Fourier-Laplace Transforms of Some Special Functions

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ABSTRACT— During the last decade or so there have been significant generalizations of the idea of integral transforms. Many new uses of the transform method in engineering and physics applications are found. Some of these new applications have prompted the development of very specialized transforms, their roots, knowledge of the properties and uses of classical integral transforms, such as Fourier Transform and Laplace transform and having considered that they are as important today they have been for the last century or so, they have been given a more extensive treatment. The main aim of this paper is to find the Fourier-Laplace transforms of some special functions and this will be used for solving various differential and integral equations.

KEYWORDS— Fourier transform, Laplace transform, Fourier-Laplace transform, Integral Transform, Differential Equation, Parameter, Generalized function.

1. INTRODUCTION

The transform is a method to convert a signal from one domain to another domain for extracting some other information contained in the signal which cannot be extracted from the signal in first domain. One of the important families of transforms is 'Integral Transform'. Actually, integral transform is an operator used to transform a signal into its equivalent form with the help of a 'kernel' function by integrating the kernel multiplied signal. The integration process involved in transformation has conferred the name as 'Integral Transform'.

Fourier transforms play an important part in the theory of many branches of science. A waveform-optical, electrical or acoustical- and its spectrum are appreciated equally as physically picturable and measurable entities, an oscilloscope enables us to see an electrical waveform and a spectroscope or spectrum analyzer enables us to see optical or electrical spectra [1]. Our acoustical appreciation is even more direct, since the ear hears spectra. Wave forms and spectra are Fourier transforms of each other; the Fourier transformation is thus an eminently physical relationship.

The theory of Laplace transforms referred to as operational calculus has in recent years become an essential part of the mathematical background required of engineers, physicists, mathematicians and other scientist. This is because in addition to being of great theoretical interest in itself, Laplace transform methods provide easy and effective means for the solution of many problems arising in various fields of science and engineering [2].

So these Fourier and Laplace transforms have various uses in many fields separately. On combining these two transforms i.e. Fourier-Laplace transforms also used for solving differential and integral equations. In this paper we find the Fourier-Laplace transform of some special functions which is help for solving differential equations. This paper is planned as follows:

Preliminary results are given in section 2. In section 3, we have find the Fourier-Laplace transform of some special functions. Lastly conclusions are given in section 4. Notations and terminology as per Zemanian. [3], [4].

2. PRELIMINARY RESULTS

The Fourier transform with parameter s of $f(t)$ denoted by $F\left[f(t)\right]=F(s)$ and is given by

$$F\left[f(t)\right]=F(s)=\int_{-\infty}^{\infty}e^{-ist}f(t)dt\ , \quad \text{for parameter } s>0\,. \tag{2.1}$$

The Laplace transform with parameter p of $f(x)$ denoted by $L\left[f(x)\right]=F(p)$ and is given by

$$L\left[f(x)\right]=F(p)=\int_{0}^{\infty}e^{-px}f(x)dx\ , \text{ for parameter } p>0\,. \tag{2.2}$$

The Conventional Fourier-Laplace transform is defined as

$$FL\{f(t,x)\} = F(s,p) = \int_{-\infty}^{\infty}\int_{0}^{\infty} f(t,x)K(t,x)dtdx \ , \tag{2.3}$$

where, $K(t,x) = e^{-i(st-ipx)}$.

3. FOURIER-LAPLACE TRANSFORMS OF SOME SPECIAL FUNCTIONS

3.1. If $FL\{f(t,x)\}(s,p)$ denotes generalized Fourier-Laplace transform of $f(t,x)$ then

$$FL\{1\} = \frac{-i}{sp}$$

Proof:- We have

$$FL\{f(t,x)\} = \int_0^\infty\int_0^\infty e^{-i(st-ipx)} f(t,x)dtdx$$

$$FL[1] = \int_0^\infty\int_0^\infty e^{-i(st-ipx)}(1)dtdx$$

$$= \int_0^\infty e^{-ist}dt.\int_0^\infty e^{-px}dx$$

$$= \left[\frac{e^{-ist}}{-is}\right]_0^\infty \cdot \left[\frac{e^{-px}}{-p}\right]_0^\infty = \frac{-1}{is}\left[e^{-ist}\right]_0^\infty \cdot \left(\frac{-1}{p}\right)\left[e^{-px}\right]_0^\infty$$

$$= \frac{1}{isp}[0-1][0-1] = \frac{1}{isp} = \frac{-i}{sp}$$

$$\therefore FL[1] = \frac{-i}{sp}$$

3.2. If $FL\{f(t,x)\}(s,p)$ denotes generalized Fourier-Laplace transform of $f(t,x)$ then

$$FL\{(\delta(t-a)\delta(x-b))\} = K(a,b,s,p)$$

Proof: $FL\{(\delta(t-a)\delta(x-b))\} = \int_0^\infty\int_0^\infty e^{-i(st-ipx)}\{(\delta(t-a)\delta(x-b))\}dtdx$

$$= \int_0^\infty \delta(t-a)e^{-ist}dt\int_0^\infty \delta(x-b)e^{-px}dx$$

$$= e^{-isa}.e^{-pb} = e^{-i(sa-ipb)} = K(a,b,s,p)$$

We know that $\int_0^\infty \delta(t-a)\phi(t)\,dt = \phi(a)$ Also $\int_0^a \delta(t-a)\phi(t)\,dt = \phi(a)$

$$\therefore FL\left\{\left(\delta(t-a)\delta(x-b)\right)\right\} = K(a,b,s,p)$$

3.3. If $FL\{f(t,x)\}(s,p)$ denotes generalized Fourier-Laplace transform of $f(t,x)$ then

$$FL\{tx\} = \frac{-1}{s^2p^2}$$

Proof:
$$FL\{tx\} = \int_0^\infty\int_0^\infty e^{-i(st-ipx)}(tx)\,dtdx$$

$$= \int_0^\infty e^{-ist}t\,dt.\int_0^\infty e^{-px}x\,dx$$

$$= \left[\left(t\frac{e^{-ist}}{-is}\right)_0^\infty - \int_0^\infty 1\frac{e^{-ist}}{-is}dt\right].\left[\left(x\frac{e^{-px}}{-p}\right)_0^\infty - \int_0^\infty 1\frac{e^{-px}}{-p}dx\right]$$

$$= \left[(0-0)+\frac{1}{is}\int_0^\infty e^{-ist}dt\right].\left[(0-0)+\frac{1}{p}\int_0^\infty e^{-px}dx\right]$$

$$= \frac{1}{isp}\left[\frac{e^{-ist}}{-is}\right]_0^\infty\left[\frac{e^{-px}}{-p}\right]_0^\infty$$

$$= -\frac{1}{s^2p^2}(0-1)(0-1) = -\frac{1}{s^2p^2}$$

$$\therefore FL\{tx\} = \frac{-1}{s^2p^2}$$

3.4. If $FL\{f(t,x)\}(s,p)$ denotes generalized Fourier-Laplace transform of $f(t,x)$ then

$$FL\{t^nx^n\} = \frac{(-1)^{-n-1}(i)^{-n-1}(n!)^2}{s^{n+1}p^{n+1}}$$

Proof: $FL\{t^nx^n\} = \int_0^\infty\int_0^\infty e^{-i(st-ipx)}(t^nx^n)\,dtdx$

$$=\int_0^\infty e^{-ist}t^n dt.\int_0^\infty e^{-px}x^n dx$$

$$=\left[\left(t^n\frac{e^{-ist}}{-is}\right)_0^\infty-\int_0^\infty nt^{n-1}\frac{e^{-ist}}{-is}dt\right].\left[\left(x^n\frac{e^{-px}}{-p}\right)_0^\infty-\int_0^\infty nx^{n-1}\frac{e^{-px}}{-p}dx\right]$$

$$=\left[0+\frac{n}{is}\int_0^\infty t^{n-1}e^{-ist}dt\right].\left[0+\frac{n}{p}\int_0^\infty x^{n-1}e^{-px}dx\right]$$

$$=\frac{n}{is}\left[\left(t^{n-1}\frac{e^{-ist}}{-is}\right)_0^\infty-\int_0^\infty (n-1)t^{n-2}\frac{e^{-ist}}{-is}dt\right].\frac{n}{p}\left[\left(x^{n-1}\frac{e^{-px}}{-p}\right)_0^\infty-\int_0^\infty (n-1)x^{n-2}\frac{e^{-px}}{-p}dx\right]$$

$$=\left[\frac{n}{is}\frac{(n-1)}{is}\int_0^\infty t^{n-2}e^{-ist}dt\right].\left[\frac{n}{p}\frac{(n-1)}{p}\int_0^\infty x^{n-2}e^{-px}dx\right]$$

$$\vdots$$

$$\vdots$$

$$=\left[\frac{n(n-1)(n-2)(n-3)\cdots\cdots 2.1}{(is)^n}\int_0^\infty e^{-ist}dt\right]\left[\frac{n(n-1)(n-2)(n-3)\cdots\cdots 2.1}{(p)^n}\int_0^\infty e^{-px}dx\right]$$

$$=\frac{n!}{(is)^n}\left(\frac{e^{-ist}}{-is}\right)_0^\infty\frac{n!}{(p)^n}\left(\frac{e^{-px}}{-p}\right)_0^\infty$$

$$=\frac{(n!)^2}{(is)^{n+1}p^{n+1}}=\frac{(-1)^{-n-1}(i)^{-n-1}(n!)^2}{s^{n+1}p^{n+1}}$$

$$\therefore FL\left\{t^n x^n\right\}=\frac{(-1)^{-n-1}(i)^{-n-1}(n!)^2}{s^{n+1}p^{n+1}}$$

3.5. If $FL\left\{f(t,x)\right\}(s,p)$ denotes generalized Fourier-Laplace transform of $f(t,x)$ then

$$FL\left\{e^{at+bx}\right\}=\frac{1}{(is-a)(p-b)}$$

Proof: $FL\left\{e^{at+bx}\right\}=\int_0^\infty\int_0^\infty e^{-i(st-ipx)}\left(e^{at+bx}\right)dtdx$

$$= \int_0^\infty e^{-ist} e^{at} dt . \int_0^\infty e^{-px} e^{bx} dx$$

$$= \int_0^\infty e^{-(is-a)t} dt . \int_0^\infty e^{-(p-b)x} dx$$

$$= \left(\frac{e^{-(is-a)t}}{-(is-a)} \right)_0^\infty \left(\frac{e^{-(p-b)x}}{-(p-b)} \right)_0^\infty = \frac{1}{(is-a)(p-b)}$$

$$\therefore FL\left\{ e^{at+bx} \right\} = \frac{1}{(is-a)(p-b)}$$

3.6. If $FL\{f(t,x)\}(s,p)$ denotes generalized Fourier-Laplace transform of $f(t,x)$ then

$$FL\{\sin at \sin bx\} = \frac{ab}{(a^2-s^2)(b^2+p^2)}$$

Proof: $$FL\{\sin at \sin bx\} = \int_0^\infty \int_0^\infty e^{-i(st-ipx)} (\sin at \sin bx) dtdx$$

$$= \int_0^\infty e^{-ist} \sin at dt . \int_0^\infty e^{-px} \sin bx dx$$

$$= \left(e^{-ist} \frac{(-is \sin at - a\cos at)}{a^2-s^2} \right)_0^\infty \left(e^{-px} \frac{(-p \sin bx - b\cos bx)}{b^2+p^2} \right)_0^\infty$$

$$= \left(0 + \frac{a}{a^2-s^2} \right) \left(0 + \frac{b}{b^2+p^2} \right) = \frac{ab}{(a^2-s^2)(b^2+p^2)}$$

$$\therefore FL\{\sin at \sin bx\} = \frac{ab}{(a^2-s^2)(b^2+p^2)}$$

3.7. If $FL\{f(t,x)\}(s,p)$ denotes generalized Fourier-Laplace transform of $f(t,x)$ then

$$FL\{\cos at \cos bx\} = \frac{isp}{(a^2-s^2)(b^2+p^2)}$$

Proof: $$FL\{\cos at \cos bx\} = \int_0^\infty \int_0^\infty e^{-i(st-ipx)} (\cos at \cos bx) dtdx$$

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$$= \int_0^\infty e^{-ist} \cos at dt . \int_0^\infty e^{-px} \cos bx dx$$

$$= \left(e^{-ist} \frac{(-is\cos at + a\sin at)}{a^2 - s^2} \right)_0^\infty \left(e^{-px} \frac{(-p\cos bx + b\sin bx)}{b^2 + p^2} \right)_0^\infty$$

$$= \left(0 + \frac{is}{a^2 - s^2} \right)\left(0 + \frac{p}{b^2 + p^2} \right) = \frac{isp}{(a^2 - s^2)(b^2 + p^2)}$$

$$\therefore FL\{\cos at \cos bx\} = \frac{isp}{(a^2 - s^2)(b^2 + p^2)}$$

FOURIER-LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

Sr. No.	$f(t,x)$	$FL\{f(t,x)\} = F(s,p)$
1	1	$\frac{-i}{sp}$
2	$\delta(t-a)\delta(x-b)$	$e^{-i(sa-ipb)} = K(a,b,s,p)$
3	tx	$\frac{-1}{s^2 p^2}$
4	$t^n x^n$	$\frac{(-1)^{-n-1}(i)^{-n-1}(n!)^2}{s^{n+1} p^{n+1}}$
5	e^{at+bx}	$\frac{1}{(is-a)(p-b)}$
6	$\sin at \sin bx$	$\frac{ab}{(a^2-s^2)(b^2+p^2)}$
7	$\cos at \cos bx$	$\frac{isp}{(a^2-s^2)(b^2+p^2)}$
8	$\sinh(at)\sinh(bx)$	$\frac{-ab}{(s^2+a^2)(p^2-b^2)}$
9	$\cosh(at)\cosh(bx)$	$\frac{-isp}{(s^2+a^2)(p^2-b^2)}$

4. CONCLUSION

In this work we have find the Fourier-Laplace transforms of some special functions and this will be used for solving various differential and integral equations.

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