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Higher Arithmetic Sequence and Its Implicit Common Difference

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Abstract. The concept of *k*-order sequence of first order arithmetic sequence has been defined by mathematical induction based on finite difference theory. It has been proved this sequence is higher arithmetic sequence. Meanwhile the sum formula and the derivation of its implicit common difference have been given.

Finite Difference and Finite Difference Sequence

Definition 1. Let Z denote set of integers. Function $y_t = f(t)$ is defined belong to Z, then

$$\Delta y_t = y_{t+1} - y_t = f(t+1) - f(t) \tag{1}$$

is first finite difference of y_t when time is t, where Δ is finite difference operator. And

$$\Delta(\Delta y_t) = \Delta y_{t+1} - \Delta y_t = \Delta f(t+1) - \Delta f(t) = f(t+2) - 2f(t+1) + f(t) \tag{2}$$

is first finite difference of Δy_t when time is t, also second finite difference of y_t, which is noted by $\Delta^2 y_t$ [1, 2]. It is easy to verify that third order finite difference of y_t when time is t as

$$\Delta^3 y_t = \Delta^2 y_{t+1} - \Delta^2 y_t = y_{t+3} - 3y_{t+2} + 3y_{t+1} - y_t \tag{3}$$

Generally, k^{th} order finite difference is defined as

$$\Delta^k y_t = \Delta(\Delta^{k-1} y_t) = \Delta^{k-1} y_{t+1} - \Delta^{k-1} y_t = \sum_{i=0}^{k} (-1)^i C_k^i y_{t+k-i} \tag{4}$$

where $k \in N$.

Particularly, in case $k = 0$, it is obtained

$$\Delta^0 y_t = \sum_{i=0}^{0} (-1)^i C_0^i y_{t+0-i} = (-1)^0 C_0^0 y_t = y_t \tag{5}$$

Definition 2. Let N denote set of natural numbers, $y_t = f(t)$ is defined belong to N. If t runs nonnegative integers, it can be obtained $Y: \{y_t = f(t)\}_{t \in N}$. Then

$$Y_{\Delta^1} : \{\Delta y_t = \Delta f(t)\}_{t \in N} \tag{6}$$

is first finite difference sequence of Y [3]. And

$$Y_{\Delta^2} : \{\Delta^2 y_t = \Delta^2 f(t)\}_{t \in N} \tag{7}$$

is first finite difference sequence of Y_{Δ^1}, also second finite difference sequence of Y.

Similarly, it could be verified that third order finite difference sequence of Y:

$$Y_{\Delta^3} : \{\Delta^3 y_t = \Delta^3 f(t)\}_{t \in N} \tag{8}$$

Generally, k^{th} order finite difference sequence of Y is defined as

$$Y_{\Delta^k} : \{\Delta^k y_t = \Delta^k f(t)\}_{t \in N, k \in N} \tag{9}$$

It is easy to verify from definition 1 that the zero order finite difference sequence of Y is itself: $Y_{\Delta^0} = Y$.

Finite Deference Sequence of Arithmetic Sequence

Definition 3. If $\forall t$, $\exists Y_{\Delta^1}$ can satisfy $\Delta y_{t+i} = \Delta y_t \neq 0$, then Y is strict first arithmetic sequence at set of nonnegative integers and

$$Y_{\Delta^{1+i}} : \{\Delta^{1+i} y_t = \Delta^{1+i} f(t) = 0\}_{t \in N, i \in Z^+} \tag{10}$$

As for Y, the common difference of first arithmetic sequence is defined and noted as

$$d_{\Delta^1} = \Delta y_t \tag{11}$$

If $\forall t$, $\exists Y_{\Delta^2}$ can satisfy $\Delta^2 y_{t+i} = \Delta^2 y_t \neq 0$, then Y_{Δ^1} is strict first arithmetic sequence at set of nonnegative integers. In another word, Y is strict second arithmetic sequence at set of nonnegative integers and

$$Y_{\Delta^{2+i}} : \{\Delta^{2+i} y_t = \Delta^{2+i} f(t) = 0\}_{t \in N, i \in Z^+} \tag{12}$$

As for Y, the implicit common difference of second arithmetic sequence is defined and noted as[4]

$$d_{\Delta^2} = \Delta^2 y_t \tag{13}$$

Generally, if $\forall t$, $\exists Y_{\Delta^k}$ can satisfy $\Delta^k y_{t+i} = \Delta^k y_t \neq 0$, then Y is strict k^{th} order arithmetic sequence at set of nonnegative integers and

$$Y_{\Delta^{k+i}} : \left\{\Delta^{k+i} y_t = \Delta^{k+i} f(t) = 0\right\}_{t\in N, i\in Z^+} \tag{14}$$

As for Y, the implicit common difference of k^{th} order arithmetic sequence is defined and noted as[5]

$$d_{\Delta^k} = \Delta^k y_t \tag{15}$$

Implicit Common Difference and Sum of k^{th} Order Sequence for First Arithmetic Sequence

k^{th} Order Sequence of First Arithmetic Sequence. According to definition 3, $d_{\Delta^1} = \Delta y_t$. As for $Y : \{y_t\}_{t\in N}$, $\Delta y_t = y_{t+1} - y_t$. It can be obtained that:

$$y_{t+1} = y_t + \Delta y_t \tag{16}$$

$$\Delta y_{t+1} = \Delta y_t \tag{17}$$

Definition 4. If each member of Y^k is the power of the corresponding member of $Y : \{y_t\}_{t\in N}$, $Y^k : \{y_t^k\}_{t\in N, k\in N}$, then Y^k is defined as k^{th} order sequence of Y.

Implicit Common Difference of k^{th} Order Sequence for First Arithmetic Sequence. Let m denote the order of the sequence. In case $m = 1$, $Y^1 : \{y_t^1\}_{t\in N}$, $\Delta y_t^1 = y_{t+1}^1 - y_t^1$, substitute (16) into it and rearrange to obtain:

$$\Delta y_t^1 = (y_t + \Delta y_t)^1 - y_t^1 = C_1^0 y_t^1 + C_1^1 (\Delta y_t)^1 - y_t^1 = C_1^1 (\Delta y_t)^1 \tag{18}$$

From (18), to deduce that

$$\Delta y_{t+1}^1 = C_1^1 (\Delta y_{t+1})^1 \tag{19}$$

Substitute (17) into (19) and rearrange to obtain:

$$\Delta y_{t+1}^1 = C_1^1 (\Delta y_{t+1})^1 = \Delta y_t^1 \tag{20}$$

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It is proved that Y^1 is first arithmetic sequence. And its common difference is

$$d_1 = \Delta y_t^1 = C_1^1(\Delta y_t)^1 = 1!(\Delta y_t)^1 \tag{21}$$

In case $m = 2$, $Y^2 : \{y_t^2\}_{t \in N}$, $\Delta y_t^2 = y_{t+1}^2 - y_t^2$, substitute (16) into it and rearrange to obtain:

$$\Delta y_t^2 = (y_t + \Delta y_t)^2 - y_t^2 = C_2^0 y_t^2 + C_2^1 y_t(\Delta y_t) + C_2^2(\Delta y_t)^2 - y_t^2 = C_2^1 y_t(\Delta y_t) + C_2^2(\Delta y_t)^2 \tag{22}$$

It is deduced From (22) that

$$\Delta y_{t+1}^2 = C_2^1 y_{t+1}(\Delta y_{t+1}) + C_2^2(\Delta y_{t+1})^2 \tag{23}$$

Substitute (17) into (23) to obtain:

$$\Delta y_{t+1}^2 = C_2^1 y_{t+1}(\Delta y_t) + C_2^2(\Delta y_t)^2 \tag{24}$$

Let (24) subtract (22) to get:

$$\Delta y_t^2 = C_2^1 y_{t+1}(y_{t+1} - y_t)(\Delta y_t) = C_2^1(\Delta y_t)^2 \tag{25}$$

It can be deduced from (25)

$$\Delta^2 y_{t+1}^2 = C_2^1(\Delta y_{t+1})^2 \tag{26}$$

Substitute (17) into (26) and rearrange to get:

$$\Delta^2 y_{t+1}^2 = C_2^1(\Delta y_t)^2 = \Delta^2 y_t^2 \tag{27}$$

It is proved that Y^2 is second arithmetic sequence. And its implicit common difference is

$$d_2 = \Delta^2 y_t^2 = C_2^1(\Delta y_t)^2 = C_2^1(\Delta y_t)C_1^1(\Delta y_t)^1 = C_2^1(\Delta y_t)d_1 = 2!(\Delta y_t)^2 \tag{28}$$

In case $m = k - 1$,

$$d_{k-1} = \Delta^{k-1} y_t^{k-1} = C_{k-1}^1(\Delta y_t)d_{k-2} = (k-1)!(\Delta y_t)^{k-1} \tag{29}$$

Let j denote the order of finite difference. In case $m = k$ and $j = 1$,

$$\Delta y_t^k = y_{t+1}^k - y_t^k \tag{30}$$

Substitute (16) into (30) to obtain:

$$\Delta y_t^k = (y_t + \Delta y_t)^k - y_t^k = C_k^1 y_t^{k-1}(\Delta y_t) + C_k^2 y_t^{k-2}(\Delta y_t)^2 + \cdots + C_k^{k-1} y_t(\Delta y_t)^{k-1} + C_k^k(\Delta y_t)^k \tag{31}$$

It is deduced from (31):

$$\Delta y_{t+1}^k = C_k^1 y_{t+1}^{k-1}(\Delta y_{t+1}) + C_k^2 y_{t+1}^{k-2}(\Delta y_{t+1})^2 + \cdots + C_k^{k-1} y_{t+1}(\Delta y_{t+1})^{k-1} + C_k^k(\Delta y_{t+1})^k \tag{32}$$

Substitute (17) into (30) to get:

$$\Delta y_{t+1}^{k} = C_k^1 y_{t+1}^{k-1}(\Delta y_t) + C_k^2 y_{t+1}^{k-2}(\Delta y_t)^2 + \cdots + C_k^{k-1} y_{t+1}(\Delta y_t)^{k-1} + C_k^k (\Delta y_t)^k \quad (33)$$

Let (33) subtract (31) to get the difference relation when $j = 2$:

$$\Delta^2 y_t^k = C_k^1(\Delta y_t^{k-1})(\Delta y_t) + C_k^2(\Delta y_t^{k-2})(\Delta y_t)^2 + \cdots + C_k^{k-2}(\Delta y_t^2)(\Delta y_t)^{k-2} + C_k^{k-1}(\Delta y_t)^k \quad (34)$$

From (34), it can be obtained that

$$\Delta^2 y_{t+1}^k = C_k^1(\Delta y_{t+1}^{k-1})(\Delta y_{t+1}) + C_k^2(\Delta y_{t+1}^{k-2})(\Delta y_{t+1})^2 + \cdots + C_k^{k-2}(\Delta y_{t+1}^2)(\Delta y_{t+1})^{k-2} + C_k^{k-1}(\Delta y_{t+1})^k \quad (35)$$

Substitute (17) into (35) to get:

$$\Delta^2 y_{t+1}^k = C_k^1(\Delta y_{t+1}^{k-1})(\Delta y_t) + C_k^2(\Delta y_{t+1}^{k-2})(\Delta y_t)^2 + \cdots + C_k^{k-2}(\Delta y_{t+1}^2)(\Delta y_t)^{k-2} + C_k^{k-1}(\Delta y_t)^k \quad (36)$$

Let (36) subtract (34) to obtain the difference relation when $j = 3$:

$$\Delta^3 y_t^k = C_k^1(\Delta^2 y_t^{k-1})(\Delta y_t) + C_k^2(\Delta^2 y_t^{k-2})(\Delta y_t)^2 + \cdots + C_k^{k-3}(\Delta y_t^3)(\Delta y_t)^{k-3} + C_k^{k-2}(\Delta^2 y_t^2)(\Delta y_t)^{k-2} \quad (37)$$

From (37), it can be obtained that

$$\Delta^3 y_{t+1}^k = C_k^1(\Delta^2 y_{t+1}^{k-1})(\Delta y_{t+1}) + C_k^2(\Delta^2 y_{t+1}^{k-2})(\Delta y_{t+1})^2 + \cdots + C_k^{k-3}(\Delta^2 y_{t+1}^3)(\Delta y_{t+1})^{k-3} + C_k^{k-2}(\Delta^2 y_{t+1}^2)(\Delta y_{t+1})^{k-2} \quad (38)$$

Substitute (17) and (25), and rearranged to obtain:

$$\Delta^3 y_{t+1}^k = C_k^1(\Delta^2 y_{t+1}^{k-1})(\Delta y_t) + C_k^2(\Delta^2 y_{t+1}^{k-2})(\Delta y_t)^2 + \cdots + C_k^{k-3}(\Delta^2 y_{t+1}^3)(\Delta y_t)^{k-3} + C_k^{k-2}(\Delta^2 y_t^2)(\Delta y_t)^{k-2} \quad (39)$$

In case $j = k-1$,

$$\Delta^{k-1} y_t^k = C_k^1(\Delta^{k-2} y_t^{k-1})(\Delta y_t) + C_k^2(\Delta^{k-2} y_t^{k-2})(\Delta y_t)^2 \quad (40)$$

To deduce that

$$\Delta^{k-1} y_{t+1}^k = C_k^1(\Delta^{k-2} y_{t+1}^{k-1})(\Delta y_t) + C_k^2(\Delta^{k-2} y_{t+1}^{k-2})(\Delta y_t)^2 \quad (41)$$

Let (41) subtract (40) to get:

$$\begin{aligned}\Delta^k y_t^k &= C_k^1(\Delta^{k-2} y_{t+1}^{k-1} - \Delta^{k-2} y_t^{k-1})(\Delta y_t) \\ &= C_k^1(\Delta^{k-1} y_t^{k-1})(\Delta y_t) \\ &= C_k^1(\Delta y_t) d_{k-1} \\ &= C_k^1(\Delta y_t)(k-1)!(\Delta y_t)^{k-1} \\ &= k!(\Delta y_t)^k\end{aligned} \quad (42)$$

It can be obtained from (42):

$$\Delta^k y_{t+1}^k = k!(\Delta y_{t+1})^k \tag{43}$$

Substitute (17) into (43) and rearrange to get:

$$\Delta^k y_{t+1}^k = k!(\Delta y_t)^k = \Delta^k y_t^k \tag{44}$$

It is proved that Y^k is k^{th} order arithmetic sequence. And its implicit common difference is

$$d_k = \Delta^k y_t^k = k!(\Delta y_t)^k = k!(d_1)^k \tag{45}$$

Sum of k^{th} Order Sequence for First Arithmetic Sequence. As for first arithmetic sequence $Y:\{y_t\}_{t\in N}$, if its common difference is d_1, then

$$y_{t+i} = y_t + id_1 \tag{46}$$

where i is the difference between y_{t+i} to y_t.

In case $t=0$, y_0 is the first member of Y. It can be obtained:

$$\begin{cases} y_i = y_0 + id_1 \\ y_i^k = (y_0 + i\Delta y_t)^k \end{cases} \tag{47}$$

So the sum of $Y^k:\{y_t^k\}_{t\in N, k\in N}$ is:

$$S_n(Y^k) = \sum_{i=0}^{n} (y_0 + id_1)^k \tag{48}$$

Conclusion

The k^{th} order sequence of first arithmetic sequence is k^{th} order arithmetic sequence. The implicit common difference of higher arithmetic sequence is formulated. The relation between common difference of first arithmetic sequence and the implicit common difference of its k^{th} order sequence is formulated as $k!(d_1)^k$ where d_1 is common difference of first arithmetic sequence. The formula indicates the varying pattern of implicit common difference of higher arithmetic sequence and its degree of variation with order increasing.

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