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Z_k-Magic Labeling of Open Star of Graphs

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Abstract. For any non-trivial abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An *A-magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k and these graphs are referred as *k-magic* graphs. In this paper we prove that the graphs such as open star of shell, flower, double wheel, cylinder, wheel, generalised Petersen, lotus inside a circle and closed helm are Z_k-magic graphs. Also we prove that super subdivision of any graph is Z_k-magic.

1. Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A detailed survey is maintained by Gallian [8]. The concept of an A-magic graph was introduced by Sedlacek [14] as follows: A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all the vertices. Stanley [16,17] noted that Z-magic graphs can be viewed in the more general context of linear homogeneous diophantine equations. Doop [1,2,3] studied the generalization of magic graphs and characterization of regular magic graphs. Lee et al. [10,11,12,18] studied the construction of magic graphs, V_4-group magic graphs, group magic graphs and group magic Eulerian graphs. For four classical products Low and Lee [13] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu et al. [15] proved that the product and composition of A-magic graphs were also A-magic. For any non-trivial abelian group A under addition a graph G is said to be A-magic if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined

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as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-magic graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k and these graphs are referred as k-magic graphs. Motivated by the concept of A-magic graph in [14] and the results in [13,15] Jeyanthi and Jeya Daisy [4,5,6,7] proved that the square graph, splitting graph, middle graph, $m\Delta_n$-snake graph, some standard subdivision graphs, cycle of some standard graphs and various families of graphs admit Z_k-magic labeling. In this paper we show that some open star of graphs admit Z_k-magic labeling. We use the following definitions in the subsequent sequel.

Definition 1.1. [9] Let G be a graph with a vertex u. The graph obtained from a star $K_{1,n}$ and $n(\geqslant 2)$ copies of G by identifying the i^{th} end vertex of $K_{1,n}$ to vertex u of the i^{th} copy of G is known as an open star of G, denoted by $OS(n.G)$.

Definition 1.2. A shell graph S_n is obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called an apex.

Definition 1.3. The flower graph Fl_n is obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.

Definition 1.4. A double wheel graph DW_n of size n can be composed of $2C_n + K_1$, that is, it consists of two cycles of size n, where the vertices of the two cycles are all connected to a common hub.

Definition 1.5. The graph $(C_n \times P_2)$ is called cylinder graph.

Definition 1.6. The wheel graph W_n is obtained by joining the vertices $v_1, v_2, \dots v_n$ of a cycle C_n to an extra vertex v called the centre.

Definition 1.7. A generalised Petersen graph $P(n, m)$, $n \geqslant 3, 1 \leqslant m < \frac{n}{2}$ is a 3 regular graph with $2n$ vertices $\{u_1, u_2, \dots u_n, v_1, v_2 \dots v_n\}$ and edges $(u_i v_i)$, $(u_i u_{i+1}), (v_i v_{i+m})$ for all $1 \leqslant i \leqslant n$, where the subscripts are taken modulo n.

Definition 1.8. The lotus inside a circle graph LC_n is obtained from the cycle $C_n : u_1, u_2, \dots u_n, u_1$ and a star $K_{1,n}$ with the central vertex v_0 and the end vertices $v_1, v_2, \dots, v_n$ by joining each u_i and $u_{i+1}(mod\ n)$.

Definition 1.9. The closed helm graph CH_n is obtained from a helm H_n by joining each pendent vertex to form a cycle.

Definition 1.10. The super subdivision graph $S^*(G)$ is obtained from G by replacing every edge e of G by a complete bipartite graph $K_{2,m}(m \geqslant 2)$ in such a way that the ends of e are merged with the two vertices of the 2-vertices part of $K_{2,m}$ after removing the edge e from G.

2. Main Results

In this section we prove that the graphs such as open star of shell, flower, double wheel, cylinder, wheel, generalised Petersen, lotus inside a circle and closed helm are Z_k-magic graphs. Also we prove that super subdivision of any graph is Z_k-magic.

THEOREM 2.1. An open star of shell graph $OS(n.S_r)$ is Z_k-magic for positive integer a and $k > (n-1)(r-2)2a$ if n is odd and for $k > (r-2)2a$ if n is even.

PROOF. Let $OS(n.S_r)$ be an open star of shell graph. Let $V(OS(n.S_r)) = \{u, u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.S_r)) = \{uu_1^j : 1 \leqslant j \leqslant n\} \cup \{u_i^j u_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{u_r^j u_1^j : 1 \leqslant j \leqslant n\} \cup \{u_1^j u_{i+2}^j : 1 \leqslant i \leqslant r-3, 1 \leqslant j \leqslant n\}$.
We consider the following two cases.
Case(i): n is odd.
Define the edge labeling $f : E(OS(n.S_r)) \to Z_k - \{0\}$ as follows:
$f(uu_1^j) = k - (r-2)2a$ for $1 \leqslant j \leqslant n-1$,
$f(uu_1^n) = (n-1)(r-2)2a$,
$f(u_1^j u_2^j) = f(u_r^j u_1^j) = a$ for $1 \leqslant j \leqslant n-1$,
$f(u_i^j u_{i+1}^j) = k - a$ for $2 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n-1$,
$f(u_1^j u_{i+2}^j) = 2a$ for $1 \leqslant i \leqslant r-3, 1 \leqslant j \leqslant n-1$,
$f(u_1^n u_{i+2}^n) = k - (n-1)2a$ for $1 \leqslant i \leqslant r-3$,
$$f(u_i^n u_{i+1}^n) = \begin{cases} k-(n-1)a, & \text{if } i = 1, r, \\ (n-1)a, & \text{if } 2 \leqslant i \leqslant r-1. \end{cases}$$
Then the induced vertex labeling $f^+ : V(OS(n.S_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.S_r))$.
Case(ii): n is even.
$$f(uu_1^j) = \begin{cases} k-(r-2)2a, & \text{if } j \ \ is\ odd, \\ (r-2)2a, & \text{if } j \ \ is\ even, \end{cases}$$
$$f(u_1^j u_2^j) = f(u_r^j u_1^j) = \begin{cases} k-a, & \text{if } j \ \ is\ odd, \\ a, & \text{if } j \ \ is\ even, \end{cases}$$
$$f(u_i^j u_{i+1}^j) = \begin{cases} a, & \text{if } i \ \ is\ odd,\ 2 \leqslant i \leqslant r-1, \\ k-a, & \text{if } i \ \ is\ even,\ 2 \leqslant i \leqslant r-1, \end{cases}$$
$$f(u_1^j u_{i+2}^j) = \begin{cases} 2a, & \text{if } j \ \ is\ odd,\ 2 \leqslant i \leqslant r-3, \\ k-2a, & \text{if } j \ \ is\ even,\ 2 \leqslant i \leqslant r-3. \end{cases}$$
Then the induced vertex labeling $f^+ : V(OS(n.S_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.S_r))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $OS(n.S_r)$ admits Z_k-magic labeling. □

EXAMPLE 2.1. Z_{25}-magic labeling of $OS(3.S_5)$ is shown in Figure 1.

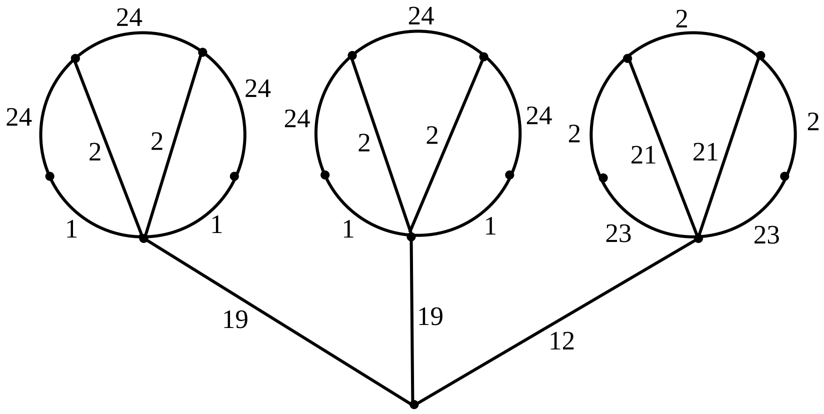

Figure 1: Z_{25}-magic labeling of $OS(3.S_5)$

THEOREM 2.2. An open star of flower graph $OS(n.Fl_r)$ is Z_k-magic for positive integer a and $k > (n-1)a$ if n is odd and for $k > 2a$ if n is even.

PROOF. Let $OS(n.Fl_r)$ be an open star of flower graph. Let $V(OS(n.Fl_r)) = \{u, w_j, v_i^j, u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.Fl_r)) = \{uv_1^j : 1 \leqslant j \leqslant n\} \cup \{v_i^j v_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{v_r^j v_1^j : 1 \leqslant j \leqslant n\} \cup \{w_j v_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\} \cup \{v_i^j u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\} \cup \{w_j u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$.
We consider the following two cases.
Case(i): n is odd.
Define the edge labeling $f : E(OS(n.Fl_r)) \to Z_k - \{0\}$ as follows:
$f(w_j v_1^j) = f(v_1^j u_1^j) = 2a$ for $1 \leqslant j \leqslant n-1$,
$f(w_j u_1^j) = k - 2a$ for $1 \leqslant j \leqslant n-1$,
$f(w_j v_i^j) = f(v_i^j u_i^j) = a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,
$f(w_j u_i^j) = k - a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,
$f(u_i^j u_{i+1}^j) = k - a$ for $1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n$,
$f(u_r^j u_1^j) = k - a$ for $1 \leqslant j \leqslant n$,
$f(uv_1^j) = k - 2a$ for $1 \leqslant j \leqslant n-1$,
$f(uv_1^n) = (n-1)2a$,
$f(w_n v_1^n) = f(v_1^n u_1^n) = k - (n-2)a$,
$f(u_1^n w_n) = (n-2)a$.
Then the induced vertex labeling $f^+ : V(OS(n.Fl_r)) \to Z_k$ is $f^+(v) \equiv 0(mod k)$ for all $v \in V(OS(n.Fl_r))$.
Case(ii): n is even.
Define the edge labeling $f : E(OS(n.Fl_r)) \to Z_k - \{0\}$ as follows:

$$f(w_j v_1^j) = f(v_1^j u_1^j) = \begin{cases} 2a, & \text{if } j \ \ is\ odd, \\ k-2a, & \text{if } j \ \ is\ even, \end{cases}$$

$$f(w_j u_1^j) = \begin{cases} k-2a, & \text{if } j \ \ is\ odd, \\ 2a, & \text{if } j \ \ is\ even, \end{cases}$$

$$f(w_jv_i^j) = f(v_i^ju_i^j) = \begin{cases} a, & \text{if } j \ \ is\ odd,\ 2 \leqslant i \leqslant r, \\ k-a, & \text{if } j \ \ is\ even,\ 2 \leqslant i \leqslant r, \end{cases}$$

$$f(w_ju_i^j) \ = \begin{cases} k-a, & \text{if } j \ \ is\ odd,\ 2 \leqslant i \leqslant r, \\ a, & \text{if } j \ \ is\ even,\ 2 \leqslant i \leqslant r, \end{cases}$$

$$f(u_i^ju_{i+1}^j) \ = \begin{cases} k-a, & \text{if } j \ \ is\ odd, \\ a, & \text{if } j \ \ is\ even, \end{cases}$$

$$f(uv_1^j) \ = \begin{cases} k-2a, & \text{if } j \ \ is\ odd, \\ 2a, & \text{if } j \ \ is\ even. \end{cases}$$

Then the induced vertex labeling $f^+ : V(OS(n.Fl_r)) \to Z_k$ is $f^+(v) \equiv \ 0(mod\ k)$ for all $v \in V(OS(n.Fl_r))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $OS(n.Fl_r)$ admits Z_k-magic labeling. □

EXAMPLE 2.2. Z_5-magic labeling of $OS(3.Fl_6)$ is shown in Figure 2.

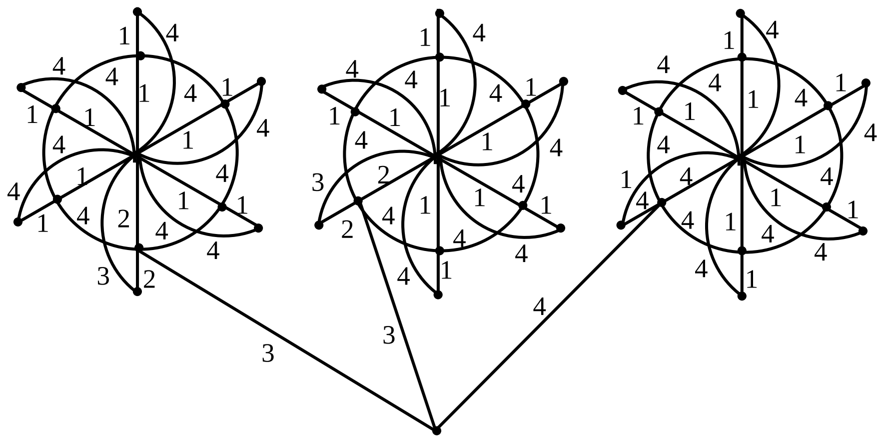

Figure 2: Z_5-magic labeling of $OS(3.Fl_6)$

THEOREM 2.3. An open star of double wheel graph $OS(n.DW_r)$ is Z_k-magic for positive integer a and $k > (n-1)4a$ if r is odd.

PROOF. Let $OS(n.DW_r)$ be an open star of double wheel graph. Let $V(OS(n.DW_r)) = \{u, w_j, v_i^j, u_i^j\ :\ 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.DW_r)) = \{uu_1^j : 1 \leqslant j \leqslant n\} \cup \{v_i^jv_{i+1}^j, u_i^ju_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{v_r^jv_1^j, u_r^ju_1^j : 1 \leqslant j \leqslant n\} \cup \{w_jv_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\} \cup \{w_ju_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$.
Define the edge labeling $f : E(OS(n.DW_r)) \to Z_k - \{0\}$ as follows:
$f(w_jv_i^j) \ = 2a$ for $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n-1$,
$f(w_nv_i^n) = k-2a$ for $1 \leqslant i \leqslant r$,
$f(w_ju_i^j) \ = k-2a$ for $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n-1$,
$f(w_nu_i^n) = 2a$ for $1 \leqslant i \leqslant r$,
$f(v_i^jv_{i+1}^j) = k-a$ for $1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n-1$,

$f(v_i^n v_{i+1}^n) = a$ for $1 \leqslant i \leqslant r-1$,
$f(v_r^j v_1^j) \ = k-a$ for $1 \leqslant j \leqslant n-1$,
$f(v_r^n v_1^n) \ = a$,

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a, & \text{if } i \ \ is\ odd,\ 1 \leqslant j \leqslant n-1, \\ 3a, & \text{if } i \ \ is\ even,\ 1 \leqslant j \leqslant n-1, \end{cases}$$

$$f(u_i^n u_{i+1}^n) = \begin{cases} (2n-3)a, & \text{if } i \ \ is\ odd, \\ k-(2n-1)a, & \text{if } i \ \ is\ even, \end{cases}$$

$f(uu_1^j) \ = 4a$ for $1 \leqslant j \leqslant n-1$,
$f(uu_1^n) \ = k-(n-1)4a$.
Then the induced vertex labeling $f^+ : V(OS(n.DW_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.DW_r))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $OS(n.DW_r)$ admits Z_k-magic labeling. □

Example 2.3. Z_9-magic labeling of $OS(3.DW_5)$ is shown in Figure 3.

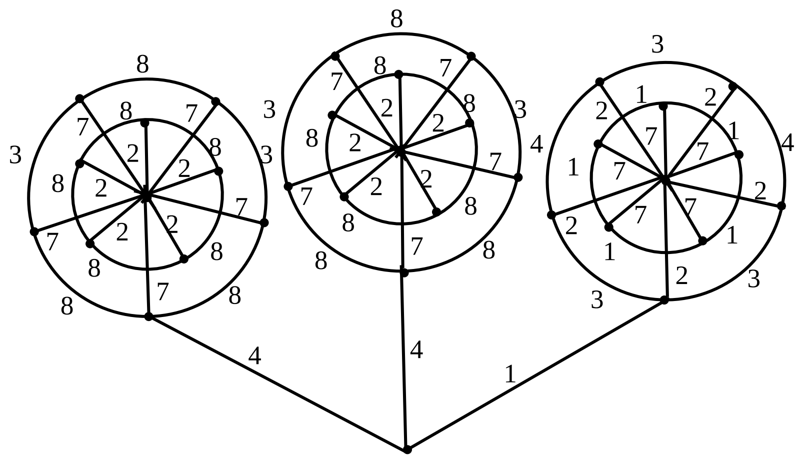

Figure 3: Z_9-magic labeling of $OS(3.DW_5)$

Theorem 2.4. An open star of cyclinder graph $OS(n.C_r \times P_2)$ is Z_k-magic for positive integer a and $k > (n-1)4a$ if r is odd.

Proof. Let $OS(n.C_r \times P_2)$ be an open star of cyclinder graph. Let $V(OS(n.C_r \times P_2)) = \{u, v_i^j, u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.C_r \times P_2)) = \{uu_1^j : 1 \leqslant j \leqslant n\} \cup \{v_i^j v_{i+1}^j, u_i^j u_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{v_r^j v_1^j, u_r^j u_1^j : 1 \leqslant j \leqslant n\} \cup \{u_i^j v_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$.
Define the edge labeling $f : E(OS(n.C_r \times P_2)) \to Z_k - \{0\}$ as follows:
$f(uu_1^j) \ = 4a$ for $1 \leqslant j \leqslant n-1$,
$f(v_i^j v_{i+1}^j) = a$ for $1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n$,
$f(v_r^j v_1^j) \ = a$ for $1 \leqslant j \leqslant n-1$,
$f(v_i^j u_i^j) \ = k-2a$ for $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,

$f(u_i^j u_{i+1}^j) = \begin{cases} k-a, & \text{if } i \ \ is\ odd,\ 1 \leqslant j \leqslant n-1, \\ 3a, & \text{if } i \ \ is\ even,\ 1 \leqslant j \leqslant n-1, \end{cases}$

$f(u_i^n u_{i+1}^n) = \begin{cases} (2n-1)a, & \text{if } i \ \ is\ odd, \\ k-(2n-3)a & \text{if } i \ \ is\ even, \end{cases}$

$f(uu_1^n) \ = k-(n-1)4a.$

Then the induced vertex labeling $f^+ : V(OS(n.C_r \times P_2)) \rightarrow Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.C_r \times P_2))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $OS(n.C_r \times P_2)$ admits Z_k-magic labeling. □

EXAMPLE 2.4. Z_{17}-magic labeling of $OS(4.C_5 \times P_2)$ is shown in Figure 4.

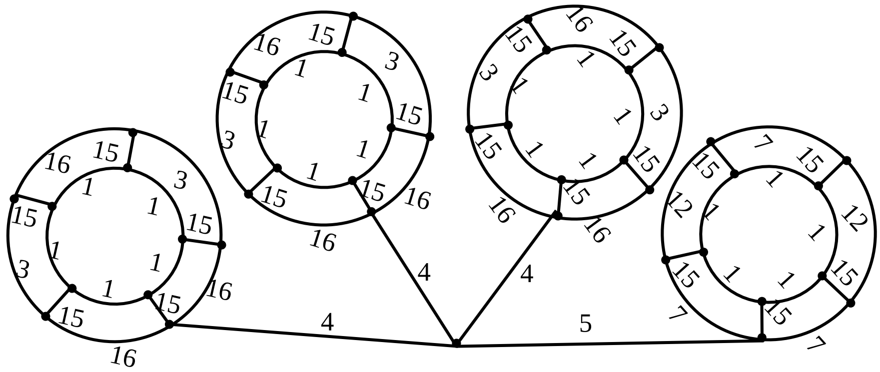

Figure 4: Z_{17}-magic labeling of $OS(4.C_5 \times P_2)$

THEOREM 2.5. An open star of wheel graph $OS(n.W_r)$ is Z_k-magic for positive integer a and $k > (n-1)(r-3)a$ if r is odd and for $k > (n-1)(r-1)a$ if r is even.

PROOF. Let $OS(n.W_r)$ be an open star of wheel graph. Let $V(OS(n.W_r)) = \{u, w_j, u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.W_r)) = \{uu_1^j : 1 \leqslant j \leqslant n\} \cup \{u_i^j u_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{u_r^j u_1^j : 1 \leqslant j \leqslant n\} \cup \{w_j u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$.
We consider the following two cases.

Case(i): r is odd.

Define the edge labeling $f : E(OS(n.W_r)) \rightarrow Z_k - \{0\}$ as follows:

$f(w_j u_1^j) \ = k-(r-1)a$ for $\ 1 \leqslant j \leqslant n-1,$

$f(w_j u_i^j) \ = a$ for $\ 2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n-1,$

$f(u_i^j u_{i+1}^j) = \begin{cases} a, & \text{if } i \ \ is\ odd,\ 1 \leqslant j \leqslant n-1, \\ k-2a, & \text{if } i \ \ is\ even,\ 1 \leqslant j \leqslant n-1, \end{cases}$

$f(uu_1^j) \ = (r-3)a$ for $\ 1 \leqslant j \leqslant n-1,$

$f(uu_1^n) \ = k-(n-1)(r-3)a,$

$f(w_n u_i^n) \ = a$ for $\ 2 \leqslant i \leqslant r,$

$f(w_n u_1^n) \ = k-(r-1)a,$

$f(u_i^n u_{i+1}^n) = \begin{cases} \frac{(n-1)(r-3)a+(r-1)a}{2}, & \text{if } i \ \ is\ odd, \\ k - \frac{(n-1)(r-3)a+(r-1)a}{2} - a, & \text{if } i \ \ is\ even. \end{cases}$

Then the induced vertex labeling $f^+ : V(OS(n.W_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.W_r))$.

Case(ii): r is even.

Define the edge labeling $f : E(OS(n.W_r)) \to Z_k - \{0\}$ as follows:

$f(w_j u_1^j) \ = k - (r-1)a$ for $1 \leqslant j \leqslant n-1$,

$f(w_j u_i^j) \ = a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n-1$,

$f(u_i^j u_{i+1}^j) = \begin{cases} a, & \text{if } i \ \ is\ odd, 1 \leqslant j \leqslant n-1, \\ k-2a, & \text{if } i \ \ is\ even, 1 \leqslant j \leqslant n-1, \end{cases}$

$f(uu_1^j) \ = ra$, for $1 \leqslant j \leqslant n-1$,

$f(uu_1^n) \ = k-(n-1)ra$,

$f(w_n u_1^n) \ = (r-1)(n-1)a$,

$f(w_n u_i^n) \ = k-(n-1)a$ for $2 \leqslant i \leqslant r$,

$f(u_i^n u_{i+1}^n) = \begin{cases} na, & \text{if } i \ \ is\ odd, \\ k-a, & \text{if } i \ \ is\ even. \end{cases}$

Then the induced vertex labeling $f^+ : V(OS(n.W_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.W_r))$. Thus f^+ is constant and it is equal to $0(mod\ k.)$ Hence $OS(n.W_r)$ admits Z_k-magic labeling. □

Example 2.5. Z_{30}-magic labeling of $OS(5.W_9)$ is shown in Figure 5.

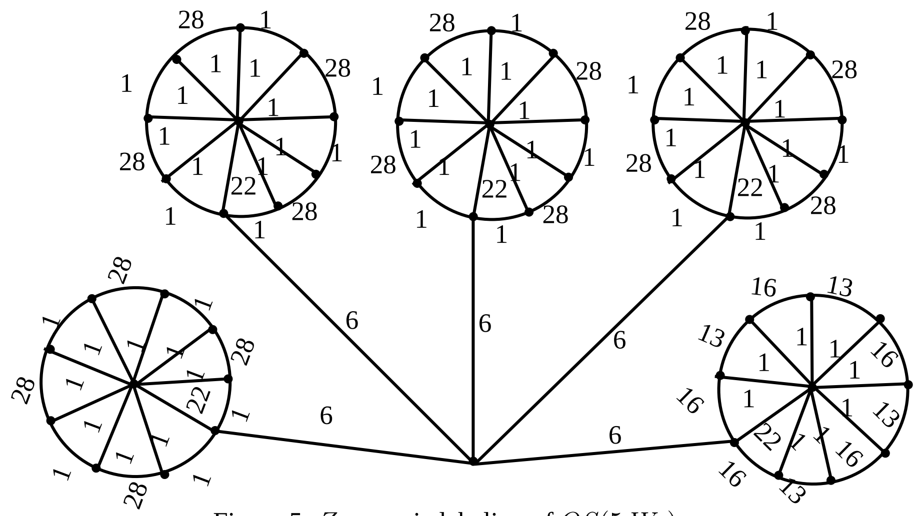

Figure 5: Z_{30}-magic labeling of $OS(5.W_9)$

Theorem 2.6. An open star of generalised Petersen graph $OS(n.P(r,m))$ is Z_k-magic for positive integer a and $k > (n-1)4a$ if r is odd and $1 \leqslant m \leqslant [\frac{r}{2}]$.

Proof. Let $OS(n.P(r,m))$ be an open star of generalised Petersen graph. Let $V(OS(n.P(r,m))) = \{u, v_i^j, u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.P(r,m))) =$

$\{uu_1^j : 1 \leqslant j \leqslant n\} \cup \{v_i^j v_{i+m}^j, u_i^j u_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{u_r^j u_1^j : 1 \leqslant j \leqslant n\} \cup \{u_i^j v_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$.
Define the edge labeling $f : E(OS(n.P(r,m))) \to Z_k - \{0\}$ as follows:
$f(uu_1^j) \ = 4a$ for $1 \leqslant j \leqslant n-1$,
$f(v_i^j v_{i+m}^j) = a$ for $1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n$,
$f(v_i^j u_i^j) \ = k - 2a$ for $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a, & \text{if } i \ \ is\ odd, 1 \leqslant j \leqslant n-1, \\ 3a, & \text{if } i \ \ is\ even, 1 \leqslant j \leqslant n-1, \end{cases}$$

$$f(u_i^n u_{i+1}^n) = \begin{cases} k-(2n-3)a, & \text{if } i \ \ is\ odd, \\ (2n-1)a, & \text{if } i \ \ is\ even, \end{cases}$$

$f(uu_1^n) \ = k - (n-1)4a$.
Then the induced vertex labeling $f^+ : V(OS(n.P(r,m))) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.P(r,m)))$. Thus f^+ is constant and it is equal to $0(mod\ k.)$ Hence $OS(n.P(r,m))$ admits Z_k-magic labeling. ☐

EXAMPLE 2.6. Z_{15}-magic labeling of $OS(4.P(5,2))$ is shown in Figure 6.

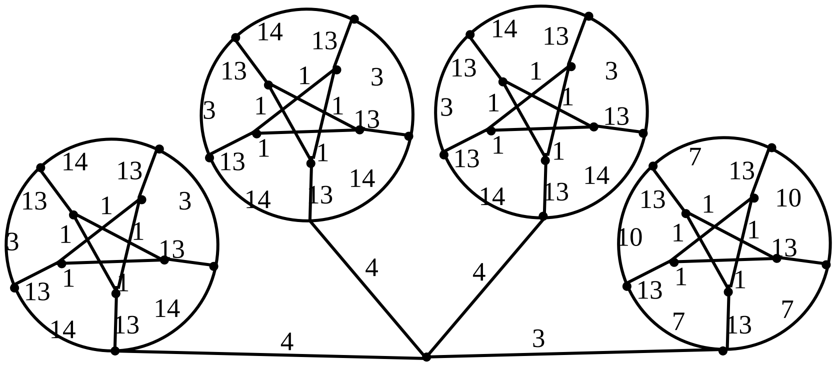

Figure 6: Z_{15}-magic labeling of $OS(4.P(5,2))$

THEOREM 2.7. An open star of lotus incide a circle graph $OS(n.LC_r)$ is Z_k-magic for positive integer a and $k > (n-1)(r-3)a$ if r is odd and for $k > (n-1)ra$ if r is even and $r > n$.

PROOF. Let $OS(n.LC_r)$ be an open star of lotus incide a circle graph. Let $V(OS(n.LC_r)) = \{u, w_j, v_i^j, u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.LC_r)) = \{uu_1^j : 1 \leqslant j \leqslant n\} \cup \{w_j v_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\} \cup \{v_i^j u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\} \cup \{v_i^j u_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{v_r^j u_1^j : 1 \leqslant j \leqslant n\} \cup \{u_i^j u_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{u_r^j u_1^j : 1 \leqslant j \leqslant n\}$.
We consider the following two cases.
Case(i): r is odd.

Define the edge labeling $f : E(OS(n.LC_r)) \to Z_k - \{0\}$ as follows:
$f(uu_1^j) \ = k-(r-3)a$ for $1 \leqslant j \leqslant n-1$,
$f(uu_1^n) \ = (n-1)(r-3)a$,
$f(w_jv_i^j) \ = a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,
$f(w_jv_1^j) \ = k-(r-1)a$ for $1 \leqslant j \leqslant n$,
$f(u_1^jv_1^j) \ = (r-2)a$ for $1 \leqslant j \leqslant n$,
$f(u_i^jv_i^j) \ = k-2a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,
$f(v_i^ju_{i+1}^j) = a$ for $1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n$,
$f(v_r^ju_1^j) \ = a$ for $1 \leqslant j \leqslant n$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} k-a, & \text{if } i \ \ is\ odd,\ 1 \leqslant j \leqslant n-1, \\ 2a, & \text{if } i \ \ is\ even,\ 1 \leqslant j \leqslant n-1, \end{cases}$$

$$f(u_i^nu_{i+1}^n) = \begin{cases} k-\frac{(n+1)(r-3)a}{2}, & \text{if } i \ \ is\ odd, \\ \frac{(n+1)(r-3)a}{2}+a, & \text{if } i \ \ is\ even. \end{cases}$$

Then the induced vertex labeling $f^+ : V(OS(n.LC_r)) \to Z_k$ is $f^+(v) \equiv 0(modk)$ for all $v \in V(OS(n.LC_r))$.

Case(ii): r is even and $r > n$

Define the edge labeling $f : E(OS(n.LC_r)) \to Z_k - \{0\}$ as follows:
$f(uu_1^j) \ = k-ra$ for $1 \leqslant j \leqslant n-1$,
$f(uu_1^n) \ = (n-1)ra$,
$f(w_jv_i^j) \ = a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n-1$,
$f(w_jv_1^j) \ = k-(r-1)a$ for $1 \leqslant j \leqslant n-1$,
$f(u_1^jv_1^j) \ = (r-2)a$ for $1 \leqslant j \leqslant n-1$,
$f(u_i^jv_i^j) \ = k-2a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n-1$,
$f(v_i^ju_{i+1}^j) = a$ for $1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n-1$,
$f(v_r^ju_1^j) \ = a$ for $1 \leqslant j \leqslant n-1$,

$$f(u_i^ju_{i+1}^j) = \begin{cases} k-a, & \text{if } i \ \ is\ odd,\ 1 \leqslant j \leqslant n-1, \\ 2a, & \text{if } i \ \ is\ even,\ 1 \leqslant j \leqslant n-1, \end{cases}$$

$f(w_nv_i^n) \ = k-(n-1)a$ for $2 \leqslant i \leqslant r$,
$f(w_nv_1^n) \ = (n-1)(r-1)a$,
$f(u_1^nv_1^n) \ = k-[(n-1)(r-1)-1]a$,
$f(u_i^nv_i^n) \ = na$ for $2 \leqslant i \leqslant r$,
$f(v_i^nu_{i+1}^n) = k-a$ for $1 \leqslant i \leqslant r-1$,
$f(v_r^nu_1^n) \ = k-a$,

$$f(u_i^nu_{i+1}^n) = \begin{cases} k-ra, & \text{if } i \ \ is\ odd, \\ (r-n+1)a, & \text{if } i \ \ is\ even. \end{cases}$$

Then the induced vertex labeling $f^+ : V(OS(n.LC_r)) \to Z_k$ is $f^+(v) \equiv \ 0(mod\ k)$ for all $v \in V(OS(n.LC_r))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $OS(n.LC_r)$ admits Z_k-magic labeling. □

Example 2.7. Z_{20}-magic labeling of $OS(4.LC_6)$ is shown in Figure 7.

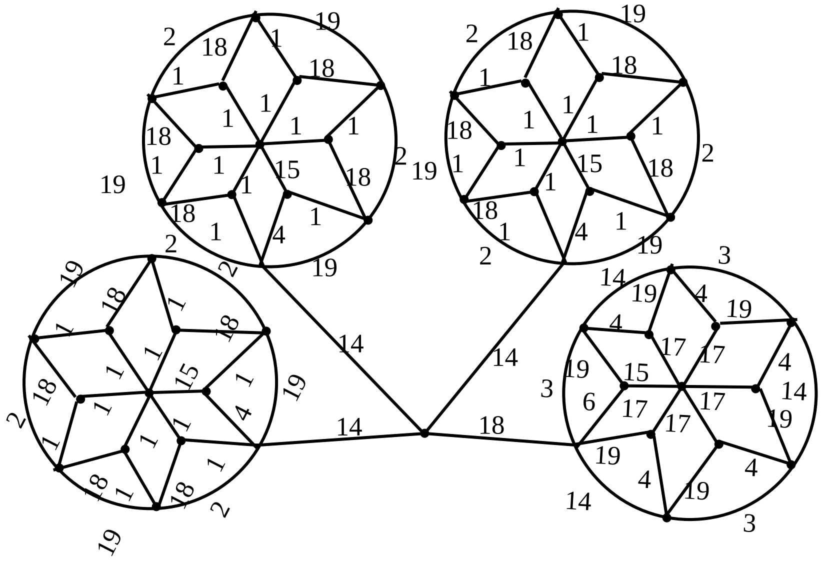

Figure 7: Z_{20}-magic labeling of $OS(4.LC_6)$

THEOREM 2.8. An open star of closed helm graph $OS(n.CH_r)$ is Z_k-magic for positive integer a and $k > (n-1)(r-1)a$ if r is odd.

PROOF. Let $OS(n.CH_r)$ be an open star of closed helm graph. Let $V(OS(n.CH_r)) = \{u, w_j, v_i^j, u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ and $E(OS(n.CH_r)) = \{uu_1^j : 1 \leqslant j \leqslant n\} \cup \{w_j v_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\} \cup \{v_i^j u_i^j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\} \cup \{v_i^j v_{i+1}^j, u_i^j u_{i+1}^j : 1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n\} \cup \{v_r^j v_1^j, u_r^j u_1^j : 1 \leqslant j \leqslant n\}$.
Define the edge labeling $f : E(OS(n.CH_r)) \to Z_k - \{0\}$ as follows:
$f(uu_1^j) \ = k-(r-1)a$ for $1 \leqslant j \leqslant n-1$,
$f(uu_1^n) \ = (n-1)(r-1)a$,
$f(w_j v_i^j) \ = a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,
$f(w_j v_1^j) \ = k-(r-1)a$ for $1 \leqslant j \leqslant n$,
$f(u_1^j v_1^j) \ = k-(r-1)a$ for $1 \leqslant j \leqslant n$,
$f(u_i^j v_i^j) \ = k-a$ for $2 \leqslant i \leqslant r, 1 \leqslant j \leqslant n$,

$$f(v_i^j v_{i+1}^j) = \begin{cases} (r-1)a, & \text{if } i \ \ is\ odd,\ 1 \leqslant j \leqslant n, \\ k-(r-1)a, & \text{if } i \ \ is\ even,\ 1 \leqslant j \leqslant n, \end{cases}$$

$$f(u_i^j u_{i+1}^j) = \begin{cases} (r-1)a, & \text{if } i \ \ is\ odd,\ 1 \leqslant j \leqslant n-1, \\ k-(r-2)a, & \text{if } i \ \ is\ even,\ 1 \leqslant j \leqslant n-1, \end{cases}$$

$$f(u_i^n u_{i+1}^n) = \begin{cases} k - \frac{(r-1)(n-2)a}{2}, & \text{if } i \ \ is\ odd, \\ \frac{(r-1)(n-2)a}{2} + a, & \text{if } i \ \ is\ even. \end{cases}$$

Then the induced vertex labeling $f^+ : V(OS(n.CH_r)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(OS(n.CH_r))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $OS(n.CH_r)$ admits Z_k-magic labeling. □

EXAMPLE 2.8. Z_{13}-magic labeling of $OS(4.CH_5)$ is shown in Figure 8.

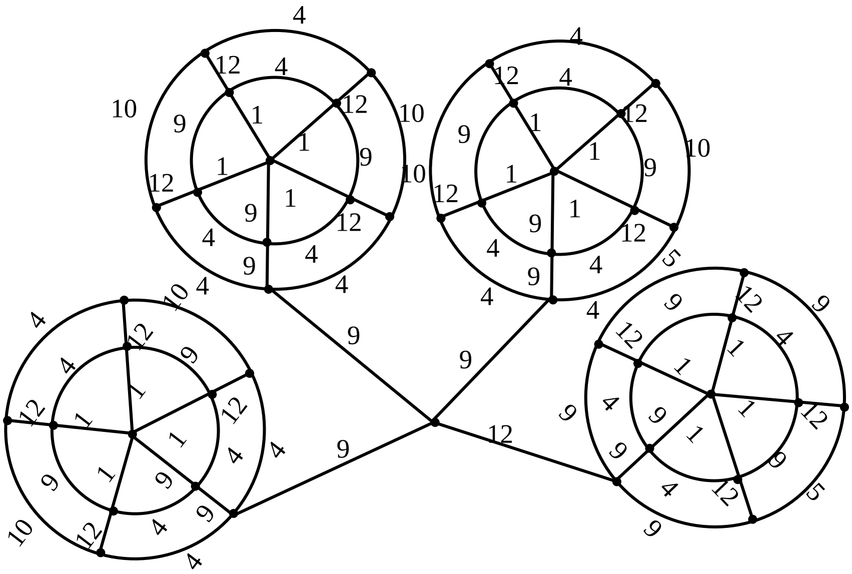

Figure 8: Z_{13}-magic labeling of $OS(4.CH_5)$

THEOREM 2.9. The super subdivision of any graph G is Z_k-magic for positive integer a and $k > (m-1)a$.

PROOF. Let G be a graph and $S^*(G)$ be the super subdivision of graph G. If uv be the edge of G then $uu_1^j, u_1^j v : 1 \leqslant j \leqslant m$ be the edges of $S^*(G)$ corresponding to uv.
Define the edge labeling $f : E(S^*(G)) \to Z_k - \{0\}$ as follows:
$f(uu_1^1) = k - (m-1)a$; $f(u_1^1 v) = (m-1)a$,
$f(uu_1^j) = a$, for $2 \leqslant j \leqslant m$,
$f(u_1^j v) = k - a$, for $2 \leqslant j \leqslant m$.
Then the induced vertex labeling $f^+ : V(S^*(G)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(S^*(G))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $S^*(G)$ admits Z_k-magic labeling. □

We conclude this paper with the following open problem for further research.
Open problem. Determine the Z_k-magic labeling of an open star of regular graphs.

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