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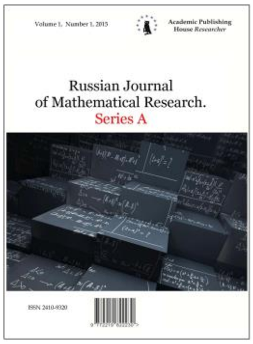

Articles and statements

UDC 51

Unsteady Visco-elastic Boundary Layer Flow Past a Stretching Plate and Heat Transfer

[1] Naseem Ahmad
[2] Ravins

[1] Department of Mathematics, Jamia Millia Islamia, Jamia Nagar New Delhi-25, India
PhD, professor
E-mail: nahmad4@jmi.ac.in
[2] Centre for Interdisciplinary Research in Basic Sciences, Jamia Millia Islamia, Ramanujan Block, Mujeeb Bag, Jamia Nagar, New Delhi-25, India
PhD, assistant professor
E-mail: ravinsdohare@gmail.com

Abstract

A closed form solution to the unsteady boundary layer flow of visco-elastic fluid (Walter's Liquid B Model) past a stretching plate has been obtained. Using the obtained velocity components u and v, the heat transfer problem has been studied. The behaviour of velocity components and temperature field has been studied though the graphs drawn for various randomly chosen values of time duration and visco-elasticity. Boundary layer thickness, skin friction and the Nusselt number have also been obtained and studied through graphs.

Keywords: walters liquid B model, stretching plate, boundary layer equations, nusselt number.

Introduction

Due to the number of applications in the industrial manufacturing, the problem of boundary layer flow past a stretching sheet has been considered as one of the interested problems during few decades. There are number of examples where this problem has significant applications such as hot rolling, wire drawing, glass-fibre production and paper production. The role of heat transfer comes in to the picture in the process of drawing the artificial fibres from the polymer solution that emerges from the orifice with a speed which increases from almost zero at the orifice up to a plateau at which it remains constant. In this case, the moving fibre is of technical interest because it is governed by the rate of cooling at which it is cooled for the quality of yarn. A number of researches are available that follow the pioneering classical work of Sakiadis [1], F.K. Tsou et al. [2] and Crane [3]. Number of scholars such as A. Naseem [4], N. Ahmad [5] [6], D. Kelly, K. Vijravelu, L. Andrews [7], N. Ahmad and K. Marwah [8], A. M. Subhas and A Joshi [9], Sidhdheshar and Mahabaleswar [10], M. Subhas Abel, P. G. Siddheshwar, Mahantesh M. Nandappaanavar [11] and

N. Ahmad, M. Mishra [12] studied the visco-elastic fluid flow past a stretching plate in various variants.

In the present paper, the flow of Visco-elastic Fluid (Walters Liquid B model) has been considered to study the convectional heat transfer within boundary layer induced by the stretching character of the plate in quiescent fluid. A closed form solution has been obtained and the results have been drawn with the help of graphs.

Method

Let x-axis be along the stretching plate and y-axis be the normal to the stretching plate. The flow has been created due the stretching character of the plate. Assuming u and v as horizontal and normal components of velocity field, the boundary layer equation governing the flow of Walters Liquid B model be:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-K_0^*\left\{u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right\} \tag{2}$$

where ρ is density, υ is kinematic viscosity, and K_0^* is visco-elasticity of the fluid.

The relevant boundary conditions are:

$$y=0, u=\frac{bx}{1-at},\ \ v=0 \tag{3}$$

$$y\to\infty,\ \ u=0,\ \ u_y=0 \tag{4}$$

Defining the dimensionless variables by

$x^*=\frac{x}{L}$, $y^*=\frac{y}{L}$, $u^*=\frac{uL}{\upsilon}$, $v^*=\frac{vL}{\upsilon}$ and $t*=\frac{t\upsilon}{L^2}$ for characteristic length L. Using all these dimensionless variables in the equations (1) through (4) , we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}-K_0\left\{u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right\} \tag{6}$$

$$y=0, u=\frac{b_0}{1-a_0 t}\ x,\ \ v=0 \tag{7}$$

$$y\to\infty,\ \ u=0,\ \ u_y=0 \tag{8}$$

where $K_0=\frac{K_0^*}{L^2}$, $b_0=b\upsilon$, $a_0=\frac{a_0 h^2}{\upsilon}$ and the star $(^*)$ has been supressed for our convenience. Setting the similarity solution of the form, $u=\frac{b_0}{1-a_0 t}xf'(y)$, we have $v=-\frac{b_0}{1-a_0}\sqrt{\frac{1-b_0K_0-a_0t}{a_0+b_0}}\left\{1-e^{-\sqrt{\frac{1-b_0K_0-a_0t}{a_0+b_0}}}\right\}$, where without loss of generality $f(0)=0$. Substituting u and v in the momentum equation(6), we get

$$a_0 f'+b_0 f'^2-b_0 ff''=(1-a_0t)f'''-b_0K_0\left\{2f'f'''-f\ f''''-f''^2\right\} \tag{9}$$

Observing the boundary conditions for stretching plate, we assume $f'(y) = \exp(-ry)$ where $r = \sqrt{\frac{1-b_0K_0-a_0t}{a_0+b_0}}$, is a function of t. From the expressions of u and v, we put a condition $b_0K_0 + a_0t < 1 \Rightarrow t < \frac{1-b_0K_0}{a_0}$

Boundary Layer Thickness

In the present case, a plate is stretching with $u = \frac{b_0}{1-a_0t}x$ which may be considered as free stream velocity. Therefore, the boundary layer thickness may be defined as the value of ordinate such that $u_\delta = \frac{1}{100}u_0$. Hence, the boundary layer thickness is

$$\delta = 4.605\sqrt{\frac{1-b_0K_0-a_0t}{a_0+b_0}} \tag{10}$$

where $u_0 = u$ at $y = 0$ and $b_0K_0 + a_0t < 1$.

Skin Friction

The wall shear stress at the stretching plate is given by

$$\begin{aligned}
\tau_p &= -\mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \\
&= -\mu\left(\frac{\partial}{\partial y}\left(\frac{b_0}{1-a_0t}x\right)e^{-\sqrt{\frac{a_0+b_0}{1-b_0K_0-a_0t}}y}\right)_{y=0} \\
&= \mu\left(\frac{b_0}{1-a_0t}x\sqrt{\frac{a_0+b_0}{1-b_0K_0-a_0t}}\right)
\end{aligned}$$

Thus, the skin friction is

$$\begin{aligned}
C_f &= \frac{\tau_p}{\rho u_0^2 L} \\
&= \frac{(1-a_0t)^2}{\rho L b_0^2 x^2}\left(\mu\frac{b_0}{1-a_0t}x\sqrt{\frac{a_0+b_0}{1-b_0K_0-a_0t}}\right) \\
&= \upsilon\frac{(1-a_0t)}{Lb_0x}\left(\sqrt{\frac{a_0+b_0}{1-b_0K_0-a_0t}}\right) \\
&= \frac{\upsilon}{u_0L}\left(\sqrt{\frac{a_0+b_0}{1-b_0K_0-a_0t}}\right) = \frac{1}{\mathrm{Re}}\left(\sqrt{\frac{a_0+b_0}{1-b_0K_0-a_0t}}\right)
\end{aligned} \tag{11}$$

where $Re = \frac{v}{u_0L}$ is Reynolds number.

Heat Transfer

The thermal boundary layer problem is governed by

$$\rho C_v\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = K\frac{\partial^2 T}{\partial y^2}$$

Boundary conditions are:

$$y = 0, \quad T = T_p \tag{12}$$

$$y \to \infty, T \to T_\infty \tag{13}$$

As the temperature varies with y only, so we have

$$v\frac{\partial T}{\partial y} = \frac{\upsilon}{\Pr}\frac{\partial^2 T}{\partial y^2} \tag{14}$$

Defining the following dimensionless variables

$$\theta = \frac{T - T_\infty}{T_p - T_\infty}, \quad \eta = ry$$

The equation becomes

$$v\frac{\partial \theta}{\partial \eta} = \frac{1}{\Pr} r \frac{\partial^2 \theta}{\partial \eta^2} \tag{15}$$

where Pr (= $\mu C_p / K$) is Prandtl number. The equation (15) can be written as

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(1 - a_0 t)(a_0 + b_0)}\{1 - e^{-\eta}\}\frac{\partial \theta}{\partial \eta} = 0$$

$$\eta = 0, \ \theta = 1: \quad \eta \to \infty, \ \theta \to 0 \tag{16}$$

The solution of the equation (16) is

$$\theta(\eta) = \frac{\int_\eta^\infty \exp\left\{-\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)}\left(\eta + e^{-\eta}\right)\right\} d\eta}{\int_0^\infty \exp\left\{-\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)}\left(\eta + e^{-\eta}\right)\right\} d\eta} \tag{17}$$

Putting $\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)} e^{-\eta} = t \Rightarrow -\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)} e^{-\eta} d\eta = dt$

$$\theta(\eta) = \frac{\gamma\left\{\left(\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)}\right), \left(\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)} e^{-\eta}\right)\right\}}{\gamma\left\{\left(\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)}\right), \left(\frac{\Pr b_0 (1 - b_0 K_0 - a_0 t)}{(a_0 + b_0)(1 - a_0 t)}\right)\right\}}$$

Nusselt Number

The Nusselt number is defined as

$$Nu = -\frac{1}{T_p - T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= -\frac{1}{T_p - T_\infty}\left\{(T_p - T_\infty)\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0} r\right\} \tag{18}$$

$$= -\frac{\exp\left\{-\dfrac{\Pr b_0(1-b_0K_0-a_0t)}{(1-a_0t)(a_0+b_0)}\right\}}{\gamma\left\{\left(\dfrac{\Pr b_0(1-b_0K_0-a_0t)}{(a_0+b_0)(1-a_0t)}\right),\left(\dfrac{\Pr b_0(1-b_0K_0-a_0t)}{(a_0+b_0)(1-a_0t)}\right)\right\}}\cdot\sqrt{\frac{a_0+b_0}{1-b_0K_0-a_0t}} \tag{19}$$

Results and Discussion

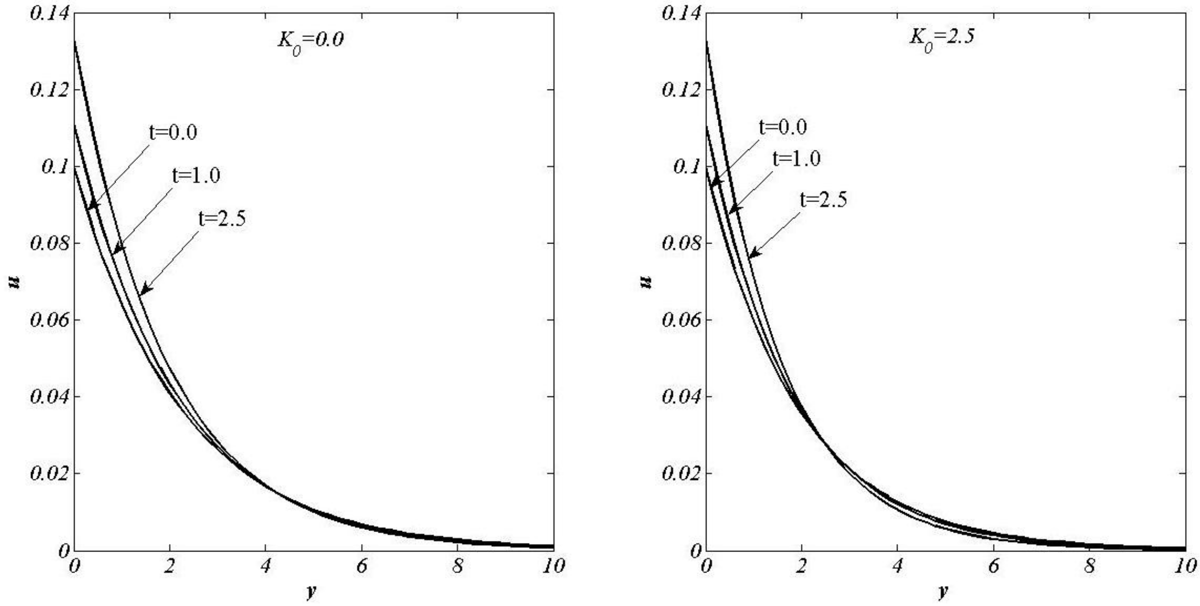

Fig. 1(a). Component of velocity $\boldsymbol{u}$ versus $\boldsymbol{y}$ for different time slot $\boldsymbol{t}$, when keeping $\boldsymbol{K_0}$ constant.

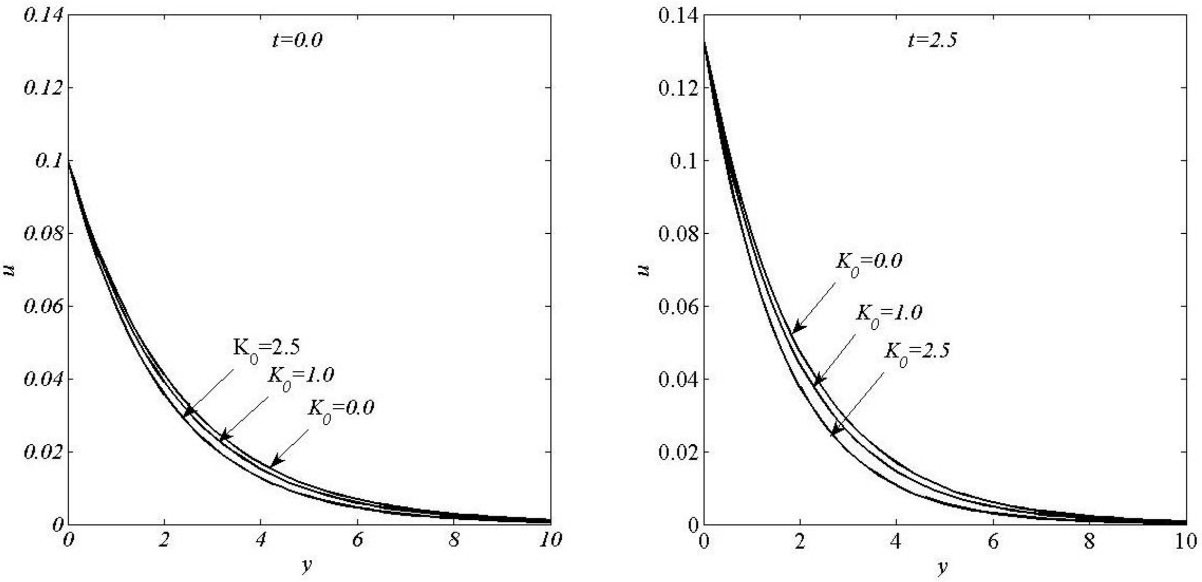

Fig. 1(b). Component of velocity $\boldsymbol{u}$ versus $\boldsymbol{y}$ for different values of visco-elastic parameter $\boldsymbol{K_0}$ with keeping time constant.

In the course of analysis this studies, we come across the following results:

Figure 1(a) is a graph of u versus y at different instant of time. It has been observed that the velocity component u increases as time increases but at the upper end of the boundary layer velocity component attains zero value. This pattern exists due visco-elastic nature of fluid.

Figure 1(b) is a graph plotted for u against y for different values of visco-elastic parameter K_0 at an instant of time t. The velocity component u attains the maximum value at the stretching plate and the fluid comes to the rest at the end of boundary layer. It is noted that the action of visco-elasticity eases within $y \in [0.5, 5]$ and we observe that as visco-elasticity K_0 increases, u decreases.

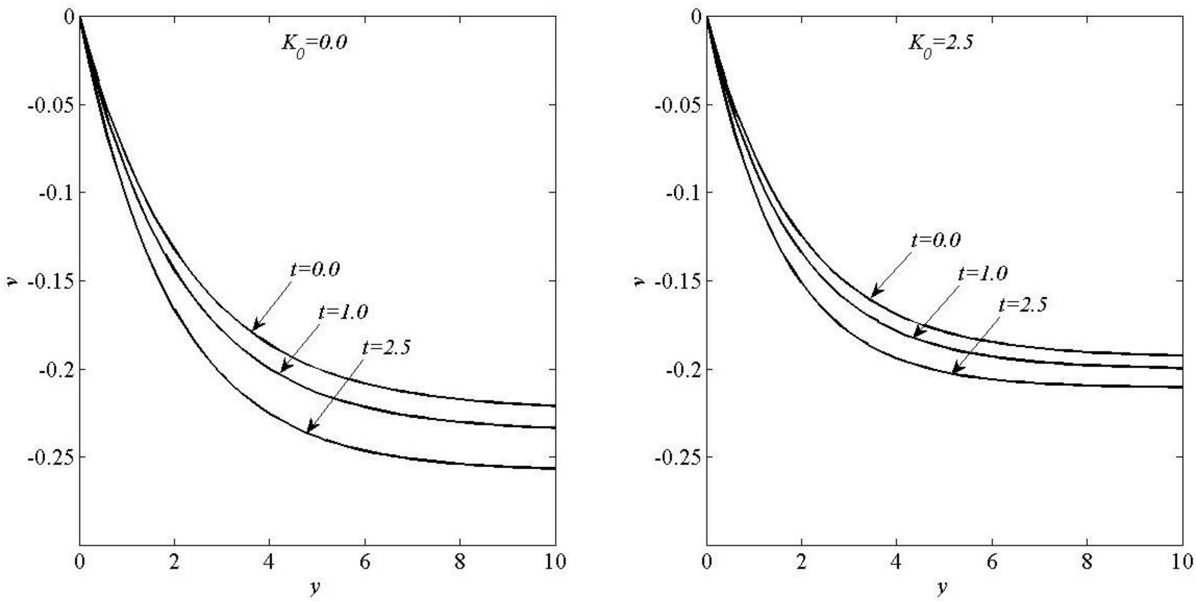

Fig. 2(a). A velocity component $\boldsymbol{v}$ versus $\boldsymbol{y}$ for various time slots with keeping constant $\boldsymbol{K_0}$.

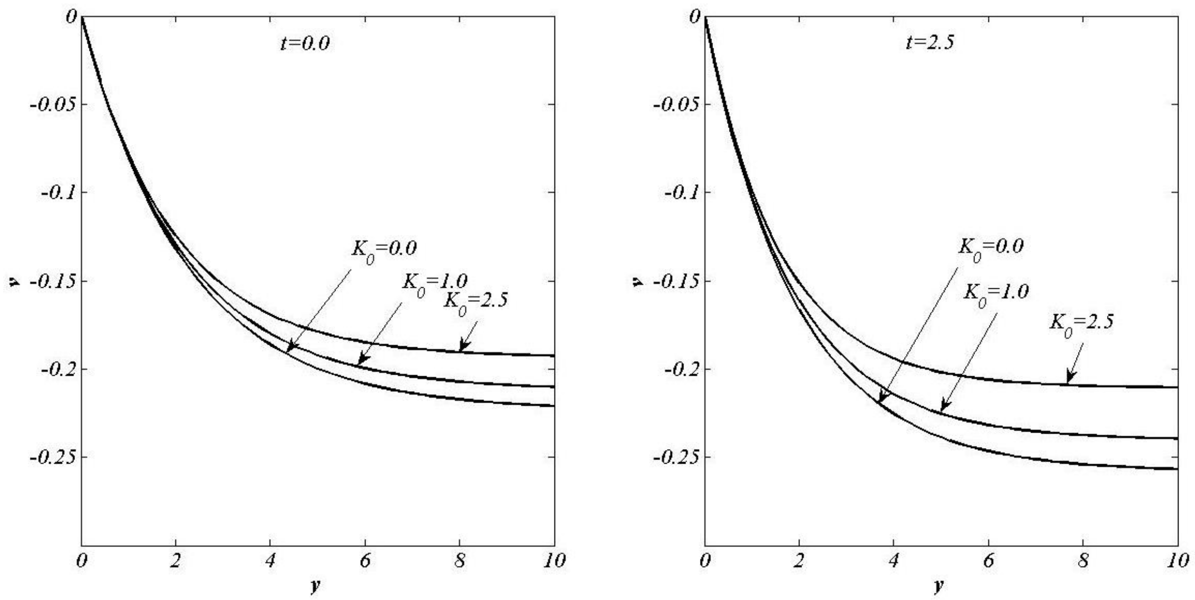

Fig. 2(b). A velocity component $\boldsymbol{v}$ versus for various values of visco-elasticity $\boldsymbol{K_0}$ with keeping constant $\boldsymbol{t}$.

Figures 2(a) and 2(b) are for vertical component v of velocity $\bar{q} = (u, v)$. In this case the magnitude of v increases as time t increases but according to figure 2 (b), the magnitude decreases as visco-elastic parameter K_0 increases.

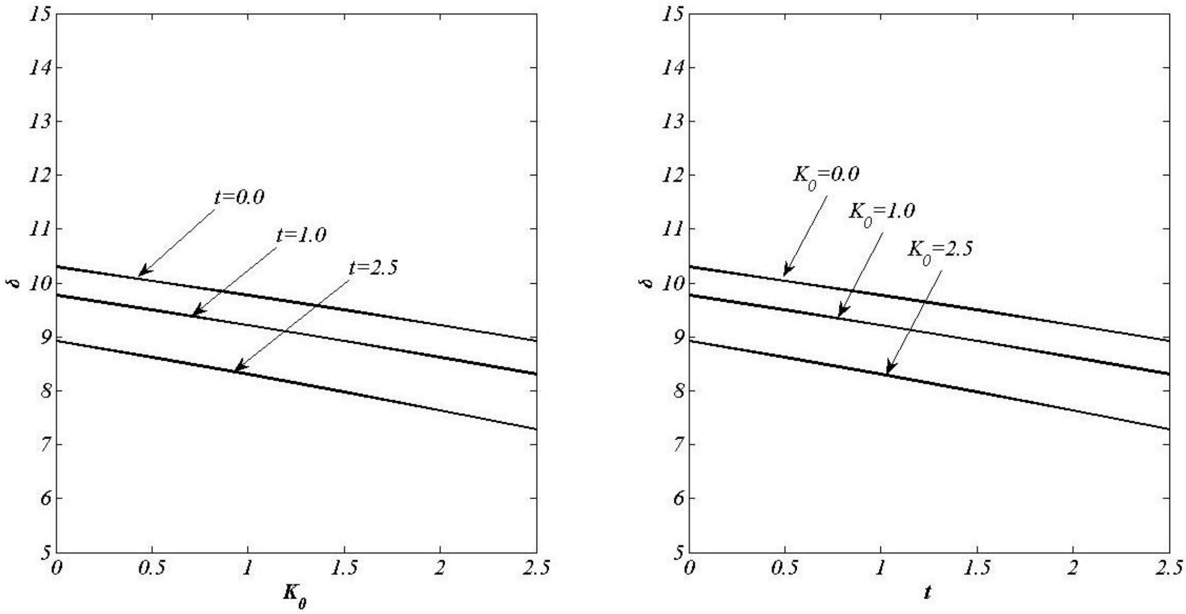

Fig. 3. Left panel is the graph of boundary layer thickness $\boldsymbol{\delta}$ versus visco-elastic parameter $\boldsymbol{K_0}$ at different time slot $\boldsymbol{t}$ while right panel is the graph of boundary layer thickness $\boldsymbol{\delta}$ versus time $\boldsymbol{t}$ for different value of visco-elasticity.

In Figure 3; left panel, shows the boundary layer thickness and it is about 10. As time t progresses, the boundary layer thickness gets suppressed. This fact agrees with the boundary layer behaviour, i.e. the boundary layer thickness decreases as time of the flow increases while it increases if the velocity increases tremendously within short span of time. According to the Figure 3; right panel, the boundary layer thickness decreases with the increase of visco-elastic parameter K_0.

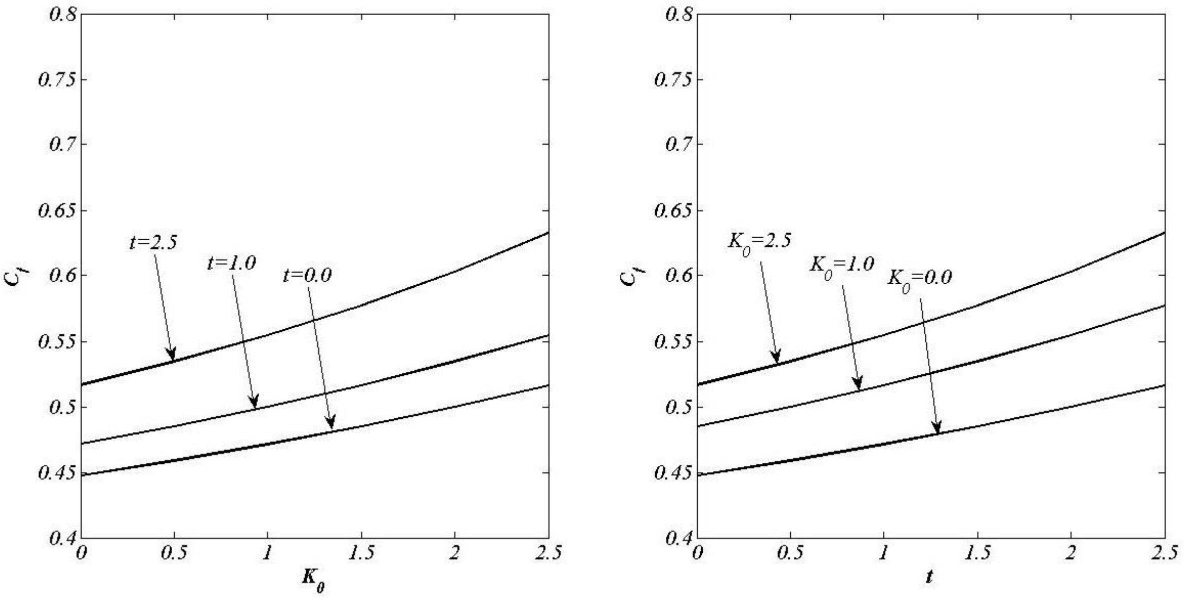

Fig. 4. Skin friction $\boldsymbol{c_f}$ versus visco-elasticity $\boldsymbol{K_0}$ for different time slots in left panel while in the right panel, $\boldsymbol{c_f}$ versus $\boldsymbol{t}$ for different values of visco-elasticity.

In Figure 4, we note that the skin friction increases with the increase of time t and visco-elasticity K_0.

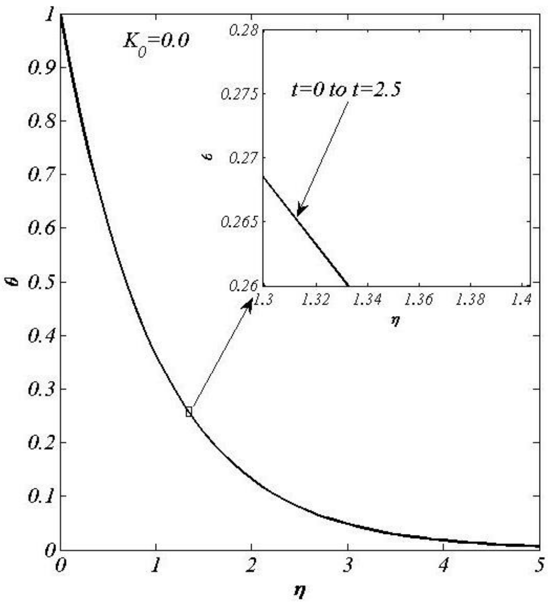

Fig. 5(a). Temperature profile $\boldsymbol{\theta}$ versus $\boldsymbol{\eta}$ for different time slot $\boldsymbol{t}$ for $\boldsymbol{K_0 = 0.0}$.

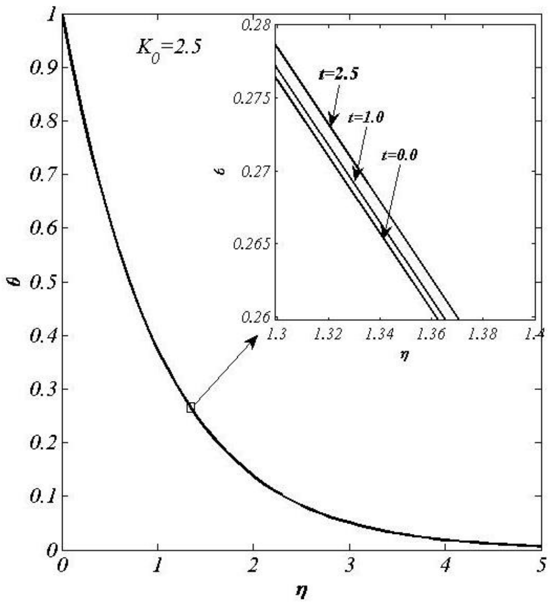

Fig. 5(b). Temperature profile $\boldsymbol{\theta}$ versus $\boldsymbol{\eta}$ for different time slot $\boldsymbol{t}$ for $\boldsymbol{K_0 = 2.5}$.

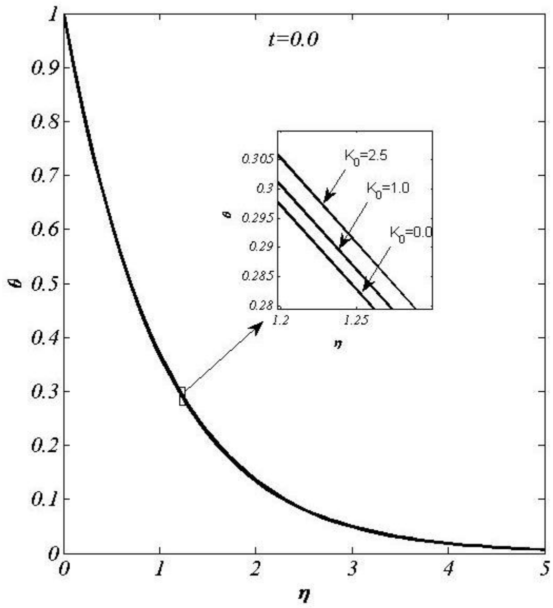

Fig. 5(c). Temperature profile $\boldsymbol{\theta}$ versus $\boldsymbol{\eta}$ for various value of visco-elasticity $\boldsymbol{K_0}$ at $\boldsymbol{t = 0.0}$.

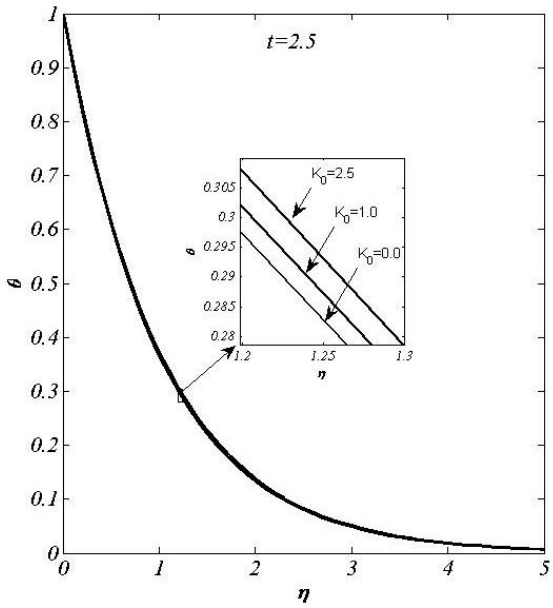

Fig. 5(d). Temperature profile $\boldsymbol{\theta}$ versus $\boldsymbol{\eta}$ for various value of visco-elasticity $\boldsymbol{K_0}$ at $\boldsymbol{t = 2.5}$.

In Figures 5(a) and 5 (b), the temperature profile θ has been plotted against η for different time slot t. We see that visco-elasticity has no significant influence on temperature field. The same pattern has also been obtained for different values of K_0 for fixed time slot in Figures $5(c)$ and $5(d)$. Here, we see that the heat transfer is behaving similar to Figures $5(a)$ and $5(b)$.

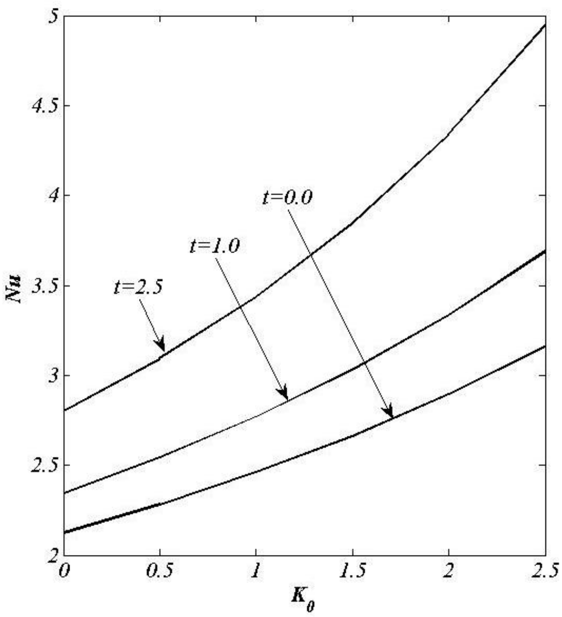

Fig. 6(a). Nusselt number **Nu** versus visco-elasticity $\mathbf{K_0}$ for different time slot **t**.

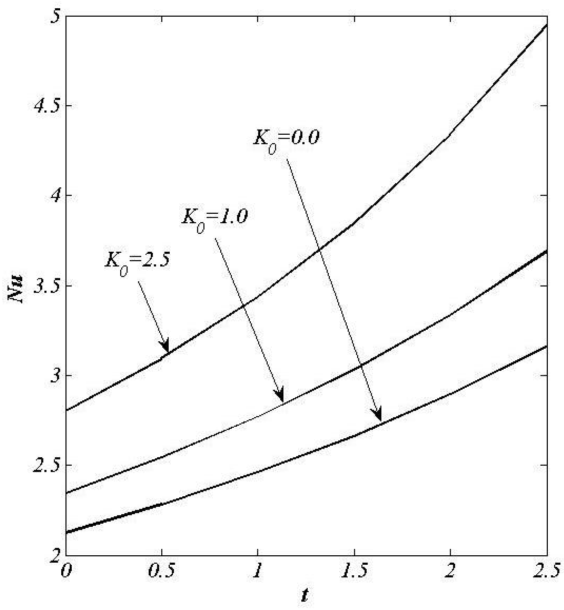

Fig.6(b). Nusselt number **Nu** versus time **t** for different values of $\mathbf{K_0}$.

The graphs obtained in Figures 6(a) and 6(b) are to show the behaviour of convectional heat transfer coefficient Nusselt number $Nu(t, K_0)$. It is seen that the Nusselt number is a function of t and K_0. We observe here that the Nusselt number Nu increases as any one of t or K_0 increases.

Conclusion

An exact solution has been obtained to the problem of heat transfer in case of visco-elastic fluid flow past a stretching plate. This paper covers two aspects: one methodology by which one can deal with unsteady heat transfer problems, and another is the flow pattern of temperature profile in case of convectional heat transfer. This problem reduces to the problem solved for unsteady boundary layer flow of viscous incompressible fluid when $K_0 = 0$ [12].

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