International Journal of Engineering Research and General Science Volume 3, Issue 6, November-December, 2015
ISSN 2091-2730

Analysis of Analytical and Numerical Methods of Epidemic Models

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Abstract— In this paper, we study SIR epidemic models for a given constant population. These mathematical models are described by nonlinear first order differential equations. First, we find the analytical solution by using the Differential Transformation Method. We then compute the numerical solution by using fourth-order Runge-Kutta Method and compare the analytical solution with the numerical. The profiles of the solutions are provided, from which we infer that the analytical and numerical solutions agreed very well.

Keywords— SIR epidemic model, Susceptible class, Infective class. Recovered class, Nonlinear Ordinary Differential Equations, Differential Transformation Method, Runge-Kutta Method

Introduction

Epidemic models are tools to analyse the spread and control of infectious diseases. The models that describe what happens on the average at the population scale are called deterministic or compartmental models. They fit well large populations.

In this paper, we study the SIR epidemic model, which is a standard compartmental model used to describe many epidemiological diseases [1]. This model was formulated by A. G. McKendrick and W. O. Kermack in 1927 [2]. We solve the resulting differential equations of the model by analytical as well as numerical methods. To find the analytical solution, we use Differential Transformation Method, which expresses the dependent variable explicitly as function of independent variable in the form of a convergent series with easily computable components.

Objective

The purpose of this article is to translate the real world problem of the spread of infectious diseases into mathematical vocabulary and to find the solution of it with the help of Mathematics. Simply formation of mathematical model of infectious diseases is not enough for a disease control. Unless we know an efficient method to solve the mathematical model, we cannot help in any detection or therapy program.

This work is an effort to help in various infectious disease control programs by providing a practical, efficient and accurate method to solve a mathematical model.

Formulation of SIR Epidemic Model

In this model, a fixed population with only the following three compartments is considered :

1. **s (t)** : It represents the number of susceptible at time t, i.e., the number of individuals who do not have the disease at time t but could get it.
2. ***i* (t)** : It represents the number of infective at time t, i.e., the number of individuals who have the disease at time t and can transmit it to others.
3. ***r* (t)** : It represents the number of individuals who have been infected but recovered from the disease at time t. The individuals in this category are not able to be infected again or transmit the infection to others, i.e., they acquire permanent immunity or they have been placed in isolation or they have died.

If *n* be the size of the population at any time t, then the differential equations for the SIR model are:

$$\frac{d\,s(t)}{d\,t} = -\frac{\beta\, s(t)\, i(t)}{n} \qquad \text{.........(1)}$$

$$\frac{d\,i(t)}{d\,t} = \frac{\beta\, s(t)\, i(t)}{n} - \gamma\, i(t) \qquad \text{.........(2)}$$

ISSN 2091-2730

$$\frac{d\,r(t)}{d\,t} = -\gamma\, i\,(t) \qquad \text{.........(3)}$$

with the initial conditions

$$\left.\begin{array}{l} s(0) = s_0 > 0 \\ i\,(0) = i_0 > 0 \\ r(0) = r_0 = 0 \end{array}\right\} \qquad \text{.........(4)}$$

It can be seen that $\frac{d}{d\,t}\left[s(t) + i\,(t) + r(t)\right] = 0$, therefore it is true that the population size is constant, i.e., s(t) + i(t) + r(t) = *n*.

Analytical Solution of SIR Epidemic Model

To find the analytical solution, we use Differential Transformation Method, which was first introduced by Zhou [3] for solving linear and nonlinear initial value problems in electrical circuit analysis. But, now a days, the method has been applied to solve a variety of problems that are modelled with differential equations.

The concept of differential transformation is derived from the Taylor series expansion. In this method, given system of differential equations and related initial conditions are transformed into a system of recurrence equations that finally leads to a system of algebraic equations whose solutions are the coefficients of a power series solution.

Taylor series expansion of a function $f\,(x)$ about the point $x = 0$ is as follows :

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k\,!}\left[\frac{d^k f}{dx^k}\right]_{x=0} \qquad \text{......(5)}$$

Definition 1. The differential transformation F(k) of a function $f\,(x)$ is defined as follows :

$$F(k) = \frac{1}{k\,!}\left[\frac{d^k f}{dx^k}\right]_{x=0} \qquad \text{......(6)}$$

Definition 2. It follows from equations (5) and (6) that the differential inverse transformation $f\,(x)$ of F (k) is given by :

$$f(x) = \sum_{k=0}^{\infty} x^k F(k) \qquad \text{......(7)}$$

Using equations (6) and (7), the following mathematical operations can be obtained :

1. If $f\,(x) = g(x) \pm h(x)$, then $F\,(k) = G(k) \pm H(k)$
2. If $f\,(x) = \text{c}\,g(x)$, then $F\,(k) = \text{c}\,G(k)$, where c is a constant
3. If $f(x) = \frac{dg(x)}{dx}$, then $F\,(k) = (k+1)G(k+1)$
4. If $f(x) = \frac{d^m g(x)}{dx^m}$, then $Y\,(k) = (k+1)(k+2).......(k+m)G(k+m)$
5. If $f(x) = 1$, then $F(k) = \delta\,(k)$
6. If $f(x) = x$, then $F(k) = \delta\,(k-1)$
7. If $f(x) = x^m$, then $F(k) = \delta\,(k\text{-}m) = \begin{cases} 1, \text{ if } k = m \\ 0, \text{ if } k \neq m \end{cases}$
8. If $f\,(x) = g(x)\,h(x)$, then $F(k) = \sum_{m=0}^{k} H(m)G(k-m)$

9. If $f(x) = e^{mx}$, then $F(k) = \dfrac{m^k}{k!}$

10. If $f(x) = (1+x)^m$, then $F(k) = \dfrac{m(m-1)(m-2)\dots\dots(m-k+1)}{k!}$

Now, we consider the SIR Model given by equations (1), (2) and (3) with the initial conditions

s(0) = 100, i(0) = 30, (0) = 20, n = 150, β = 0.1 and γ = 0.2 …..(8)

If S(k), I(k) and R(k) denote the differential transformation of s(t), i(t) and r(t) respectively, then the following recurrence relations of equations (1), (2) and (3) can be obtained :

$$S(k+1) = \frac{1}{k+1}\left[-\frac{\beta}{n}\sum_{m=0}^{k} S(m)\,I(k-m)\right] \quad ......(9)$$

$$I(k+1) = \frac{1}{k+1}\left[\frac{\beta}{n}\sum_{m=0}^{k} S(m)\,I(k-m) - \gamma\, I(k)\right] \quad ......(10)$$

$$R(k+1) = \frac{1}{k+1}\left[\gamma\, I(k)\right] \quad ......(11)$$

with the initial conditions

S(0) = 100, I(0) = 30, R(0) = 20, n = 150, β = 0.1 and γ = 0.2 …..(12)

$$\therefore\ \mathrm{S}(1) = -\frac{0.1}{150} S(0) I(0) = -\frac{0.1}{150}\times 100\times 30 = -2$$

$$\mathrm{I}(1) = \frac{0.1}{150} S(0) I(0) - 0.2(I(0)) = \frac{0.1}{150}\times 100\times 30 - 0.2(30) = 2-6 = -4$$

$$\mathrm{R}(1) = 0.2(I(0)) = 0.2\,(30) = 6$$

$$\mathrm{S}(2) = \frac{1}{2}\left[-\frac{0.1}{150}\{S(0)I(1) + S(1)I(0)\}\right] = \frac{1}{2}\left[-\frac{0.1}{150}\{100\,(-4) + (-2)30\}\right] = 0.153333$$

$$\mathrm{I}(2) = \frac{1}{2}\left[\frac{0.1}{150}\{S(0)I(1) + S(1)I(0)\} - 0.2 I(1)\right] = \frac{1}{2}\left[\frac{0.1}{150}\{100(-4) + (-2)30\} - 0.2(-4)\right] = 0.246667$$

$$\mathrm{R}(2) = \frac{1}{2} 0.2\, I(1) = \frac{1}{2} 0.2(-4) = -0.4$$

$$\mathrm{S}(3) = \frac{1}{3}\left[-\frac{0.1}{150}\{S(0)\,I(2) + S(1)\,I(1) + S(2)\,I(0)\}\right]$$

$$= \frac{1}{3}\left[-\frac{0.1}{150}\{100\,(0.246667) + (-2)(-4) + 0.153333(30)\}\right]$$

$$= -0.00828148666$$

$$\mathrm{I}(3) = \frac{1}{3}\left[\frac{0.1}{150}\{S(0)\,I(2) + S(1)\,I(1) + S(2)\,I(0)\} - 0.2\, I(2)\right]$$

$$= \frac{1}{3}\left[\frac{0.1}{150}\{100\,(0.246667) + (-2)(-4) + 0.153333(30)\} - 0.2\,(0.246667)\right]$$

$$= -0.00816298$$

$$R(3) = \frac{1}{3} 0.2\, I(2) = \frac{1}{3} 0.2\,(0.24667) = 0.0164444666$$

Therefore, using the equations

$$s(t) = \sum_{k=0}^{\infty} t^k S(k) = S(0) + t\,S(1) + t^2 S(2) + t^3 S(3) + \ldots\ldots\ldots\ldots$$

$$i(t) = \sum_{k=0}^{\infty} t^k I(k) = I(0) + t\,I(1) + t^2 I(2) + t^3 I(3) + \ldots\ldots\ldots\ldots$$

$$r(t) = \sum_{k=0}^{\infty} t^k R(k) = R(0) + t\,R(1) + t^2 R(2) + t^3 R(3) + \ldots\ldots\ldots\ldots$$

the closed form of the solution when k = 3 can be written as follows :

$$s(t) = 100 - 2t + 0.153333t^2 - 0.0082814866\,t^3 + \ldots\ldots\ldots\ldots$$

$$i(t) = 30 - 4t + 0.246667t^2 - 0.00816298t^3 + \ldots\ldots\ldots\ldots$$

$$r(t) = 20 + 6t - 0.4t^2 + 0.0164444666\,t^3 + \ldots\ldots\ldots\ldots$$

Numerical Solution of SIR Epidemic Model

Using equations (1), (2), (3) and the initial conditions (8), the values of s(t), *i*(t) and *r*(t) are calculated by fourth-order Runge-Kutta Method at t = 0.2, 0.4, 0.6, 0.8 and 1.0, taking the interval of differencing h = 0.2.The numerical results are compared with the results obtained by DTM.

To evaluate s(t) at t = 0.2, we have

$$k_1 = h\left[-\frac{\beta\, s(0)\, i(0)}{n}\right] = 0.2\left[-\frac{0.1\times 100\times 30}{150}\right] = -0.4$$

$$k_2 = -\frac{h\beta}{n}\left[\left\{s(0) + \frac{k_1}{2}\right\}\left\{i(0) + \frac{h}{2}\right\}\right] = -\frac{0.2\times 0.1}{150}\left[99.8\times 30.1\right] = -0.40053066$$

$$k_3 = -\frac{h\beta}{n}\left[\left\{s(0) + \frac{k_2}{2}\right\}\left\{i(0) + \frac{h}{2}\right\}\right] = -\frac{0.2\times 0.1}{150}\left[99.79973467\times 30.1\right] = -0.4005296018$$

$$k_4 = -\frac{h\beta}{n}\left[\{s(0) + k_3\}\{i(0) + h\}\right] = -\frac{0.2\times 0.1}{150}\left[99.5994704\times 30.2\right] = -0.4010538675$$

$$\therefore \Delta s = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}\left(-0.4 - 0.80106132 - 0.8010592036 - 0.4010538675\right)$$

$$= -0.4005290652$$

and $s(0.2) = s(0) + \Delta s = 100 - 0.4005290652 = 99.5994709348 \approx 99.59947093$

Similarly, to evaluate *i* (t) at t = 0.2, we have

$$k_1 = h\left[\frac{\beta\, s(0)\, i(0)}{n} - \gamma\, i(0)\right] = 0.2\left[\frac{0.1\times 100\times 30}{150} - 0.2\,(30)\right] = -0.8$$

$$k_2 = h\left[\frac{\beta}{n}\left\{s(0)+\frac{h}{2}\right\}\left\{i(0)+\frac{k_1}{2}\right\}-\gamma\left\{i(0)+\frac{k_1}{2}\right\}\right]$$

$$= 0.2\left[\frac{0.1}{150}\times 100.1\times 29.6 - 0.2(29.6)\right]$$

$$= -0.78893866668$$

$$k_3 = h\left[\frac{\beta}{n}\left\{s(0)+\frac{h}{2}\right\}\left\{i(0)+\frac{k_2}{2}\right\}-\gamma\left\{i(0)+\frac{k_2}{2}\right\}\right]$$

$$= 0.2\left[\frac{0.1}{150}\times 100.1\times 29.6055306666 - 0.2(29.6055306666)\right]$$

$$= -0.789086077368$$

$$k_4 = h\left[\frac{\beta}{n}\{s(0)+h\}\{i(0)+k_3\}-\gamma\{i(0)+k_3\}\right]$$

$$= 0.2\left[\frac{0.1}{150}\times 100.2\times 29.2109139226 - 0.2(29.2109139226)\right]$$

$$= -0.7781787469$$

$$\therefore \Delta i = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.8 - 1.57787733336 - 1.578172154736 - 0.7781787469)$$

$$= -0.789038039166$$

and $i(0.2) = i(0) + \Delta i = 30 - 0.789038039166 = 29.210961960834 \approx 29.21096196$

Similarly, to evaluate *r* (t) at t = 0.2, we have

$$k_1 = h\gamma\, i(0) = 0.2\times 0.2\times 30 = 1.2$$

$$k_2 = h\gamma\left\{i(0)+\frac{h}{2}\right\} = 0.2\times 0.2(30+0.1) = 1.204$$

$$k_3 = h\gamma\left\{i(0)+\frac{h}{2}\right\} = 0.2\times 0.2(30+0.1) = 1.204$$

$$k_4 = h\gamma\{i(0)+h\} = 0.2\times 0.2(30+0.2) = 1.208$$

$$\therefore \Delta r = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(1.2 + 2.408 + 2.408 + 1.208) = 1.204$$

and $r(0.2) = r(0) + \Delta r = 20 + 1.204 = 21.204$

Using the above formulae, the values of s(t), *i*(t) and *r*(t) are calculated for other values of t.

Comparison of Analytical Solution with the Numerical Solution

The numerical results are compared with the results obtained by DTM and displayed below.

t	s(t) by DTM (4 iterate)	s(t) by fourth-order Runge -Kutta Method	Difference

0.2	99.60606707	99.59947093	0.00659614
0.4	99.22400326	99.21098305	0.01302021
0.6	98.85341108	98.83415434	0.01925674
0.8	98.4938930	98.46861578	0.02527722
1.0	98.14505151	98.11400069	0.03105082

Table 1 : Numerical Comparison of s(t)

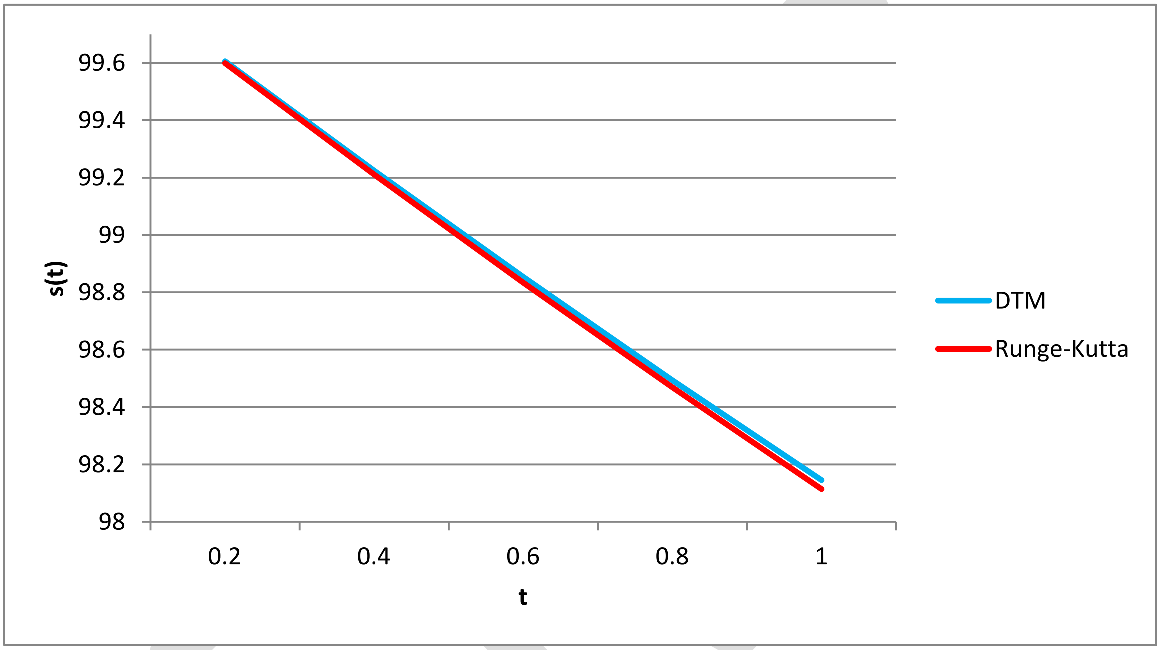

Figure 1 : Plot of s(t) versus time t

t	*i*(t) by DTM (4 iterate)	*i*(t) by fourth-order Runge -Kutta Method	Difference
0.2	29.20980138	29.21096196	0.00116058
0.4	28.43894429	28.44115771	0.00221342
0.6	27.68703692	27.69020597	0.00316905
0.8	26.95368743	26.95772764	0.00404021
1.0	26.23850402	26.24334619	0.00484217

Table 2 : Numerical Comparison of i(t)

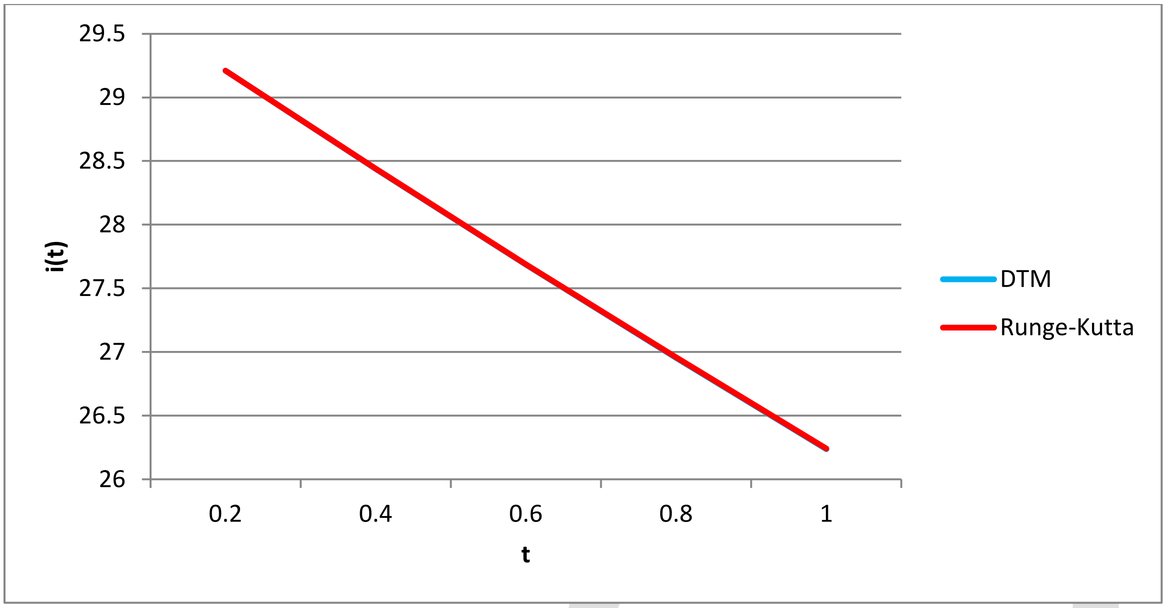

Figure 2 : Plot of *i*(t) versus time t

t	*r*(t) by DTM (4 iterate)	*r*(t) by fourth-order Runge -Kutta Method	Difference
0.2	21.18413156	21.20400000	0.01986844
0.4	22.33705245	22.37643853	0.03938608
0.6	23.45955200	23.51808484	0.05853284
0.8	24.55241957	24.62969308	0.07727351
1.0	25.61644447	25.71200219	0.09555772

Table 3 : Numerical Comparison of r(t)

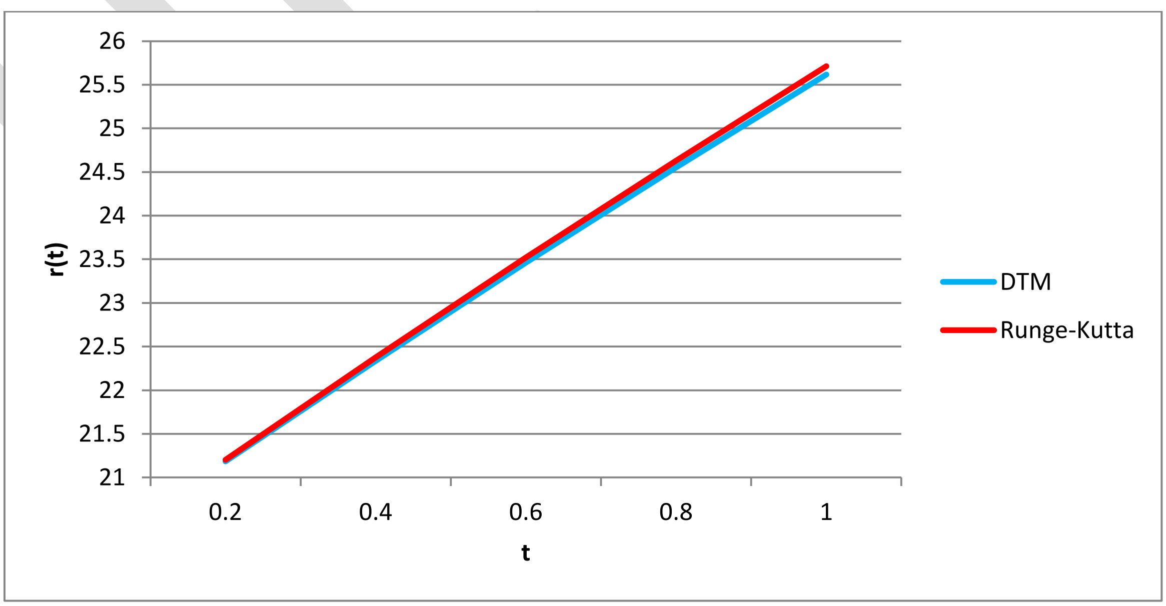

Figure 3 : Plot of *r*(t) versus time t

It can be seen that the value of $s(t) + i(t) + r(t)$ calculated by DTM is exactly equal to n for every t, whereas the value of $s(t) + i(t) + r(t)$ calculated by fourth-order Runge -Kutta Method differs slightly from n for various values of t.This confirms the ability of DTM as a powerful tool for solving non linear equations.

Conclusion

In this paper, Differential Transformation Method (DTM) has been used to solve SIR Epidemic Model with given initial conditions. As this method provides an explicit solution of the model, it is very useful in understanding and analysing an epidemic. The numerical comparison of this method with the fourth-order Runge -Kutta Method proves the efficiency and accuracy of the method. Moreover, this method provides a direct scheme for solving differential equations without the need for linearization, perturbation or any transformation. It may, therefore, be concluded that it is a powerful mathematical tool for solving epidemic models.

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International Journal of Engineering Research and General Science Volume 3, Issue 6, November-December, 2015
ISSN 2091-2730

IJERGS