delta|\beta_1, \beta_2, \alpha, data) \propto \delta^{n+m+a_4-1} exp\Big[-\delta b_4 - (\beta_1 + 1)s_1(x, \alpha, \delta) - (\beta_2 + 1)s_2(y, \alpha, \delta) \Big].$$

Clearly, the above forms of the posterior density do not lead to explicit Bayes estimates of the model parameters. For this reason, we prefer to use the Metropolis-Hasting method with normal proposal distribution. The algorithm of Gibbs sampling is summarized below:

1. Start with an initial guess $\left(\beta_1^{(0)}, \beta_2^{(0)}, \alpha^{(0)}, \delta^{(0)}\right)$.
2. Set $t = 1$.
3. Generate $\beta_1^{(t)}$ from Gamma $(n + a_1, b_1 + s_1(x, \alpha^{(t-1)}, \delta^{(t-1)}))$.
4. Generate $\beta_2^{(t)}$ from Gamma $(n + a_2, b_2 + s_2(y, \alpha^{(t-1)}, \delta^{(t-1)}))$.
5. Use the Metropolis-Hasting method to generate $\alpha^{(t)}$ from $f(\alpha^{(t-1)}|\beta_1, \beta_2, \delta, data)$ with the $N(\alpha^{(t-1)}, 0.5)$ proposal distribution.
6. Use the Metropolis-Hasting method to generate $\delta^{(t)}$ from $f(\delta^{(t-1)}|\beta_1, \beta_2, \alpha, data)$ with the $N(\delta^{(t-1)}, 0.5)$ proposal distribution.
7. Compute $R^{(t)}$ from equation (2.1).
8. Set $t = t + 1$.

9. Repeat Steps 3-8, T times.

Now the approximate posterior mean and variance of R, respectively, are calculated as

$$\hat{E}(R|data) = \frac{1}{T}\sum_{t=1}^{T} R^{(t)}, \tag{2.12}$$

and

$$v\hat{a}r\,(R|data) = \frac{1}{T}\sum_{t=1}^{T}\left(R^{(t)} - \hat{E}(R|data)\right)^2. \tag{2.13}$$

Using the result in Chen and Shao [19], we construct the $100(1-\gamma)\%$ highest posterior density (HPD) credible intervals as

$$\left(R_{[\frac{\gamma}{2}T]}, R_{[(1-\frac{\gamma}{2})T]}\right),$$

where $R_{[\frac{\gamma}{2}T]}$ and $R_{[(1-\frac{\gamma}{2})T]}$ are the $\frac{\gamma}{2}T$-th smallest integer and the $(1-\frac{\gamma}{2})T$-th smallest integer of $\{R_t, t = 1, 2, ..., T\}$, respectively.

It is worthy to point out that the Metropolis algorithm adopts only symmetric proposal distributions. Therefore the normal distribution is appropriate. It is checked here that the normal proposal distribution with variance $\sigma^2 = 0.5$ is best for the rapid convergence of the Metropolis algorithm.

2.5 Numerical Simulations

The comparisons between the the MLEs and Bayes estimators of R cannot be done theoretically. Thus, we present some simulations to compare the performance of the obtained results. We compare the MLEs and Bayes estimators in terms of their biases and mean squared errors (MSE). We also compare different confidence intervals, namely; the confidence intervals obtained by using asymptotic distribution of the MLEs, bootstrap confidence intervals and the HPD credible intervals in terms of the average confidence lengths.

The Bayes estimates are computed under the squared error loss function. To assess the performance of the methods used in estimation, we use different parameter values, different hyper-parameters and different sample sizes. Further, we assume two priors to obtain the Bayes estimators and HPD credible intervals. These priors are typical and given as: Prior 1: $a_j = 0.0001$, $b_j = 0.0001$, where $j = 1, 2, 3, 4$, and Prior 2: $a_j = 1$, $b_j = 3$, where $j = 1, 2, 3, 4$. In Table 1, we present the average biases and MSEs of the MLEs and Bayes estimators based on 1000 replications. The results are given under different sample sizes. The average confidence/credible lengths and the corresponding coverage percentages are given in Table 2. It should be noted that for the two bootstrap methods, we compute the confidence intervals based on 1000 bootstrap iterations. The Bayes estimates and the corresponding credible intervals are based on $T = 1000$ samples. Also, the confidence level, γ used in finding the confidence intervals or the credible intervals is 0.95.

It is observed that the MLE compares very well with the Bayes estimators in terms of biases and MSEs, see Table 1. Also, it is observed that the Bayes estimators based on Prior 2 perform better than those obtained based on Prior 1. Table 2 shows that, the confidence intervals based on the asymptotic distributions of the MLEs work well when n and m are getting larger, the Boot-p confidence intervals perform better than the Boot-t confidence intervals, the bootstrap method provide the smallest average lengths, the HPD credible intervals are wider than the other confidence intervals.

Table 1. Biases and MSEs of MLE and Bayes estimators of R for different values of parameters

			BS			BS	
(n,m)	Method	MLE	prior1	prior2	MLE	prior1	prior2
		$\beta_1=1.5,\beta_2=2,\alpha=\delta=1$			$\beta_1=2,\beta_2=1.5,\alpha=\delta=1$		
(15,15)	MSE	0.0106	0.0161	0.0161	0.0108	0.0176	0.0173
	Bias	-0.0054	0.5012	0.5011	-0.0030	0.5489	0.5410
(15,25)	MSE	0.0089	0.0160	0.0156	0.0087	0.0175	0.0170
	Bias	-0.0023	0.4980	0.4877	-0.0009	0.5462	0.5305
(15,50)	MSE	0.0083	0.0158	0.0152	0.0081	0.0173	0.0166
	Bias	-0.0017	0.4934	0.4768	-0.0002	0.5411	0.5213
(25,15)	MSE	0.0091	0.0158	0.0162	0.0091	0.0173	0.0174
	Bias	-0.0156	0.4932	0.5048	-0.0133	0.5417	0.5452
(25,25)	MSE	0.0071	0.0157	0.0157	0.0070	0.0172	0.0170
	Bias	-0.0143	0.4908	0.4914	-0.0095	0.5383	0.5337
(25,50)	MSE	0.0065	0.0157	0.0154	0.0063	0.0172	0.0168
	Bias	-0.0076	0.4926	0.4846	-0.0051	0.5394	0.5285
(50,15)	MSE	-0.0081	0.0161	0.0166	0.0082	0.0176	0.0179
	Bias	-0.0046	0.5029	0.5214	-0.0020	0.5516	0.5619
(50,25)	MSE	0.0067	0.0160	0.0162	0.0066	0.0174	0.0175
	Bias	-0.0015	0.5013	0.5096	0.0001	0.5484	0.5516
(50,50)	MSE	0.0056	0.0159	0.0159	0.0055	0.0174	0.0173
	Bias	0.0012	0.5002	0.5000	0.0026	0.5478	0.5448

Table 2. Average confidence credible length and the coverage percentage

		MLEs			Boot		BS	
(n,m)		Untransformed	Logit	Arcsin	boot-p	boot-t	prior1	prior2
					$\beta_1=1.5,\ \beta_2=2,\ \alpha=\delta=1$			
(15,15)	Len	0.6355	0.5421	0.4353	0.2291	0.2487	0.3482	0.3340
	CP	0.9930	0.9830	0.9610	0.8570	0.8700	0.9780	0.9920
(15,25)	Len	0.6052	0.5279	0.3779	0.1951	0.1966	0.3140	0.3045
	CP	0.9990	0.9960	0.9660	0.9470	0.9480	0.9840	0.9920
(15,50)	Len	0.6329	0.5516	0.3594	0.1839	0.1843	0.2847	0.2881
	CP	0.9990	0.9990	0.9780	0.9250	0.9260	0.9820	0.9920
(25,15)	Len	0.5097	0.4543	0.3194	0.2121	0.2159	0.3160	0.3071
	CP	0.9940	0.9790	0.9250	0.5550	0.5630	0.9940	0.9940
(25,25)	Len	0.4923	0.4454	0.2705	0.1774	0.1806	0.2752	0.2658
	CP	0.9960	0.9940	0.9280	0.7560	0.7610	0.9900	0.9920
(25,50)	Len	0.4627	0.4258	0.2225	0.1444	0.1465	0.2401	0.2380
	CP	0.9990	0.9990	0.9190	0.5730	0.5790	0.9780	0.9900
(50,15)	Len	0.3624	0.3354	0.2074	0.1923	0.1990	0.2893	0.2924
	CP	0.9590	0.9500	0.7960	0.7810	0.7890	0.9860	0.9820
(50,25)	Len	0.3626	0.3400	0.1744	0.1601	0.1649	0.2426	0.2387
	CP	0.9880	0.9840	0.8070	0.6330	0.6450	0.9860	0.9820
(50,50)	Len	0.3481	0.3300	0.1376	0.1255	0.1257	0.2007	0.1940
	CP	0.9990	0.9990	0.8110	0.9180	0.9190	0.9860	0.9860
					$\beta_1=2,\ \beta_2=1.5,\ \alpha=\delta=1$			
(15,15)	Len	0.6673	0.5629	0.4549	0.2405	0.2494	0.3476	0.3338
	CP	0.9960	0.9890	0.9710	0.7630	0.7720	0.9780	0.9860
(15,25)	Len	0.5528	0.4864	0.3451	0.2301	0.2305	0.3163	0.3074
	CP	0.9970	0.9890	0.9360	0.9650	0.9660	0.9900	0.9920
(15,50)	Len	0.4006	0.3689	0.2292	0.2125	0.2130	0.2888	0.2929
	CP	0.9850	0.9680	0.8430	0.9520	0.9530	0.9800	0.9780
(25,15)	Len	0.5920	0.5186	0.3702	0.1909	0.1920	0.3142	0.3045
	CP	0.9980	0.9970	0.9610	0.5100	0.5130	0.9900	0.9980
(25,25)	Len	0.5169	0.4653	0.2839	0.1863	0.1868	0.2756	0.2658
	CP	0.9980	0.9950	0.9300	0.8640	0.8650	0.9880	0.9900
(25,50)	Len	0.3960	0.3694	0.1905	0.1748	0.1752	0.2435	0.2389
	CP	0.9970	0.9940	0.8500	0.7810	0.7820	0.9880	0.9860
(50,15)	Len	0.4595	0.4213	0.2637	0.1335	0.1346	0.2860	0.2887
	CP	0.9960	0.9940	0.9040	0.8340	0.8360	0.9880	0.9960
(50,25)	Len	0.4335	0.4023	0.2087	0.1353	0.1356	0.2407	0.2387
	CP	0.9990	0.9970	0.8970	0.6420	0.6430	0.9860	0.9920
(50,50)	Len	0.3655	0.3460	0.1446	0.1317	0.1320	0.2014	0.1941
	CP	0.9990	0.9990	0.8210	0.9180	0.9190	0.9760	0.9800

3 Multicomponent Stress-Strength Model

In this section, we study the multicomponent stress-strength reliability for Dagum distribution when both stress and strength variates follow the same population. We also obtain the asymptotic confidence intervals for the multicomponent stress-strength reliability using MLE. We run a simulation study to compare the estimators.

Let the random samples $Y, X_1, X_2, \cdots, X_k$ be independent, $G(y)$ be the continuous distribution function of Y and $F(x)$ be the common continuous distribution function of $X_1, X_2, \cdots, X_k$. The reliability in a multicomponent stress-strength model developed by Bhattacharyya and Johnson [20] is

$$\begin{aligned} R_{s,k} &= P\left(\text{at least } s \text{ of the} \left(x_1, x_2, \cdots, x_k\right) \text{exceed} \, Y\right). \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1-G(y)]^i [G(y)]^{k-i} dF(y). \end{aligned} \tag{3.1}$$

3.1 Maximum Likelihood Estimator of $R_{s,k}$

Let $X \sim Dagum(\beta_1, \alpha, \delta)$ and $Y \sim Dagum(\beta_2, \alpha, \delta)$, where X and Y are two independent random variables with unknown shape parameters β_1 and β_2 and common scale parameters δ and shape parameter α. The reliability in multicomponent stress-strength for Dagum distribution using (1.1) results in:

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1-(\alpha y^{-\delta})^{-\beta_1}]^i [(1+\alpha t^{-\delta})^{-\beta_1}]^{k-i} \beta_2 \alpha \delta t^{-\delta-1} (1+\alpha t^{-\delta})^{-\beta_2-1} dt. \\ &= v \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} (1-t)^i t^{k-i+v-1} dt. \\ &= v \sum_{i=s}^{k} \binom{k}{i} B(k+v-i, i+1). \end{aligned} \tag{3.2}$$

where $t = (1+\alpha y^{-\delta})^{-\beta_1}$, $v = \beta_2/\beta_1$ and $B(\cdot, \cdot)$ is the incomplete beta function. After some simplification equation (3.2) is reduced to

$$R_{s,k} = v \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=i}^{k} (k+v-j) \right]^{-1}. \tag{3.3}$$

The MLE of $R_{s,k}$ becomes

$$\hat{R}_{s,k} = \hat{v} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=i}^{k} (k+\hat{v}-j) \right]^{-1}, \quad \text{where } \hat{v} = \frac{\hat{\beta}_1}{\hat{\beta}_2}. \tag{3.4}$$

To obtain the asymptotic confidence intervals for $R_{s,k}$, we proceed as follows: The asymptotic variances (AV) of the $\hat{\beta}_1$ and $\hat{\beta}_2$ are given by

$$AV(\hat{\beta}_1) = \left[E\left(\frac{\partial^2 L}{\partial \beta_1} \right) \right]^{-1} = \frac{\beta_1^2}{n}, \text{ and } AV(\hat{\beta}_2) = \left[E\left(\frac{\partial^2 L}{\partial \beta_2} \right) \right]^{-1} = \frac{\beta_2^2}{m}.$$

The asymptotic variance of an estimate of $R_{s,k}$ is given by (For details see Rao [21]):

$$AV(\hat{R}_{s,k}) = AV(\hat{\beta}_1) \left(\frac{\partial R_{s,k}}{\partial \beta_1} \right)^2 + AV(\hat{\beta}_2) \left(\frac{\partial R_{s,k}}{\partial \beta_2} \right)^2. \tag{3.5}$$

Thus from equation (3.5), the asymptotic variance of $\hat{R}_{s,k}$ can be obtained. To avoid the difficulty of derivation of $R_{s,k}$, we obtain $\hat{R}_{s,k}$ and their derivatives for $(s, k) = (1, 3)$ and $(2, 4)$ separately. They are given as

$$\hat{R}_{1,3} = \frac{3}{(\hat{v}+3)}, \quad \hat{R}_{2,4} = \frac{12}{(\hat{v}+4)(\hat{v}+3)},$$
$$\frac{\partial \hat{R}_{1,3}}{\partial \beta_1} = \frac{1}{\beta_1}\left[\frac{3\hat{v}}{(\hat{v}+3)^2}\right], \quad \frac{\partial \hat{R}_{1,3}}{\partial \beta_2} = -\frac{1}{\beta_1}\left[\frac{3}{(\hat{v}+3)^2}\right],$$
$$\frac{\partial \hat{R}_{2,4}}{\partial \beta_1} = \frac{1}{\beta_1}\left[\frac{12\hat{v}[2\hat{v}+7]}{(\hat{v}+4)(\hat{v}+3)^2}\right], \quad \text{and} \quad \frac{\partial \hat{R}_{2,4}}{\partial \beta_2} = -\frac{1}{\beta_1}\left[\frac{12[2\hat{v}+7]}{(\hat{v}+4)(\hat{v}+3)^2}\right].$$

Therefore,

$$AV(\hat{R}_{1,3}) = \frac{9\hat{v}^2}{(\hat{v}+3)^4}\left(\frac{1}{n}+\frac{1}{m}\right),$$

and

$$AV(\hat{R}_{2,4}) = \frac{144\hat{v}^2[2\hat{v}+7]^2}{[(\hat{v}+4)(\hat{v}+3)]^4}\left(\frac{1}{n}+\frac{1}{m}\right).$$

As $n \to \infty$ and $m \to \infty$, we have $(R_{s,k} - \hat{R}_{s,k})/AV(\hat{R}_{s,k}) \to N(0,1)$ for $s,k = 1,2,\cdots$. The asymptotic 95% confidence interval (CI) of system reliability $R_{s,k}$ is

$$\hat{R}_{s,k} \mp 1.96\sqrt{AV(\hat{R}_{s,k})}.$$

Thus, the asymptotic 95% confidence intervals for $R_{1,3}$ and $R_{2,4}$ are, respectively, given by

$$\hat{R}_{1,3} \mp 1.96\frac{3\hat{v}}{(\hat{v}+3)^2}\sqrt{\frac{1}{n}+\frac{1}{m}},$$

and

$$\hat{R}_{2,4} \mp 1.96\frac{12\hat{v}[2\hat{v}+7]}{[(\hat{v}+4)(\hat{v}+3)]^2}\sqrt{\frac{1}{n}+\frac{1}{m}}.$$

3.2 Simulation Study

We generate 3000 random samples of size 10, 15, 20, 25, 30 each from stress and strength populations for $(\beta_1, \beta_2) = (3.0, 1.5), (2.5, 1.5), (2.0, 1.5), (1.5, 1.5), (1.5, 2.0), (1.5, 2.5)$ and (1.5,3.0) as proposed by Bhattacharyya and Johnson [20]. The MLEs of β, α and δ, say $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\delta}$ are estimated by solutions of the nonlinear equation. These ML estimators of β_1 and β_2 are then substituted in v to get the multicomponent reliability for $(s,k) = (1,3), (2,4)$. The average bias and average MSE of the reliability estimates over the 3000 replications are given in Table 3. Average confidence length of the simulated 95% confidence intervals of, $R_{s,k}$ are given in Table 3. The true values of reliability in multicomponent stress-strength with the given combinations for $(s,k) = (1,3)$ are 0.857, 0.833, 0.800, 0.750, 0.692, 0.643, 0.600 and for $(s,k) = (2,4)$ are 0.762, 0.725, 0.674, 0.600, 0.519, 0.454, 0.400. Thus the true value of reliability in multicomponent stress-strength decreases as β_2 increases for a fixed β_1 whereas reliability in multicomponent stress-strength increases as β_1 increases for a fixed β_2 in both cases of (s,k). Therefore, the true value of reliability decreases as v increases and vice versa. The average bias and average MSE decrease as sample size increases for both situations of (s,k). Also, it is noted that the bias is negative and relatively small in all the combinations of the parameters in both situations of (s,k) which leads MLE to underestimate the parameters, and thus the $R_{s,k}$. This generally proves the consistency property of the MLE of $R_{s,k}$. Whereas, among the parameters the absolute bias and MSE decrease as β_1 increases for a fixed β_2 in both cases of (s,k). The absolute bias and MSE increase as β_2 increases for a fixed β_1 in both cases of (s,k). It is also clear that as the sample size increases the length of CI decreases in all cases.

Table 3. In each cell the first row represents the average bias, the second row represents the average MSE and the third row represents the average confidence length of the simulated 95% confidence intervals of $R_{s,k}$ using MLE

(s,k)	(n,m)	(β_1,β_2) (3.0,1.5)	(2.5,1.5)	(2.0,1.5)	(1.5,1.5)	(1.5,2.0)	(1.5,2.5)	(1.5,3.0)
	(10,10)	-0.0100	-0.0106	-0.0119	-0.0143	-0.0171	-0.0177	-0.0176
		0.0043	0.0053	0.0069	0.0094	0.0124	0.0147	0.0167
		0.2197	0.2466	0.2809	0.3247	0.3633	0.3856	0.3976
	(15,15)	-0.0039	-0.0045	-0.0056	-0.0080	-0.0108	-0.0121	-0.0122
		0.0027	0.0033	0.0044	0.0060	0.0080	0.0094	0.0106
		0.1754	0.1983	0.2276	0.2655	0.2994	0.3200	0.3318
(1,3)	(20,20)	-0.0011	-0.0015	-0.0025	-0.0045	-0.0071	-0.0091	-0.0097
		0.0019	0.0024	0.0031	0.0043	0.0057	0.0069	0.0079
		0.1504	0.1705	0.1964	0.2299	0.2604	0.2792	0.2901
	(25,25)	-0.0003	-0.0009	-0.0018	-0.0037	-0.0061	-0.0081	-0.0092
		0.0015	0.0018	0.0024	0.0033	0.0044	0.0053	0.0061
		0.1344	0.1526	0.1760	0.2063	0.2339	0.2512	0.2614
	(30,30)	-0.0007	-0.0011	-0.0021	-0.0042	-0.0066	-0.0086	-0.0097
		0.0013	0.0016	0.0020	0.0028	0.0037	0.0044	0.0050
		0.1232	0.1398	0.1612	0.1891	0.2145	0.2304	0.2398
	(10,10)	-0.0131	-0.0129	-0.0133	-0.0139	-0.0141	-0.0114	-0.0081
		0.0101	0.0117	0.0143	0.0173	0.0196	0.0205	0.0208
		0.3376	0.3713	0.4098	0.4506	0.4742	0.4765	0.4675
	(15,15)	-0.0044	-0.0048	-0.0055	-0.0073	-0.0088	-0.0083	-0.0064
		0.0065	0.0076	0.0093	0.0115	0.0132	0.0139	0.0140
		0.2728	0.3022	0.3363	0.3735	0.3962	0.4004	0.3945
(2,4)	(20,20)	-0.0006	-0.0009	-0.0018	-0.0036	-0.0055	-0.0064	-0.0056
		0.0047	0.0055	0.0068	0.0085	0.0099	0.0106	0.0107
		0.2353	0.2614	0.2921	0.3258	0.3470	0.3516	0.3468
	(25,25)	-0.0004	-0.0002	-0.0012	-0.0031	-0.0051	-0.0064	-0.0065
		0.0036	0.0048	0.0053	0.0066	0.0077	0.0082	0.0084
		0.2108	0.2345	0.2626	0.2934	0.3130	0.3175	0.3132
	(30,30)	-0.0003	-0.0008	-0.0019	-0.0042	-0.0063	-0.0077	-0.0079
		0.0031	0.0037	0.0045	0.0055	0.0064	0.0069	0.0070
		0.1933	0.2150	0.2407	0.2692	0.2872	0.2914	0.2876

4 Concluding Remarks

In this paper, we have considered the problem of estimation of a single and multicomponent stress-strength reliability for a three-parameter Dagum distribution when both stress and strength variates follow the same distribution. The maximum likelihood and Bayes estimators of the stress-strength parameter, R have been derived. We have compared the MLEs and Bayes estimators in terms of their biases and mean squared errors. It may be noted, from Table 1 that the maximum likelihood estimates have the smallest mean squared errors as compared with their corresponding Bayes estimates. We observed the Bayesian method is sensitive to the choice of the priors. Generally the Bayes estimators based on Prior 2 perform better than the Bayes estimators based on Prior 1. For all sample sizes (n, m), the bootstrap methods provide the smallest average lengths. It is observed that boot-p confidence intervals perform better than those obtained by the boot-t methods. Further, the coverage probability is quite close to the given value in all sets of parameters. For the multicomponent stress-strength reliability, we have estimated the asymptotic confidence intervals. The simulation result in Table 3 indicates that the averages of bias and MSE decrease as sample size increases. Also, the absolute bias and MSE decrease (increase) as β_1 increases (β_2 increases) in both situations of (s, k).

Acknowledgments.

The authors are grateful to the anonymous referees for their valuable comments and suggestions which improved the presentation of the paper. This study is a part of the Master Thesis of the second named author whose work was supervised by the first named author.

References

1. C. Dagum, A New Model for Personal Income Distribution: Specification and Estimation, *Economic Applique*, **33** (1977) 413–437.

2. F. Domma, L'andamento della Hazard function nel modello di Dagum a tre parametri, Quaderni di Statistica, 4 (2002), 103–114.
3. F. Domma S. Giordano and M. M. Zenga, Maximum likelihood estimation in Dagum distribution with censored sample, *Journal of Applied Statistics*, **38** (2011), 2971–2985.
4. F. Domma and F. Condino, The beta-Dagum distribution: definition and properties, *Communications in Statistics-Theory and Methods*, **42** (2013) 4070–4090.
5. C. Kleiber and S. Kotz, Statistical Size Distributions in Economics and Actuarial Sciences. (John Wiley, Hoboken, New Jersey, 2003).
6. C. Kleiber, A guide to the Dagum distributions. (Springer, New York, 2008).
7. M. N. Shahzad and Z. Asghar, Comparing TL-Moments, L-Moments and Conventional Moments of Dagum Distribution by Simulated data, *Revista Colombiana de Estadistica* **36** (2013) 79–93.
8. B. O. Oluyede and Y. Ye, Weighted Dagum and related distributions, *Afrika Matematika*, **25** (2014) 1125–1141.
9. S. Kotz, Y. Lumelskii, and M. Pensky, The Stress-Strength Model and its Generalizations-Theory and Applications. (World Scientific, New York, 2003)
10. M. Pandey and B. Uddin, . Estimation of reliability in multicomponent stress-strength model following Burr distribution. (Proceedings of the First Asian congress on Quality and Reliability, (pp. 307–312). New Delhi, India 1985)
11. R. G. Srinivasa and R. R. L. Kantam, Estimation of reliability in multicomponent stress-strength model: log-logistic distribution, *Electronic Journal of Applied Statistical Analysis*, **3** (2010) 75–84.
12. S. Nadarajah and S. Kotz, Moments of some J-shaped distributions, *Journal of Applied Statistics*, **30** (2003) 311–317.
13. G. Casella and B. L. Roger, Statistical inference. (Vol. 2. Pacific Grove, CA: Duxbury, 2002)
14. A. Asgharzadeh, R. Valiollahi and M. Z. Raqab, Estimation of the stress-strength reliability for the generalized logistic distribution, *Statistical Methodology* **15** (2013) 73–94.
15. L. Held and D. S Bove, Applied Statistical Inference: Likelihood and Bayes. (Springer Science & Business Media, 2013)
16. J. F. Lawless, Statistical Models and Methods for Lifetime Data, (Second ed., John Wiley and Sons, New York, 2003).
17. S. P. Mukherjee and S. S. Maiti, Stress-strength reliability in the Weibull case, *Frontiers in Reliability* **4** (1998) 231–248.
18. P. Hall, Theoretical comparison of bootstrap confidence intervals, *Annals of Statistics* **16** (1988) 927–953.
19. M. H. Chen, Q. M. Shao, Monte Carlo estimation of Bayesian credible and HPD intervals, *Journal of Computational and Graphical Statistics* **8** (1999) 69–92.
20. G. K. Bhattacharyya and R. A. Johnson, Estimation of reliability in multicomponent stress-strength model. *JASA*, **69** (1974) 966–970.
21. C. R. Rao, Linear Statistical Inference and its Applications, (Wiley Eastern Limited, India 1973)

