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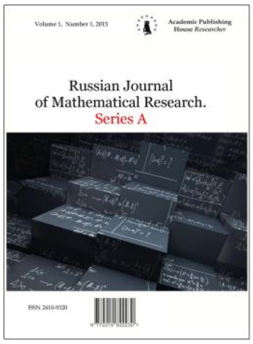

UDC 51

On Factorization of Fundamental Polynomials of Two and Three Variables

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Abstract

A set of knots is called n-independent if for arbitrary data at those knots, there is a (not necessary unique) polynomial of total degree at most n that matches the given information. For an arbitrary n-independent knot set X in R^3 we are interested with n-fundamental polynomials which have simplest possible form. In the present paper we bring necessary and sufficient conditions for the set X of cardinality not exceeding $3n+1$, such that all its knots have n-fundamental polynomials in form of products of linear factors. We bring also necessary and sufficient conditions for n-independence of non-coplanar knot sets in R^3 of the mentioned cardinality .

Keywords: Polynomial interpolation, fundamental polynomial, n-independent knots.

Introduction

Let Π_n^3 be the space of polynomials of three variables and total degree at most n:

$$\Pi_n^3 = \left\{ \sum_{i+j+k\leq n} a_{ijk} x^i y^j z^k : \ a_{ijk} \in \mathrm{R} \right\}.$$

We have that

$$N := \dim \Pi_n^3 = \binom{n+3}{3}.$$

In case of two variables the corresponding space we denote by Π_n.

Denote by $\Pi_n(L)$ the set of restrictions of polynomials Π_n^3 on a plane L in R^3. Notice that if the plane L is not perpendicular to the XY coordinate plane then we may assume that polynomial $p \in \Pi_n(L)$ is given by an equation $q(x,y) = 0$, where $q \in \Pi_n$. Indeed, in this case L is given by an equation $z = ax + by + c$ and we have for the restriction of $p \in \Pi_n^3$ on L: $p|_L = p(x,y,ax+by+c) =: q(x,y)$.

Consider a set of distinct knots (points) in R^3

$X_s = \{A_1, A_2, \dots, A_s\}$.

The problem of finding a polynomial $p \in \Pi_n^3$, which satisfies the conditions

$$p(A_i) = c_i, \ \ i = 1,2,\dots,s, \tag{1.1}$$

is called interpolation problem. A polynomial $p \in \Pi_n^3$ is called an n-fundamental polynomial for a knot $A_k \in X_s$, if

$p(A_i) = \delta_{ik}, \quad i = 1,2, \dots, s,$
where δ is the Kronecker symbol. We denote this fundamental polynomial by $p_k^* = p_{A,X_s}^* = p_{A,X_s,n}^*$. Sometimes we call fundamental also a polynomial that vanishes at all knots of X but one, since it is a nonzero constant times a fundamental polynomial.

Definition 1.1. *A set of knots X is called n-independent, if each its knot has a fundamental polynomial. Otherwise, X is called n-dependent.*

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence is $\#X \le N$. Having fundamental polynomials of all knots of X we get a solution of interpolation problem (1.1), by using the Lagrange formula:

$$p = \sum_{i=1}^{s} c_i p_i^*. \tag{1.2}$$

Thus we get that the knot set X is n -independent if and only if the interpolating problem (1.1) is solvable, meaning that for any data $\{c_1, c_2, \dots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n^3$, satisfying the conditions(1.1).

In view of the Lagrange formula (1.2), it is important to find the simplest possible fundamental polynomials, to which this paper is devoted. Namely, we are interested with fundamental polynomials that are products of linear factors.

Let us bring some results on n –independence of knot sets in the plane we shall use in the sequel. Let us start with the following simple but important result of Severi:

Theorem 1.2 ([3]). *Any set of knots X, with $\#X \le n + 1$, is n-independent.*

Indeed, for each knot $A \in X$ here we can find n -fundamental polynomial which is a product of k lines with, with $k = \#X - 1 \le n$. We require just that each of these lines passes through a knot of $X \setminus \{A\}$ and does not pass through A.

We shall use the same letter, say p, to denote the polynomial $p \in \Pi_n \setminus \Pi_0$ and the algebraic curve given by the equation $p(x, y) = 0$. We do the same for the polynomial $p \in \Pi_n^3 \setminus \Pi_0^3$ and algebraic surface $p(x, y, z) = 0$.

Next two results extend the Severi theorem to the cases of sets with no more than $2n + 1$ and $3n$ knots, respectively. To present them it is convenient to introduce the following three conditions.

Suppose that X is a knot set in $R^2$and $A \in X$ is a knot.

1) We say that X satisfies the condition $\langle n \rangle_1$ if no $n + 2$ knots of X are collinear.

We say that X satisfies the condition $\langle n \rangle_{1,A}$ if no $n + 1$ knots of $X \setminus \{A\}$are collinear together with A.

2) We say that X satisfies the condition $\langle n \rangle_2$ if no $2n + 2$ knots of X are lying in a conic (reducible or irreducible).

In the corresponding condition concerned with the knot A we distinguish reducible and irreducible conics. Namely, we say that X satisfies the condition $\langle n \rangle_{2,A}$ if the following two conditions hold:

i) no 2n+1 knots of $X \setminus \{A\}$ are lying on an irreducible conic together with A,

ii) if $n + 1$ knots of $X \setminus \{A\}$ are collinear and are lying in a line α then no n knots of $X \setminus \alpha$ are collinear together with A,

3) We say that X, satisfies the condition $\langle n \rangle_3$ if $\#X < 3n$ or $\#X = 3n$ and there are no curves $\gamma \in \Pi_3$ and $\sigma_n \in \Pi_n$, such that $\gamma \cap \sigma_n = X$.

If the knot set X is in R^3 (or in R^k)then we say that the above mentioned conditions are satisfied if they are satisfied for each knot set $X \cap L$, where L is any plane in R^3 .

Theorem 1.3 ([1]). *Any knot set X, with $\#X \le 2n + 1$, is n -independent, if and only if the condition $\langle n \rangle_2$ is satisfied.*

Next theorem concerns the n-independence of at most $3n$ knots.

Theorem 1.4 ([2]). *Let X be a set of knots in R^2, with $\#X \le 3n$. Then the set X is n-independent if and only if the conditions $\langle n \rangle_1, \langle n \rangle_2, \langle n \rangle_3$ hold.*

The following is a generalization of this result to the case of R^k.

Theorem 1.5 ([4]). *Let X be a set of knots in R^k, with $\#X \le 3n$. Then the set X is n-independent if and only if the conditions $\langle n \rangle_1, \langle n \rangle_2, \langle n \rangle_3$ hold.*

Next three well-known lemmas concern the factorization of polynomials vanishing at some points of lines, reducible and irreducible conics, respectively (see e.g., [2], Corollaries 3.3 and 3.4).

Lemma 1.6 ([2]). *Suppose that α is a line. Then for any polynomial $p \in \Pi_n$ vanishing at $n+1$ points of α we have that $p = \alpha q$, where $q \in \Pi_{n-1}$.*

Lemma 1.7 ([2]). *Suppose that α_i, $i = 1, 2$ are two lines. Then for any polynomial $p \in \Pi_n$ vanishing at $n+1$ points of α_1 and n points of $\alpha_2 \backslash \alpha_1$ we have that $p = \alpha_1 \alpha_2 q$, where $q \in \Pi_{n-2}$.*

Lemma 1.8 ([2]). *Suppose that β is an irreducible conic. Then for any polynomial $p \in \Pi_n$ vanishing at $2n+1$ points of β we have that $p = \beta q$, where $q \in \Pi_{n-2}$.*

Following theorem is a special case of the Cayley-Bacharach theorem (see e.g., [1]).

Theorem 1.9. ([1]). *Let $\gamma \in \Pi_3$ and $\sigma_n \in \Pi_n$ be curves such that $\gamma \cap \sigma_n = X$ and $\#X = 3n$. Then any curve of degree n containing all but one knot of X, contains all knots of X.*

Next result from [5] concerns the factorization of fundamental polynomials into linear factors.

Theorem 1.10 ([5]). *Let X be an n-independent set of knots with $\#X \leq 2n+1$. Then for each knot of X there is an n-fundamental polynomial, which is a product of lines. Moreover, this statement is not true in general for n-independent knot sets X, with $\#X \geq 2n+2$ and $n \geq 2$.*

We have this result also in a wider setting:

Proposition 1.11 ([5]). *Let X be a set of nodes with $\#X \leq 2n+1$ and $A \in X$. Then the following three statements are equivalent:*

i) the knot A has an n-fundamental polynomial,

ii) the knot A has an n-fundamental polynomial, which is a product of linear factors,

iii) no $n+1$ nodes of $X \backslash \{A\}$ are collinear together with the node A.

Below we consider the possibility of factorization of fundamental polynomials into factors of degree at most 2.

Theorem 1.12 ([5]). *Let X be an n-independent set of nodes with $\#X \leq 2n + [n/2] + 1$. Then for each node of X there is an n-fundamental polynomial, which is a product of lines and conics. Moreover, this statement is not true in general for n-independent node sets X with $\#X \geq 2n + [n/2] + 1$ and $n \geq 3$.*

The first statement of Theorem follows from the following result which covers more wider setting.

Proposition 1.13 ([5]). *Let X be a set of knots with $\#X \leq 2n + [n/2] + 1$ and $A \in X$. Then the following three statements are equivalent:*

i) the knot A has an n-fundamental polynomial,

ii) the knot A has an n-fundamental polynomial, which is a product of lines and conics,

iii) for the knot A the conditions $\langle n \rangle_{1,A}, \langle n \rangle_{2,A}$ hold.

1. On factorization of bivariate fundamental polynomials

In this section we prove the following interesting in itself proposition which in the Section 4 will be used to establish a result on n-independence of knot sets in R^3.

Theorem 2.1. *Let X be a set of knots in R^2 with $\#X = 3n - k$, where $k, n \geq 1$. Suppose that the conditions $\langle n \rangle_1, \langle n \rangle_2$ hold. Then the knot set X is n-independent and a fundamental polynomial of each knot in X can be written in form of products of k lines and a curve of degree $n-k$: $p^*_{A,X} = \alpha_0 \alpha_1 \ldots \alpha_{k-1} q$, $\forall A \in X$.*

The statement of Theorem readily follows from the following more general proposition.

Proposition 2.2. *Let X be a set of knots in R^2 with $\#X = 3n - k$, where $k, n \geq 1$ and $A \in X$. Suppose that the conditions $\langle n \rangle_{1,A}, \langle n \rangle_{2,A}$ hold. Then there exists a fundamental polynomial of A of form $p^*_{A,X} = \alpha_0 \alpha_1 \ldots \alpha_{k-1} q$, where $\alpha_i \in$ ⍰Π_1 are lines and $q \in$ ⍰Π_{n-k}.*

Proof. The cases $n = 1, 2$ are evident. Let us prove the case $n = 3$:

First assume that $k = 1$. In this case we have 8 knots in X. Therefore $\#X = 2n + [n/2] + 1$. Thus according to Proposition 1.13 A has a fundamental polynomial which is a product of lines and conics. Since n is odd there is a line factor.

For the cases $k \geq 2$ we have $\#X \leq 1 + 2n$ and hence according to Proposition 1.11 the knot A has a fundamental polynomial which is a product of lines.

Thus we may assume from now on that $n \geq 4$.

Let us now use induction on $n + k$. Assume that the statement is true for all natural numbers n' and k', such that $n' + k' \leq n + k - 1$, and prove it for $n' = n$ and $k' = k$. Notice that it is enough to find a line α_0, $A \notin \alpha_0$, passing through two knots of X such that conditions $\langle n-1\rangle_{1,A}, \langle n-1\rangle_{2,A}$ hold for the knot set $X' := X \backslash \alpha_0$.

Indeed, after the choice of such a line we will have at most $3(n-1) - (k-1) = 3n - k - 2$ knots in X'. Therefore in view of induction assumption there exists a fundamental polynomial of form $p^*_{A,X',n-1} = \alpha_1 \dots \alpha_{k-1} q$, where $\alpha_i \in \Pi_1$ and $q \in \Pi_{n-k}$. Hence we get a desired fundamental polynomial $p^*_{A,X} = \alpha_0 \alpha_1 \dots \alpha_{k-1} q$.

Notice that if there is a line α passing through $n + 1$ knots of $X \backslash \{A\}$, then it can be taken as a desired line. Thus we may assume from now on that any line not passing through A, passes through at most n knots of X.

Now assume that the set X satisfies the conditions $\langle n-1\rangle_{1,A}, \langle n-1\rangle_{2,A}$.

Then is easily seen that in this case as a desired line we can take any line α_0, $A \notin \alpha_0$, passing through two knots of X.

The remaining cases we consider in three steps.

Step 1. Suppose that there is an irreducible conic β passing through A and at least $2n - 1$ other knots of X. Notice that there can be at most one such conic. Indeed, if there are two such conics, then we will have at least $1 + 2(2n-1) - 3 = 4n - 4$ knots in X, where 3 stands for the possible intersection knots of the two conics different from A. On the other hand $4n - 4 > \#X = 3n - k$; if $n \geq 4$, where $k \geq 1$.

Note that according to the condition $\langle n\rangle_{2,A}$ the conic β contains at most $2n$ knots of X different from A.

Now suppose that the condition $\langle n-1\rangle_{1,A}$ is not satisfied - there is a line α_A passing through A and n other knots. Then let us verify that the following three conditions are satisfied: the line α_A and the conic β intersect at two knots of X, the conic β contains exactly $2n - 1$ knots of $X \backslash \{A\}$ and $k = 1$. Indeed, in this case we have that X contains at least $1 + n + (2n-1) - 1 = 3n - 1$ knots, where the last '-1' in the left hand side of the equality means that the line and the conic, besides A, intersect also at another knot B.

It is easily seen that in this case as a desired line we can take a line passing through B and a knot from β, different from A.

Finally suppose that the part ii) of the condition $\langle n-1\rangle_{2,A}$ is not satisfied. Therefore there is a line α, $A \notin \alpha$, passing through exactly n knots of X. Then we can verify that α intersects β at at least one knot of X and all the knots of X, except possibly a knot, lie in $\alpha \cup$ β. Indeed, as in the previous case we have that X contains at least $1 + n + (2n-1) - 1 = 3n - 1$ knots, where the last '-1' in the left hand side of the equality means that α and β intersect at a knot in X.

It is easily seen that here α is a desired line .

Step 2. Suppose that there is a line α, $A \notin \alpha$, passing through exactly n knots, and a line α_A, $A \in \alpha_A$, passing through at least $n - 1$ knots of $X \backslash \alpha$. Let us consider two cases.

First suppose that α_A passes through A and n other knots of $X \backslash \alpha$. Note that outside of these two lines there are at most $n - 2$ knots. Therefore if there is a second line α', $A \notin \alpha'$, passing through n knots, then it intersects the lines α and α_A at two different knots. Therefore in this case α' is a desired line. If there is no second line α', then it is easily seen that we can take as a desired line α_0 a line passing through one knot from α and another from α_A, different from A.

Next suppose that α_A passes through A and exactly $n - 1$ other knots of $X \backslash \alpha$. Note that the conditions $\langle n-1\rangle_{1,A}$ and $\langle n-1\rangle_{2,A}$ are satisfied, since there are at most $n - 1$ knots outside of the lines α, α_A. Therefore it is easily seen that α is a desired line.

Step 3. Suppose that there is a line α_A passing through A and n other knots of X. Notice that there can be at most two such lines. In this case it is easily seen that as a desired line α_0, $A \notin \alpha_0$, we can take any line that intersects each of the considered lines at a knot different from A.

Note that Proposition 1.11 is a special case of Proposition 3.2 where we take $k = n - 1$.

Note also that for knot sets of cardinality $2n + 2$ we get from Proposition 3.2 that each knot has a fundamental polynomial in the form of product of $n - 2$ lines and a conic, which improves Proposition 1.13 in this case.

2. **On factorization of trivariate fundamental polynomials**

Let us start this section by proving Theorem 1.4 in a more general setting.

Proposition 3.1. *Let X be a set of knots in R^2 with $\#X \le 3n$. Then a knot $A \in X$ has an n-fundamental polynomial if and only if the conditions $\langle n\rangle_{1,A}, \langle n\rangle_{2,A}, \langle n\rangle_3$ hold.*

Proof. Let us start with the 'only if' part of Proposition. Suppose that the knot A has an n-fundamental polynomial and let us prove that the conditions $\langle n\rangle_{1,A}, \langle n\rangle_{2,A}, \langle n\rangle_3$ hold.

First we show that the condition $\langle n\rangle_{1,A}$ is satisfied. Indeed, suppose by way of contradiction that a line passes through A and $n + 1$ other nodes of X. Then the fundamental polynomial $p^*_{A,X}$ vanishes at those $n + 1$ nodes. Therefore, by Lemma 1.6, it vanishes at all the points of the line including A, which is a contradiction.

Next we show that the condition$\langle n\rangle_{2,A}$ takes place. Indeed, suppose by way of contradiction that a line α passes through $n + 1$ nodes of X and another line α_A passes through A and n nodes of $X \setminus \alpha_A$. Then the fundamental polynomial $p^*_{A,X}$ vanishes at these $n + 1$ and n nodes. Therefore, by Lemma 1.7, it vanishes at all the points of the lines α and α_A, including A, which is a contradiction. Now assume on the contrary that an irreducible conic β passes through A and $2n + 1$ other nodes of X. Then the fundamental polynomial $p^*_{A,X}$ vanishes at those $2n + 1$ nodes and therefore, by Lemma 1.8, it vanishes at all the points of β, including A, which is a contradiction.

Finally we show that $\langle n\rangle_3$ is satisfied. Indeed, suppose by way of contradiction that $\#X = 3n$, and there are curves $\gamma \in \Pi_3$ and $\sigma_n \in \Pi_n$, such that $\gamma \cap \sigma_n = X$. Then the fundamental polynomial $p^*_{A,X}$ vanishes at all $3n - 1$ knots of X different from A. Therefore according to Theorem 1.9 it vanishes at all the points of X, including A, which is a contradiction.

Now let us prove the 'if' part. It is enough to verify that the conditions $\langle n\rangle_1$ and $\langle n\rangle_2$ of Theorem 1.4 are satisfied. Let us do this in two steps.

Step 1. Suppose there are $n + 1$ knots, belonging to a line α and $A \notin \alpha$: Then we take as a fundamental polynomial of A: $p^*_{A,X,n} = \alpha p^*_{A,X',n-1}$, where $X' = X \setminus \alpha$. Here the existence of the fundamental polynomial $p^*_{A,X',n-1}$ follows from Proposition 1.11, since $\#X' \le 2n - 1 = 2(n - 1) + 1$ and according to the condition $\langle n\rangle_{2,A}$ no n knots of X' are collinear together with A.

Step 2. Suppose there are $2n + 2$ knots, belonging to an irreducible conic β and $A \notin \beta$. Then we take the fundamental polynomial in the form: $p^*_{A,X,n} = \beta p^*_{A,X',n-2}$, where $X' = X \setminus \beta$. Here the existence of the fundamental polynomial $p^*_{A,X',n-2}$ follows from Theorem 1.2, since $\#X' \le n - 2$.

Notice that, in view of Step 1, we may assume that there is no line passing through $n + 1$ knots and not passing through A. Then, according to the condition $\langle n\rangle_{1,A}$, there is no line passing through A and $n + 1$ other knots. Therefore there is no reducible conic (pair of lines) passing through 2n + 2 knots.

Now in view of Steps 1, 2 and the conditions $\langle n\rangle_{1,A}, \langle n\rangle_{2,A}, \langle n\rangle_3$, we may assume that no $n + 2$ knots of X are collinear and no 2n + 2 knots of X belong to a conic(reducible or irreducible). Therefore in view of condition $\langle n\rangle_3$ and Theorem 1.4 X is n-independent.

Next we present a main result of the paper.

Theorem 3.2. *Let X be a set of knots in R^3 with $\#X \le 3n + 1$ and $A \in X$. Then the knot A has an n-fundamental polynomial, which is a product of linear factors, if and only if the condition $\langle n\rangle_{1,A}$ and the following condition hold:*

i) if at least $2n + 1$ knots of $X \setminus \{A\}$ are lying in a plane L_A passing through A, then all the knots of $X \cap L_A$ different from A lie in n lines not passing through A.

Proof. Without lose of generality we can assume that $\#X = 3n + 1$. Indeed, it suffices to verify that if $\#X < 3n + 1$ then we can add a knot B to X such that the set $X' = X \cup \{B\}$ satisfies the conditions $\langle n\rangle_{1,A}$ and i) with X replaced by X'. For this purpose we can consider all the lines passing through any two knots of X and all the planes passing through any three non-collinear knots of X. Note that both these sets are finite. Then it is easily seen that as a desired knot B one can choose any knot which does not belong to the considered lines and planes.

Let us start with the 'only if' part. Suppose that the knot A has an n -fundamental polynomial, which is a product of linear factors: $p_{A,X}^* = L_1 L_2 \dots L_n$.

Then a line passing through A intersects each plane $L_i,\ i = 1,2,\dots,n,$ in at most one point. Therefore there is no line passing through A and $n+1$ other knots of X, since all the knots of X lie in the planes $L_1, L_2, \dots, L_n$. Thus the condition $\langle n \rangle_{1,A}$ is satisfied.

Next suppose that L is a plane passing through A and at least $2n+1$ other knots of X. Then the planes $L_1, L_2, \dots, L_n$ will intersect L in at most n lines not passing through A. Thus all the knots of $X \cap L$ different from A lie in these lines. Therefore the condition $\langle n \rangle_{1,A}$ holds.

Now we will prove the 'if' part in the following three steps.

Step 1. First consider the case when there are at least $2n$ knots of $X \setminus \{A\}$ that belong to a plane L_A together with A. Let us show that all these knots lie in n lines in L_A not passing through A. Indeed, if the number of these knots is greater than $2n$ then this follows from the condition i) of Theorem. On the other hand if there are exactly $2n$ knots in the plane L_A then this follows from Proposition 1.11. Here we take into account the condition $\langle n \rangle_{1,A}$ of Theorem. Then notice that in this case there are at most n knots outside of the plane L_A, i.e., $\#(X \setminus L_A) \le n$. Thus we can take $p_{A,X}^*$ as a product of n planes each passing through one of the mentioned n lines and a knot from outside L_A. Note that if there are less than n knots outside of L_A, then we can take any other point from $\mathrm{R}^3 \setminus L_A$.

Step 2. Now consider the case when there is a plane L_A passing through A and exactly $2n-1$ other knots. Therefore there are exactly $n+1$ knots in $X \setminus L_A$. We want to choose three non-collinear knots: a knot from L_A different from A and two knots from outside of L_A such that the plane L passing through them is not passing through A, and for the remaining knots in L_A there are no n knots collinear together with A. Note that after we chose such three knots we will have for the knot set $X' := X \setminus L$, that $\#X' \le 3(n-1)+1$ and the following conditions are satisfied:

1) there are at most $2(n-1)+1$ knots in L_A,
2) no n knots are collinear together with A,
3) there are at most $n-1$ knots outside of L_A.

Thus in the same way as in the Step 1, by using Proposition 1.11, we can construct a fundamental polynomial of A with respect to X': $p_{A,X'}^*$ in form of product of planes. Finally notice that we can take $p_{A,X}^* = L p_{A,X'}^*$.

Now let us describe the choice of the mentioned three knots. Notice that in view of Step 1 we may assume that there is at most one line passing through A and n other knots, because otherwise we would have a plane passing through A and at least $2n$ other knots. Let us consider two cases.

First suppose that there is such a line l_A in L_A. In this case we chose the first knot B any knot from l_A different from A. The other two knots C and D we chose such that the knots A, B, C and D are not coplanar. This will not be possible only if all the $n+1$ knots of $X \setminus L_A$ belong to a plane L' that passes through l_A. This case was considered in Step 1, since there are $2n+1$ knots in the plane L' different from A.

Next suppose that there is no such a line l_A in L_A. It is easily seen that in this case as three knots mentioned above we can take any three knots such that they are not coplanar together with A.

Step 3. Now we may assume that no $2n-1$ knots of $X \setminus \{A\}$ belong to a plane together with A. In this case we will use induction on n.

In the case $n=1$ we have 4 knots in X. If they are not coplanar then we will take $p_{A,X}^* = L$, where L is the plane passing through the 3 knots different from A. Otherwise if the 4 knots are coplanar then according to the condition i) of Theorem the 3 knots different from A are lying in a line. Therefore as a plane L we can take any plane passing through that line and not passing through A.

Now suppose that Theorem is true for $n-1$ and let us prove it for n. Notice that it is enough to find a plane L passing through 3 knots of X which is not passing through A, and for $X' := X \setminus L$ the conditions $\langle n \rangle_{1,A}$ and i) of Theorem hold with $n-1$. Indeed, then we will take $p_{A,X}^* = L p_{A,X'}^*$, where in view of induction hypothesis, $p_{A,X'}^*$ is a product of $n-1$ planes. Now let us describe the choice of the mentioned three knots. The condition i) of Theorem holds for the set X' with $n-1$, since there are no $2(n-1)+1=2n-1$ knots of X coplanar together with A. Thus to complete the proof it remains to note that if there is a line l_A passing through A and n other knots (recall that there can be at most one such line) then we can take one of these three knots any knot from l_A different from A. Notice that the choice of the mentioned three knots will not be possible only if the whole set X lies in a plane – a case considered in Step 1.

Independence of $3n+1$ knots in R^3

The following theorem improves Theorem 1.5 in the case of R^3.

Theorem 4.1. *Let X be a set of non-coplanar knots in R^3 with $\#X \le 3n+1$. Then X is n-independent if and only if the conditions $\langle n\rangle_1, \langle n\rangle_2, \langle n\rangle_3$ hold.*

Notice that this theorem readily follows from the following result which covers more wider setting.

Proposition 4.2. *Let X be a set of non-coplanar knots in R^3 with $\#X \le 3n+1$. Then a knot $A \in X$ has an n-fundamental polynomial if and only if the conditions $\langle n\rangle_{1,A}, \langle n\rangle_{2,A}, \langle n\rangle_3$ hold.*

First let us prove the following lemma.

Lemma 4.3. *Let L be a plane in R^3 and q be a curve of degree n in L: $q \in \Pi_n(L)$. Then for any point B outside of L, there exists a surface $p \in \Pi_n^3$ passing through B, such that $p|_L = q$.*

Proof. Without lose of generality assume that the curve in L is given by an equation $q(x,y)=0$. Then we can take the surface p in form: $p(x,y,z) = q(x,y) + cL(x,y,z),$ where the constant c is chosen such that $p(A)=0$.

Now let us turn to

Proof of Proposition 4.2. The 'only if' part is obvious. Let us prove the 'if' part. If there is no plane passing through A and at least $2n+1$ other knots, then Proposition follows from Theorem 3.2. Thus assume that there is a plane L passing through A and at least $2n+1$ other knots. Assume that there are $k+1$ knots outside of L: $1 \le k+1 \le n-1$.

First suppose that $k=0$. Denote by B the knot outside of L. According to Proposition 3.1 A has a fundamental polynomial $q = p^*_{A,Y} \in \Pi_n(L)$, with respect to the set $Y = X \cap L$. Thus, in view of Lemma 4.3, we can take $p^*_{A,X} = p$, where $p \in \Pi_n^3$ is a surface, such that $p|_L = q$ and $p(B) = 0$.

Next suppose that $k \ge 1$, then according to Proposition 2.2, we have for the fundamental polynomial $p^*_{A,Y} \in \Pi_n(L)$: $p^*_{A,Y} = \alpha_0\alpha_1 \dots \alpha_{k-1} q$, where $\alpha_i \in \Pi_1(L)$, $q \in \Pi_{n-k}(L)$. So here we can take $p^*_{A,X} = L_0L_1 \dots L_{k-1}p$, where L_i is a plane passing through the line α_i and a knot from outside of L, while p is a surface passing through q and the last knot from outside, given in Lemma 4.3.

The following corollary readily follows from Theorems 4.1 and 1.4 .

Corollary 4.4. *Let X be a set of knots in R^3 with $\#X \le 3n+1$. Then X is n -independent if and only if for any plane L the set $X \cap L$ is n-independent.*

References:

1. D. Eisenbud, M. Green and J. Harris, Cayley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. (N.S.), 33 (1996), 295-324.

2. H. Hakopian and A. Malinyan, Characterization of n-independent sets with no more than 3n points, Jaen J. Approx. 4 (2012), 119 -134.

3. F. Severi, Vorlesungen Euber Algebraische Geometrie, Teubner, Berlin, 1921.

4. A. Malinyan, Characterization of n -independent sets of $\le 3n$ points in R^d , Vestnik RAU. Seria Phys.-Math. And natural sciences (in Russian), №1, 3-15 (2013).

5. V. Vardanyan, On bivariate fundamental polynomials, British Journal of Mathematics and Computer Science 10(5): 1-17, 2015.

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О факторизации фундаментальных многочленов двух и трех переменных

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Аннотация. Множество узлов называется - независимым, если для произвольных значений в этих узлах существует (не обязательно единственный) интерполяционный многочлен суммарной степени не выше n. Для произвольного n-независимого множества узлов X из R^3 мы рассматриваем задачу построения фундаментальных многочленов, имеющих простейшую форму. В настоящей статье мы приводим необходимые и достаточные условия для множества X мощности, не превышающей $3n+1$, так, чтобы каждый узел имел - фундаментальный многочлен, являющийся произведением линейных множителей. Приводится также необходимые и достаточные условия для - независимости некомпланарнымих множеств узлов из R^3 отмеченной мощности.

Ключевые слова: интерполяция с многочленами, фундаментальный многочлен, n - независимые узлы.