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WEAKLY ISOTONE AND STRONGLY REVERSE ISOTONE MAPPINGS OF RELATIONAL SYSTEMS

Daniel Abraham Romano

ABSTRACT. The setting of this article is Classical algebra and Bishop's constructive algebra (the algebra based on the Intuitionistic logic). The Esakia's concept in the classical mathematics of strongly isotone mapping between ordered sets is extended onto two different concepts of mappings: on the concept of weakly isotone and the concept of strongly reverse isotone mapping of relational systems. Some characterizations of those mappings are given and some application of those mappings in ordered semigroup theory are given.

1. Introduction

This investigation in Classical algebra and in Bishop's constructive mathematics (in sense of well-known books [**1, 2, 3, 7, 8, 13**] and papers [**9, 10, 11**]) is a continuation of the author's paper [**12**].

Bishop's constructive mathematics is develop on Constructive Logic / Intuitionistic Logic - logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in the Intuitionistic Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in the Minimal Logic. Since the Intuitionistic Logic is a part of the Classical Logic, these results in the Constructive mathematics are compatible with suitable results in the Classical mathematics. Let us recall that the following deduction principle $A \vee B, \neg B \vdash A$ is acceptable in the Intuitionistic Logic.

Let $(A, =, \neq)$ be a set in the sense of books [**1, 2, 3, 7, 8, 13**] , where '$\neq$' is a binary relation on A which satisfies the following properties:

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$\neg(x \neq x)$, $x \neq y \Longrightarrow y \neq x$, $x \neq z \Longrightarrow x \neq y \vee y \neq z$, $x \neq y \wedge y = z \Longrightarrow x \neq z$, called *apartness* (A. Heyting, [**7**]). Let Y be a subset of A and $x \in A$. The subset Y of A is *strongly extensional subset* in A if and only if $(\forall x \in X)(y \in Y \Longrightarrow y \neq x \vee x \in Y)$ ([**1**], [**8**]). For $x \in A$ the subset $C_{\neq_A}(x) = \{t \in A : t \neq_A x\}$ is the *principal* strongly extensional subset of A generated by the element x.

Let $\varphi : (A, =, \neq) \longrightarrow (B, =, \neq)$ be a mapping. We say that it is φ is *strongly extensional*, in short se-mapping, if $(\forall a, b \in A)(\varphi(a) \neq \varphi(b) \Longrightarrow a \neq b)$ holds.

Let $\alpha \subseteq A \times B$ and $\beta \subseteq B \times C$ be relations. The *filled product* ([**9**]) of relations α and β is the relation

$$\beta * \alpha = \{(a, c) \in A \times C : (\forall b \in B)((a, b) \in \alpha \vee (b, c) \in \beta)\}.$$

A relation α on A is an *co-order* ([**5, 9, 10, 11**]) on A if and only if

$$\alpha \subseteq \neq, \ \alpha \subseteq \alpha * \alpha, \ \neq \subseteq \alpha \cup \alpha^{-1}.$$

In that case, for (A, α) we say co-ordered relational system.

As in [**5, 9, 10**], a relation $\tau \subseteq A \times A$ is a *co-quasiorder* on A, and (A, τ) is a co-quasiordered relational system, if and only if

$$\tau \subseteq (\subseteq \alpha) \neq, \ \tau \subseteq \tau * \tau.$$

For undefined notions and notations we refereed to the books [**1, 2, 3, 7, 8, 13**] and to articles [**5, 9, 10, 11**].

What is specificity in this text? In the first place, it is the application of the Intuitionistic Logic instead of the Classical Logic. In mathematics based on the Intuitionistic logic, there is a need to determine the so-called negative concepts through a positive approach. This aspect fragmented classical concepts in a number of different concepts mutually inequivalent within mathematics founded on the Intuitionistic logic. For example, the classical concept of order (quasi-order) relation $\leqslant \subseteq A \times A$ on a set A in the Constructive mathematics is divided into the associated pair of concepts: order and co-order (co-quasiorder) relation $\nleqslant \subseteq A \times A$. Of course, there is a need to understand their mutual connections.

New in this article are:
- Introduction of the concept of weakly isotone mapping;
- Introduction of the concept of strongly reverse isotone mapping;
- Characterizations of weakly isotone (strongly reverse isotone) mapping: Theorem 3.2 (Theorem 3.1) and Theorem 3.3; and
- Characterization of weakly isotone (strongly reverse isotone) mapping linking this mapping with the lower and the upper cones (Theorem 3.4).

2. Definition and preliminaries

Let $\varphi : (A, \alpha) \longrightarrow (B, \beta)$ be a mapping of relational systems.
- φ is called *isotone* if

$$(\forall x, y \in A)((x, y) \in \alpha \Longrightarrow (\varphi(x), \varphi(y)) \in \beta);$$

- φ is called *reverse isotone* if and only if

$$(\forall x, y \in A)((\varphi(x), \varphi(y)) \in \beta \Longrightarrow (x, y) \in \alpha).$$

Define

$$\varphi^{-1}(\beta) = \{(x, x') \in A \times A : (\varphi(x), \varphi(x')) \in \beta\}.$$

It is easy to see that φ is:
- isotone if and only if $\alpha \subseteq \varphi^{-1}(\beta)$;
- reverse isotone if and only if $\varphi^{-1}(\beta) \subseteq \alpha$.

Let us note if $\varphi : ((A, =, \neq), \alpha) \longrightarrow ((B, =, \neq), \beta)$ is a se-mapping of co-quasiordered systems, then the relation

$$\varphi^{-1}(\beta) = \{(x, y) \in A \times A : (\varphi(x), \varphi(y)) \in \beta\}$$

is also a co-quasiorder in A. Particulary, since apartnesses are relations on sets, in case of mapping $\varphi : (A, \neq_A) \longrightarrow (B, \neq_B)$ we have:
- φ is an isotone mapping if and only if

$$(\forall x, x' \in A)(x \neq_A x' \Longrightarrow \varphi(x) \neq_A \varphi(x'))$$

holds. In Constructive mathematics, in this special case, for mapping φ we say that it is an *embedding*.
- φ is a reverse isotone mapping if and only if

$$(\forall x, x' \in A)(\varphi(x) \neq_B \varphi(x')) \Longrightarrow x \neq_A x')$$

holds. So, in this case, reverse isotone mapping is a se-mapping.

Let (A, α) be a relational system. For a binary relation α on A and $a \in A$ denote $U_\alpha(a) = \{t \in A : (a, t) \in \alpha\}$ and $L_\alpha = \{t \in A : (t, a) \in \alpha\}$. The set $U_\alpha(a)$, the left class of α generated by a, is called *upper class* generated by a; the set $L_\alpha(a)$, the right class of α generated by a, is called *down class* generated by a. In case $\alpha = \neq_A$, we have $U_{\neq_A}(x) = C_{\neq_A}(x)$ for any $x \in A$.

The following two lemmas show what kind of connections exist between mappings of relational systems and down and upper classes of those relational systems.

Lemma 2.1. *If φ is a reverse isotone surjective mapping, then $U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$.*

Proof. Indeed, let $z \in U_\beta(\varphi(x))$, i.e. let $(\varphi(x), z) \in \beta$. Since φ is a surjective mapping, then there exists an element t of A such that $z = \varphi(t)$ and $(\varphi(x), \varphi(t)) \in \beta$. Since φ is a reverse isotone mapping, we have $(x, t) \in \alpha$. Thus, $t \in U_\alpha(x)$ and $z = \varphi(t) \in \varphi(U_\alpha(x))$. $\square$

Lemma 2.2. *If φ is an isotone surjective mapping, then $U_\beta(\varphi(x)) \supseteq \varphi(U_\alpha(x))$.*

Proof. Indeed, let $z \in \varphi(U_\alpha(x))$. Then, there exists an element t of A such that $z = \varphi(t)$ and $(x, t) \in \alpha$. Since φ is a isotone mapping, we have $(\varphi(x), \varphi(t)) \in \beta$. Thus, $z = \varphi(t) \in U_\beta(\varphi(x))$. $\square$

For a strongly extensional and embedding mapping $\varphi : (A, \neq_A) \longrightarrow (B, \neq_B)$, particulary, we have

$$\varphi(C_{\neq_A}(x)) = C_{\neq_B}(\varphi(x)).$$

Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be relations. With '$\circ$' we denote composition between relations in the following sense

$$S \circ R = \{(x, z) \in X \times Z : (\exists y \in Y)((x, y) \in R \wedge (y, z) \in S)\}\ .$$

Assertions exposed in the main theorem of paper [**9**] and review's comment on article [**12**] are motivations for this investigation. The following theorem is the main result of paper [**10**] (For mapping $\varphi : A \longrightarrow B$ we say that it is U-mapping if and only if $U_\beta(\varphi(x)) = \varphi(U_\alpha(x))$ holds for any element $x \in A$):

THEOREM 2.1. ([10], Theorem 2.2) *Let (A, α) and (B, β) be co-quasiordered relational systems and $\varphi : A \longrightarrow B$ a surjective se-mapping. Then:*
(1) *φ is an U-mapping if and only if $\alpha \subseteq \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha$ holds.*
(2) *φ is an U-mapping if and only if $\varphi^{-1}(\beta) \subseteq \alpha \subseteq Ker\varphi \circ \varphi^{-1}(\beta)$ holds.*

Strongly isotone mappings, in classical mathematics, were introduced by Esakia in his well-known article [**6**]. Recall that a mapping φ of an ordered set $(A, \leqslant_A)$ into an ordered set $(B, \leqslant_B)$ is said to be *strongly isotone in classical sense* if $\varphi(x) \leqslant_B y$ holds for $(x, y) \in A \times B$ if and only if there exists $x' \in A$ such that $x \leqslant_A x'$ and $\varphi(x') = y$.

For our needs, a weakly isotone (a strongly reverse isotone) mapping of a relational system into another one is a special case of a strong homomorphism of relational systems in the sense of papers [**6, 10, 12**] modified for our needs.

DEFINITION 2.1. Let $\varphi : A \longrightarrow B$ be a mapping of a relational system (A, α) into a relational system (B, β). It is said that φ to be:
- *weakly isotone* if $(\varphi(x), y) \in \beta$ holds for $(x, y) \in A \times B$ if there exists $x' \in A$ such that $(x, x') \in \alpha$ and $\varphi(x') = y$;
- *strongly reverse isotone* if $(\varphi(x), y) \in \beta$ holds for $(x, y) \in A \times B$ than there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $\varphi(x') = y$.

Let us note that in our special case, a mapping $\varphi : (A, \neq_A) \longrightarrow (B, \neq_B)$ is:
- a weakly isotone if $\varphi(x) \neq_B y$ holds if there exists an element $x' \in A$ such that $x \neq_A x'$ and $\varphi(x') = y$;
- a strongly reverse isotone if the following implication holds

$$\varphi(x) \neq_B y \Longrightarrow (\exists x' \in A)(x \neq_A x' \wedge \varphi(x') = y).$$

In the following proposition we show connection between isotone (reverse isotone) and weakly isotone (strongly reverse isotone) mappings.

THEOREM 2.2. *Let $\varphi : A \longrightarrow B$ a mapping from relational system (A, α) into relational system (B, β). Then:*
(1) *If φ is weakly isotone mapping, then φ is an isotone mapping.*
(2) *If φ is reverse isotone mapping, then φ is a strongly reverse isotone mapping.*

PROOF. (1) Let φ be a weakly isotone mapping and let x, x' arbitrary elements of A such that $(x, x') \in \alpha$. Since φ is a correspondence, then $x, x' \in dom\varphi$ and there exists an element $y \in B$ such that $(x', y) \in \varphi$. Thus, $(x, y) \in \varphi \circ \alpha$. Since φ is a weakly isotone mapping, then $(\varphi(x), y) \in \beta$. Therefore, we have $(\varphi(x), \varphi(x')) \in \beta$. So, the mapping φ is an isotone mapping.

(2) Let φ be a reverse isotone mapping and let $(\varphi(x), y) \in \beta$ for $(x, y) \in A \times im\varphi$. Then, there exists an element $x' \in A$ such that $(x', y) \in \varphi$. Since φ

is a reverse isotone mapping, then from $(\varphi(x), \varphi(x')) \in \beta$ we have $(x, x') \in \alpha$. Therefore, there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $\varphi(x') = y$. Thus, the mapping φ is a strongly reverse isotone mapping. $\square$

So, the notion of weakly isotone mapping is weaker then notion of isotone mapping and the notion of strongly reverse isotone mapping is stronger then notion of reverse isotone mapping.

In the next section we analyze some characteristics of weakly isotone and strongly reverse isotone mappings between relational systems. In addition, we prove Theorem 3.4 as a generalization of well-known Easkia's theorem.

All material exposed here, in this paper, is acceptable in the classical mathematics. We used classical notions. The use of specific constructive notions is in Theorem 2.1 only. The difference between constructive approach and classical approach is in use the Intuitionistic logic instead the classical logic. But, since the Intuitionistic logic is a part of the Classical logic, all propositions proved here are acceptable and in the classical domain.

3. The main results

Our first characterization of strongly reverse isotone mapping is given in the following proposition:

Theorem 3.1. *Let $\varphi : A \longrightarrow B$ be a mapping between two relational systems. Then, φ is a strongly reverse isotone mapping if and only if $\varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha$ holds.*

Proof. ($\Longrightarrow$) Let (x, x') be arbitrary element of $\varphi^{-1}(\beta)$, i.e. let $(\varphi(x), \varphi(x')) \in \beta$. Since φ is a strongly reverse isotone, then there exists an element $x'' \in A$ such that $(x, x'') \in \alpha$ and $\varphi(x'') = \varphi(x')$. Thus, we have $(x, x') \in Ker\varphi \circ \alpha$.

($\Longleftarrow$) Let $(\varphi(x), y) \in \beta$ be holds for $(x, y) \in A \times im\varphi$. Since $y \in im\varphi$, there exists an element $x' \in A$ such that $\varphi(x') = y$. Out of $(x, x') \in \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha$ we conclude that there exists an element $x'' \in A$ such that $(x, x'') \in \alpha$ and $\varphi(x'') = \varphi(x') = y$. So, φ is a strongly reverse isotone mapping. $\square$

Remark 3.1. Let us note that benefiting above result, we have another proof for point (2) of Theorem 2.2. Suppose that φ is a reverse isotone mapping, i.e. let $\varphi^{-1}(\beta) \subseteq \alpha$ holds. Thus, $\varphi^{-1}(\beta) \subseteq \alpha \subseteq \triangle_A \circ \alpha \subseteq Ker\varphi \circ \alpha$. So, by Theorem 3.1, φ is a strongly isotone mapping.

Corollary 3.1. *Let $\varphi : (A, \neq_A) \longrightarrow (B, \neq_B)$ be a mapping. Then φ is a strongly reverse isotone if the following implication holds*

$$\varphi(x) \neq_B \varphi(x') \Longrightarrow (\exists x'' \in A)(x \neq_A x'' \wedge \varphi(x'') = \varphi(x')).$$

For weakly isotone mapping we have the following assertion:

Theorem 3.2. *A mapping $\varphi : (A, \alpha) \longrightarrow (B, \beta)$ between relational systems is a weakly isotone mapping if and only if $\alpha \subseteq \varphi^{-1}(\beta) \circ Ker\varphi$ holds.*

Proof. Suppose that for given $(x, y) \in A \times im\varphi$ there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $\varphi(x') = y$. Out of $(x, x') \in \alpha \subseteq \varphi^{-1}(\beta) \circ Ker\varphi$ we conclude that there exists an element $x'' \in A$ such that $(x, x'') \in Ker\varphi$ and $(x'', x') \in \varphi^{-1}(\beta)$. Since $\varphi(x) = \varphi(x'')$ and $(\varphi(x''), \varphi(x')) \in \beta$, we have $(\varphi(x), y) \in \beta$. So, φ is a weakly isotone mapping.

Opposite, let φ be a weakly isotone mapping. Then, by Theorem 2.2 (point 1), φ is an isotone mapping and $\alpha \subseteq \varphi^{-1}(\beta)$ holds. Thus, we have

$$\alpha \subseteq \varphi^{-1}(\beta) \subseteq \varphi^{-1}(\beta) \circ \triangle_A \subseteq \varphi^{-1}(\beta) \circ Ker\varphi.$$

$\square$

Corollary 3.2. *Let $\varphi : (A, \neq_A) \longrightarrow (B, \neq_B)$ be a mapping. Then φ is a weakly isotone if the following implication holds*

$$x \neq_A x' \Longrightarrow (\exists x'' \in A)(\varphi(x) = \varphi(x'') \wedge \varphi(x'') \neq_B \varphi(x')).$$

In the following proposition we give another description of strongly reverse isotone mapping (weakly isotone mapping).

Theorem 3.3. *Mapping $\varphi : (A, \alpha) \longrightarrow (B, \beta)$ is a strongly reverse isotone mapping if and only if $\beta \circ \varphi \subseteq \varphi \circ \alpha$ holds.*

Mapping φ is a weakly isotone if and only if $\varphi \circ \alpha \subseteq \beta \circ \varphi$ holds.

Proof. Let φ be a strongly reverse isotone mapping between two relational systems. Then, for arbitrary element $(x, y) \in \beta \circ \varphi$ there exists an element $y' \in im\varphi$ such that $\varphi(x) = y'$ and $(y, y') \in \beta$. Thus, $(\varphi(x), y) \in \beta$. Since φ is a strongly reverse isotone mapping, there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $(x', y) \in \varphi$. Therefore, we have $(x, y) \in \varphi \circ \alpha$.

Opposite, assume that inclusion $\beta \circ \varphi \subseteq \varphi \circ \alpha$ holds. Let $(x, y) \in A \times im\varphi$ be element such that $(\varphi(x), y) \in \beta$. Hence, $(x, y) \in \beta \circ \varphi \subseteq \varphi \circ \alpha$. Thus, we conclude that there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $(x', y) \in \varphi$. It means φ is a strongly reverse isotone mapping.

The second part of this theorem is proven by analogy to the proof of the first part. Indeed. Let φ is a weakly isotone mapping and let $(x, y) \in \varphi \circ \alpha$ be an arbitrary element. Thus, there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $\varphi(x) = y$. Since φ is a weakly isotone mapping, then $(\varphi(x), y) \in \beta$. So, $(x, y) \in \beta \circ \varphi$.

Let $\varphi \circ \alpha \subseteq \beta \circ \varphi$ be holds and let for element $(x, y) \in A \times im\varphi$ there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $(x', y) \in \varphi$. Out of $(x, y) \in \varphi \circ \alpha \subseteq \beta \circ \varphi$, we conclude that there exists an element $y' \in im\varphi$ such that $\varphi(x) = y'$ and $(y', y) \in \beta$. Hence, $(\varphi(x), y) \in \beta$. So, φ is a weakly isotone mapping. $\square$

As a consequent of this theorem we have:

Corollary 3.3. *A mapping $\varphi : A \longrightarrow B$ is a weakly isotone and strongly reverse isotone mapping if and only if $\beta \circ \varphi = \varphi \circ \alpha$.*

Previous proposition gives an explanation what happen in the special case of an embedding se-mapping $\varphi : (A, \neq_A) \longrightarrow (B, \neq_B)$: $\neq_B \circ \varphi = \varphi \circ \neq_A$.

The following proposition gives us an explanation what happen in the classical approach:

Corollary 3.4. *A mapping* $\varphi : (A, \leqslant_A) \longrightarrow (B, \leqslant_B)$ *is a strongly isotone in the classical sense if and only if* $\leqslant_B \circ \varphi = \varphi \circ \leqslant_A$ *holds.*

In the following theorem we show some characteristics of such mappings binding requests for weakly isotone and strongly reverse isotone mapping with lower and upper cones.

Theorem 3.4. *Let* $\varphi : A \longrightarrow B$ *be a mapping from relational system* (A, α) *into relational system* (B, β)*. Then:*
(1) *If* φ *is a strongly reverse isotone mapping, then holds* $\varphi^{-1}(L_\beta(y)) \subseteq L_\alpha(\varphi^{-1}(y))$ *for any element* $y \in im\varphi$ *and* $U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$ *for any element* $x \in A$*.*

Opposite, if holds $\varphi^{-1}(L_\beta(y)) \subseteq L_\alpha(\varphi^{-1}(y))$ *for any element* $y \in im\varphi$ *or* $U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$ *for any element* $x \in A$*, then* φ *is a strongly reverse isotone mapping.*

(2) *If* φ *is weakly isotone mapping, then holds* $\varphi(U_\alpha(x)) \subseteq U_\beta(\varphi(x))$ *for any element* $x \in A$ *and* $L_\alpha(\varphi^{-1}(y)) \subseteq \varphi^{-1}(L_\beta(y))$ *for any element* $y \in im\varphi$*.*

Opposite, if holds $\varphi(U_\alpha(x)) \subseteq U_\beta(\varphi(x))$ *for any element* $x \in A$*, or* $L_\alpha(\varphi^{-1}(y)) \subseteq \varphi^{-1}(L_\beta(y))$ *for any element* $y \in im\varphi$ *then* φ *is a weakly isotone mapping.*

Proof. (1) Let φ be a strongly reverse isotone mapping. Then:

$$
\begin{aligned}
x \in \varphi^{-1}(L_\beta(y)) &\Longleftrightarrow \varphi(x) \in L_\beta(y)\\
&\Longleftrightarrow (\varphi(x), y) \in \beta\\
&\Longrightarrow (x, y) \in \varphi \circ \alpha\\
&\Longrightarrow (\exists x' \in A)((x, x') \in \alpha \wedge (x', y) \in \varphi)\\
&\Longrightarrow (x, \varphi^{-1}(y)) \in \alpha\\
&\Longleftrightarrow x \in L_\alpha(\varphi^{-1}(y)).
\end{aligned}
$$

$$
\begin{aligned}
y \in U_\beta(\varphi(x)) &\Longleftrightarrow (\varphi(x), y) \in \beta\\
&\Longrightarrow (\exists x' \in A)((x, x') \in \alpha \wedge \varphi(x') = y)\\
&\Longleftrightarrow (\exists x' \in A)(x' \in U_\alpha(x) \wedge \varphi(x') = y)\\
&\Longrightarrow y = \varphi(x') \in \varphi(U_\alpha(x)).
\end{aligned}
$$

Opposite, suppose that holds $\varphi^{-1}(L_\beta(y)) \subseteq L_\alpha(\varphi^{-1}(y))$ for any element $y \in im\varphi$. Let $(x, y) \in A \times im\varphi$ such that $(\varphi(x), y) \in \beta$. Thus $\varphi(x) \in L_\beta(y)$ and $x \in \varphi^{-1}(L_\beta(y)) \subseteq L_\alpha(\varphi^{-1}(y))$, i.e. $(x, \varphi^{-1}(y)) \in \alpha$. Since there exists an element $\varphi^{-1}(y) \in A$ such that $(x, \varphi^{-1}(y)) \in \alpha$ and $(\varphi^{-1}(y), y) \in \varphi$, then φ is a strongly reverse isotone mapping.

Now, suppose that holds $U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$ for any element $x \in A$ and let $(x, y) \in A \times im\varphi$ be an arbitrary element such that $(\varphi(x), y) \in \beta$. Thus $y \in U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$, and there exists an element $x' \in A$ such that $\varphi(x') = y$ and $(x, x') \in \alpha$. Therefore, φ is a strongly reverse isotone mapping.

(2) Let φ be a weakly isotone mapping. Then φ is an isotone mapping (by point (2) of Theorem 2.2) and by Lemma 2.2, we have $\varphi(U_\alpha(x)) \subseteq U_\beta(\varphi(x))$.

Opposite, suppose that for $(x,y) \in A \times im\varphi$ there exists an element $x' \in A$ such that $(x,x') \in \alpha$ and $\varphi(x') = y$. Thus, $x' \in U_\alpha(x)$ and $y = \varphi(x') \in \varphi(U_\alpha(x) \subseteq U_\beta(\varphi(x))$. It means $(\varphi(x), y) \in \beta$. So, φ is a weakly isotone mapping.

A proof for the second part of assertion is given by the following sequence:

$$\begin{aligned} x \in L_\alpha(\varphi^{-1}(y)) &\Longleftrightarrow (x, \varphi^{-1}(y)) \in \alpha \\ &\Longrightarrow (x, \varphi^{-1}(y)) \in \alpha \wedge (\varphi^{-1}(y), y) \in \varphi \\ &\Longrightarrow (x,y) \in \varphi \circ \alpha \\ &\Longrightarrow (\varphi(x), y) \in \beta \\ &\Longleftrightarrow \varphi(x) \in L_\beta(y) \\ &\Longleftrightarrow x \in \varphi^{-1}(L_\beta(y)). \end{aligned}$$

At other side, suppose that for $(x,y) \in A \times im\varphi$ exists an element $x' \in A$ such that $(x,x') \in \alpha$ and $(x', y) \in \varphi$. Thus, $x \in L_\alpha(x') = L_\alpha(\varphi^{-1}(y)) \subseteq \varphi^{-1}(L_\beta(y))$. So, $\varphi(x) \in L_\beta(y)$, i.e. $(\varphi(x), y) \in \beta$. Therefore, φ is a weakly isotone mapping. $\square$

In view of the above natural results, we now investigate under what conditions the inverse image of a principal lower-set is also a principal lower-set. Out of above theorem immediately follows the following assertion:

Corollary 3.5. *Let $\varphi : A \longrightarrow B$ be a mapping from relational system (A, α) into relational system (B, β). Then the following conditions are equivalent:*

1. The mapping φ is a weakly isotone and strongly reverse isotone mapping.

2. $\varphi^{-1}(L_\beta(y)) = L_\alpha(\varphi^{-1}(y))$ for any element $y \in im\varphi$.

3. $U_\beta(\varphi(x)) = \varphi(U_\alpha(x))$ for any element $x \in A$.

According to previous assertion, weakly isotone and strongly reverse isotone mapping between two relational systems coincide with U-mapping.

At the end of this article, we give a classical approach of these equivalences for ordered sets:

Corollary 3.6. *Let $\varphi : A \longrightarrow B$ be a mapping from ordered set $(A, \leqslant_A)$ into ordered set $(B, \leqslant_B)$. Then the following conditions are equivalent:*

1. The mapping φ is a strongly isotone mapping in the classical sense.

2. $\varphi^{-1}(L_{\leqslant_B}(y)) = L_{\leqslant_A}(\varphi^{-1}(y))$ for any element $y \in im\varphi$.

3. $U_{\leqslant_B}(\varphi(x)) = \varphi(U_{\leqslant_A}(x))$ for any element $x \in A$.

4. Some examples

Let X be a subset of semigroup $(S, \cdot)$ and $a \in S$. We define

$$aX = \{ax : x \in X\} \text{ and } Xa = \{xa : x \in X\}.$$

This section we will start with the following consequents of Theorem 3.3 and Theorem 3.4.

4.1. Examples of strongly reverse isotone mappings.

Corollary 4.1. *For a semigroup $(S, \cdot)$ ordered under order $\leqslant$ the following conditions are equivalent:*

(1) *A left translation* $l_a : (S, \leqslant) \ni x \longmapsto ax \in (S, \leqslant)$ *is a strongly reverse isotone mapping for any* $a \in S$.
(2) $\leqslant \circ\, l_a \subseteq l_a \circ \leqslant$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $U_{\leqslant}(ab) \subseteq a \cdot U_{\leqslant}(b)$.
(4) *For any elements* $a, b, c \in S$ *the following implication*

$$ab \leqslant c \Longrightarrow (\exists c' \in S)(c = ac' \wedge b \leqslant c')$$

holds.

Corollary 4.2. *For a semigroup* $(S, \cdot)$ *ordered under order* $\leqslant$ *the following conditions are equivalent:*

(1) *A right translation* $r_a : (S, \leqslant) \ni x \longmapsto xa \in (S, \leqslant)$ *is a strongly reverse isotone mapping for any* $a \in S$.
(2) $\leqslant \circ\, r_a \subseteq r_a \circ \leqslant$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $U_{\leqslant}(ab) \subseteq U_{\leqslant}(a) \cdot b$.
(4) *For any elements* $a, b, c \in S$ *the following implication*

$$ab \leqslant c \Longrightarrow (\exists c' \in S)(c = c'b \wedge a \leqslant c')$$

holds.

Remark 4.1. Semigrous $(S, \cdot)$ which satisfies condition (4) of Corollary 4.1 (Corollary 4.2) are known ([**4**]) as LSC-ordered (RSC-ordered) semigroup.

Corollary 4.3. *For a semigroup* $(S, =, \neq, \cdot)$ *ordered under co-order* $\nleqslant$ *the following conditions are equivalent:*

(1) *A left translation* $l_a : (S, \nleqslant) \ni x \longmapsto ax \in (S, \nleqslant)$ *is a strongly reverse isotone mapping for any* $a \in S$.
(2) $\nleqslant \circ\, l_a \subseteq l_a \circ \nleqslant$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $U_{\nleqslant}(ab) \subseteq a \cdot U_{\nleqslant}(b)$.
(4) *For any elements* $a, b, c \in S$ *the following implication*

$$ab \nleqslant c \Longrightarrow (\exists c' \in S)(c = ac' \wedge b \nleqslant c')$$

holds.

Corollary 4.4. *For a semigroup* $(S, =, \neq, \cdot)$ *ordered under co-order* $\nleqslant$ *the following conditions are equivalent:*

(1) *A right translation* $r_a : (S, \nleqslant) \ni x \longmapsto xa \in (S, \nleqslant)$ *is a strongly reverse isotone mapping for any* $a \in S$.
(2) $\nleqslant \circ\, r_a \subseteq r_a \circ \nleqslant$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $U_{\nleqslant}(ab) \subseteq U_{\nleqslant}(a) \cdot b$.
(4) *For any elements* $a, b, c \in S$ *the following implication*

$$ab \nleqslant c \Longrightarrow (\exists c' \in S)(c = c'b \wedge a \nleqslant c')$$

holds.

4.2. Examples of weakly isotone mappings.

Corollary 4.5. *For a semigroup* $(S, \cdot)$ *ordered under order* $\leqslant$ *the following conditions are equivalent:*

(1) *A left translation* $l_a : (S, \leqslant) \ni x \longmapsto ax \in (S, \leqslant)$ *is a weakly isotone mapping for any* $a \in S$.
(2) $l_a \circ \leqslant \subseteq \leqslant \circ l_a$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $a \cdot U_{\leqslant}(b) \subseteq U_{\leqslant}(ab)$.
(4) *For any elements* $a, b, c \in S$ *the following implication*
$$b \leqslant c \Longrightarrow ab \leqslant ac$$
holds.

Corollary 4.6. *For a semigroup* $(S, \cdot)$ *ordered under order* $\leqslant$ *the following conditions are equivalent:*

(1) *A right translation* $r_a : (S, \leqslant) \ni x \longmapsto ax \in (S, \leqslant)$ *is a weakly isotone mapping for any* $a \in S$.
(2) $r_a \circ \leqslant \subseteq \leqslant \circ r_a$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $U_{\leqslant}(b) \cdot a \subseteq U_{\leqslant}(ba)$.
(4) *For any elements* $a, b, c \in S$ *the following implication*
$$b \leqslant c \Longrightarrow ba \leqslant ca$$
holds.

Remark 4.2. Semigrous $(S, \cdot)$ which satisfies condition (4) of Corollary 4.5 (Corollary 4.6) are known ([**4**]) as semigroups with left (right) stable order.

Corollary 4.7. *For a semigroup* $(S, =, \neq, \cdot)$ *ordered under co-order* $\nleqslant$ *the following conditions are equivalent:*

(1) *A left translation* $l_a : (S, \nleqslant) \ni x \longmapsto ax \in (S, \nleqslant)$ *is a weakly isotone mapping for any* $a \in S$.
(2) $l_a \circ \nleqslant \subseteq \nleqslant \circ l_a$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $a \cdot U_{\nleqslant}(b) \subseteq U_{\nleqslant}(ab)$.
(4) *For any elements* $a, b, c \in S$ *the following implication*
$$b \nleqslant c \Longrightarrow ab \nleqslant ac$$
holds.

Corollary 4.8. *For a semigroup* $(S, =, \neq, \cdot)$ *ordered under co-order* $\nleqslant$ *the following conditions are equivalent:*

(1) *A right translation* $r_a : (S, \nleqslant) \ni x \longmapsto ax \in (S, \nleqslant)$ *is a weakly isotone mapping for any* $a \in S$.
(2) $r_a \circ \nleqslant \subseteq \nleqslant \circ r_a$ *for any* $a \in S$.
(3) *For any pair of elements* $a, b \in S$ *we have* $U_{\nleqslant}(b) \cdot a \subseteq U_{\nleqslant}(ba)$.
(4) *For any elements* $a, b, c \in S$ *the following implication*
$$b \nleqslant c \Longrightarrow ba \nleqslant ca$$
holds.

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(Home): 6, Kordunaška Street, 78000 Banja Luka, Bosnia and Herzegovina
E-mail address: bato49@hotmail.com

(Work 1) East Sarajevo University, Faculty of Education, 76300 Bijeljina, Semberski Ratari Street, Bosnia and Herzegobina

(Work 2) Banja Luka University, Faculty of Mechanical Engineering, 78000 Banja Luka, 71, Vojvoda Stepa Stepanović Street, Bosnia and Herzegovina