A Note on Generalized Almost Contact Metric Manifold

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Abstract—In 1960, S. Sasaki [7] dicussed on differentiable manifolds with certain structures and in 1961, S. Sasaki and Y. Hatakeyama [8] studied on differentiable manifolds with certain structures which are closely related to almost contact structure. In 1963, Y. Hatakeyama, Y. Ogawa and S. Tanno [3] discussed some properties of manifolds with contact metric structure. Also in 1963, Hatakeyama [2] studied on differentiable manifolds with almost contact structures and in 2011, R. Nivas and A. Bajpai [6] studied on generalized Lorentzian Para-Sasakian manifolds. In 1975, Golab [1] discussed on quarter-symmetric connection in a differentiable manifold. In 1980, R. S. Mishra, and S. N. Pandey [4] discussed on quarter-symmetric metric F-connection and in 1982, K. Yano and T. Imai [11] studied on quarter-symmetric metric connections and their curvature tensors. Quarter-symmetric metric connection is also studied by R. N. Singh and S. K. Pandey [9] , A. K. Mondal and U. C. De [5] and many others. T. Suguri and S. Nakayama [10] considered D-conformal deformations on almost contact metric structure. In this paper D-conformal transformation in a generalized almost contact metric manifold has been discussed. Generalized induced connection in a generalized almost contact metric manifold has also been discussed.

Keywords—Generalized almost contact metric manifold, generalized D-conformal transformation, generalized induced connection.

1. INTRODUCTION

Let V_n be an odd $(n = 2m + 1)$ dimensional differentiable manifold, on which there are defined a tensor field F of type (1, 1), contravariant vector fields T_i, covariant vector fields A_i, where $i = 3,4,5, \ldots (n-1)$, and a metric tensor g, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\quad \overline{\overline{X}} = -X + \sum_{i=3}^{n-1} A_i(X) T_i, \quad \overline{T_i} = 0, \quad A_i(T_i) = 1, \quad \overline{X} \stackrel{\text{def}}{=} FX, \quad A_i(\overline{X}) = 0,$

rank $F = n - i$

(1.2) $\quad g(\overline{X}, \overline{Y}) = g(X, Y) - \sum_{i=3}^{n-1} A_i(X) A_i(Y)$, where $A_i(X) = g(X, T_i)$,

$`F(X, Y) \stackrel{\text{def}}{=} g(\overline{X}, Y) = -`F(Y, X),$

Then V_n will be called a generalized almost contact metric manifold and the structure (F, T_i, A_i, g) will be called generalized almost contact metric structure.

Let D be a Riemannian connection on V_n, then we have

(1.3) (a) $\quad (D_X `F)(\overline{Y}, Z) - (D_X `F)(Y, \overline{Z}) - \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(Z) - \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(Y) = 0$

(b) $\quad (D_X `F)(\overline{Y}, \overline{\overline{Z}}) = (D_X `F)(\overline{\overline{Y}}, \overline{Z})$

(1.4) (a) $(D_X`F)(\overline{Y},\ \overline{Z})+(D_X`F)(Y,Z)-\sum_{i=3}^{n-1}A_i(Y)(D_XA_i)(\overline{Z})+\sum_{i=3}^{n-1}A_i(Z)(D_XA_i)(\overline{Y})=0$

(b) $(D_X`F)\left(\overline{\overline{Y}},\ \overline{\overline{Z}}\right)+(D_X`F)(\ \overline{Y},\ \overline{Z}\ )=0$

2. GENERALIZED CONNECTION IN A GENERALIZED ALMOST CONTACT METRIC MANIFOLD

Let V_{2m-1} be submanifold of V_{2m+1} and let $c : V_{2m-1} \to V_{2m+1}$ be the inclusion map such that

$$d \in V_{2m-1} \to cd \in V_{2m+1}\ ,$$

Where c induces a linear transformation (Jacobian map) $J : T'_{2m-1} \to T'_{2m+1}$.

T'_{2m-1} is a tangent space to V_{2m-1} at point d and T'_{2m+1} is a tangent space to V_{2m+1} at point cd such that

$$\hat{X} \text{ in } V_{2m-1} \text{ at } d \to J\hat{X} \text{ in } V_{2m+1} \text{ at } cd$$

Let $\tilde{g}$ be the induced metric tensor in V_{2m-1} . Then we have

(2.1) $\tilde{g}(\hat{X},\hat{Y}) \stackrel{\text{def}}{=} g(J\hat{X},J\hat{Y})$

We now suppose that a generalized semi-symmetric metric connection B in a generalized almost contact metric manifold is given by

(2.2) $iB_XY=iD_XY+\sum_{i=3}^{n-1}A_i(Y)X-\sum_{i=3}^{n-1}g(X,Y)T_i$

Where X and Y are arbitrary vector fields of V_{2m+1}. If

(2.3) $T_i=Jt_i+\rho_iM+\sigma_iN$, where $i=3,4,5,\ldots\ldots(n-1)$.

Where t_i, $i=3,4,5,\ldots\ldots(n-1)$ are C^∞ vector fields in V_{2m-1} and M and N are unit normal vectors to V_{2m-1}.

Denoting by $\grave{D}$ the connection induced on the submanifold from D, we have Gauss equation

(2.4) $D_{JX}J\hat{Y}=J(\grave{D}_X\hat{Y})+p(\hat{X},\hat{Y})M+q(\hat{X},\hat{Y})N$

Where h and k are symmetric bilinear functions in V_{2m-1}. Similarly we have

(2.5) $B_{JX}J\hat{Y}=J(\grave{B}_X\hat{Y})+r(\hat{X},\hat{Y})M+s(\hat{X},\hat{Y})N$,

Where $\grave{B}$ is the connection induced on the submanifold from B and r and s are symmetric bilinear functions in V_{2m-1}

Inconsequence of (2.2), we have

(2.6) $iB_{JX}J\hat{Y}=iD_{JX}J\hat{Y}+\sum_{i=3}^{n-1}A_i(J\hat{Y})J\hat{X}-\sum_{i=3}^{n-1}g(J\hat{X},J\hat{Y})T_i$

Using (2.4), (2.5) and (2.6), we get

(2.7) $iJ(\grave{B}_X\hat{Y})+\ ir(\hat{X},\hat{Y})M+\ is(\hat{X},\hat{Y})N=\ iJ(\grave{D}_X\hat{Y})+\ ip(\hat{X},\hat{Y})M+\ iq(\hat{X},\hat{Y})N+\sum_{i=3}^{n-1}A_i(J\hat{Y})J\hat{X}-$

$$\sum_{i=3}^{n-1}g(J\hat{X},J\hat{Y})T_i$$

Using (2.3), we obtain

(2.8) $iJ(\grave{B}_X\hat{Y})+\ ir(\hat{X},\hat{Y})M+\ is(\hat{X},\hat{Y})N=\ iJ(\grave{D}_X\hat{Y})+\ ip(\hat{X},\hat{Y})M+\ iq(\hat{X},\hat{Y})N+\sum_{i=3}^{n-1}a_i(\ \hat{Y})J\hat{X}-$

$$\sum_{i=3}^{n-1}\tilde{g}(\ \hat{X},\hat{Y})(Jt_i+\rho_iM+\sigma_iN)$$

Where $\tilde{g}(\hat{Y},t_i) \stackrel{\text{def}}{=} a_i(\ \hat{Y})$, where $i=3,4,5,\ldots\ldots(n-1)$.

This gives

(2.9) $i\grave{B}_X\hat{Y} = i\grave{D}_X\hat{Y} + \sum_{i=3}^{n-1} a_i(\hat{Y})\hat{X} - \sum_{i=3}^{n-1} \tilde{g}(\hat{X},\hat{Y})t_i$

Iff

(2.10) (a) $ir(\hat{X},\hat{Y}) = ip(\hat{X},\hat{Y}) - \sum_{i=3}^{n-1} \rho_i\tilde{g}(\hat{X},\hat{Y})$

(b) $is(\hat{X},\hat{Y}) = iq(\hat{X},\hat{Y}) - \sum_{i=3}^{n-1} \sigma_i\tilde{g}(\hat{X},\hat{Y})$

Thus we have

Theorem 2.1 The connection induced on a submanifold of a generalized almost contact metric manifold with a generalized semi-symmetric metric connection with respect to unit normal vectors M and N is also semi-symmetric metric connection iff (2.10) holds.

3. GENERALIZED D-CONFORMAL TRANSFORMATION

Let the corresponding Jacobian map J of the transformation b transforms the structure $(F,\ T_i,\ A_i,\ g)$ to the structure $(F, V_i,\ \upsilon_i, h)$ such that

(3.1) (a) $J\overline{Z} = \overline{JZ}$ (b) $h(JX,JY)ob = e^{\sigma}\,g(\overline{X},\overline{Y}) + e^{2\sigma}\sum_{i=3}^{n-1} A_i(X)A_i(Y)$

(c) $V_i = e^{-\sigma}JT_i$ (d) $\upsilon_i(JX)ob = e^{\sigma}A_i(X)$

Where σ is a differentiable function on V_n, then the transformation is said to be generalized D-conformal transformation.

Theorem 3.1 The structure $(F, V_i,\ \upsilon_i, h)$ is generalized almost contact metric structure.

Proof. Inconsequence of (1.1), (1.2), (3.1) (b) and (3.1) (d), we get

$$h(J\overline{X}, J\overline{Y})ob = e^{\sigma}\,g(\overline{X},\overline{Y}) = h(JX,JY)ob - \sum_{i=3}^{n-1} e^{2\sigma}A_i(X)A_i(Y)$$

$$= h(JX,JY)ob - \sum_{i=3}^{n-1}\{\upsilon_i(JX)ob\}\{\upsilon_i(JY)ob\}$$

This gives

(3.2) $h(J\overline{X}, J\overline{Y}) = h(JX,JY) - \sum_{i=3}^{n-1} \upsilon_i(JX)\,\upsilon_i(JY)$

Using (1.1), (3.1) (a), (3.1) (c) and (3.1) (d), we get

(3.3) $\overline{\overline{JX}} = J\overline{\overline{X}} = -JX + \sum_{i=3}^{n-1} A_i(X)JT_i = -JX + \sum_{i=3}^{n-1}\{\upsilon_i(JX)ob\}V_i$

Also

(3.4) $\overline{V_i} = e^{-\sigma}\overline{JT_i} = 0$

Proof follows from equations (3.2), (3.3) and (3.4).

Theorem 3.2 Let E and D be the Riemannian connections with respect to h and g such that

(3.5) (a) $E_{JX}JY = JD_XY + JH(X,Y)$ and

(b) $`H(X,Y,Z) \stackrel{\text{def}}{=} g(H(X,Y),Z)$

Then

(3.6) $2E_{JX}JY = 2JD_XY + J[2e^{\sigma}\{\sum_{i=3}^{n-1}(X\sigma)A_i(Y)\,T_i + \sum_{i=3}^{n-1}(Y\sigma)A_i(X)\,T_i - \sum_{i=3}^{n-1}({}^{-1}G\nabla\sigma)\,A_i(X)\,A_i(Y)\} + (e^{\sigma} - 1)\sum_{i=3}^{n-1}\{(D_XA_i)(Y) + (D_YA_i)(X) - 2A_i(H(X,Y))\}T_i + (e^{\sigma} - 1)\sum_{i=3}^{n-1}\{A_i(X)(D_YT_i) + A_i(Y)(D_XT_i) - A_i(X)({}^{-1}G\nabla A_i)(Y) - A_i(Y)({}^{-1}G\nabla A_i)(X)\}]$

Proof. Inconsequence of (3.1) (b), we have

$$JX(h(JY,JZ))ob = X\{e^{\sigma}\,g(\overline{Y},\overline{Z}) - \sum_{i=3}^{n-1} e^{2\sigma}A_i(Y)A_i(Z)\}$$

We have

(3.7) $h(E_{JX}JY, JZ)ob + h(JY, E_{JX}JZ)ob = (X\sigma)e^{\sigma}\, g\,(\overline{Y}, \overline{Z}) + e^{\sigma}\, g\,(D_X\overline{Y}, \overline{Z}) + e^{\sigma}\, g\,(\overline{Y}, D_X\overline{Z}) + \sum_{i=3}^{n-1}\{2(X\sigma)e^{2\sigma} A_i(Y)A_i(Z) + e^{2\sigma}(D_XA_i)(Y)A_i(Z) + e^{2\sigma}(D_XA_i)(Z)A_i(Y) + e^{2\sigma} A_i(D_XY)A_i(Z) + e^{2\sigma} A_i(D_XZ)A_i(Y)\}$

Also

(3.8) $h(E_{JX}JY, JZ)ob + h(JY, E_{JX}JZ)ob = e^{\sigma}\, g\,(\overline{D_XY}, \overline{Z}) + e^{\sigma}\, g\,(\overline{H(X,Y)}, \overline{Z}) + e^{\sigma}\, g\,(\overline{Y}, \overline{H(X,Z)}) + e^{\sigma}\, g\,(\overline{Y}, \overline{D_XZ}) + \sum_{i=3}^{n-1}\{e^{2\sigma} A_i(D_XY)A_i(Z) + e^{2\sigma} A_i(Y)A_i(H(X,Z)) + e^{2\sigma} A_i(D_XZ)A_i(Y) + e^{2\sigma} A_i(H(X,Y))A_i(Z)\}$

Inconsequence of (1.3) (a), (3.7) and (3.8), we have

(3.9) $(X\sigma) g\,(\overline{Y}, \overline{Z}) + 2(X\sigma)e^{\sigma} \sum_{i=3}^{n-1}\{A_i(Y)A_i(Z)\} + (e^{\sigma} - 1)\sum_{i=3}^{n-1}\{(D_XA_i)(Y)A_i(Z) + (D_XA_i)(Z)A_i(Y)\} - (e^{\sigma} - 1)\sum_{i=3}^{n-1}\{A_i(H(X,Y))A_i(Z) + A_i(H(X,Z))A_i(Y)\} = `H(X,Y,Z) + `H(X,Z,Y)$

Writing two other equations by cyclic permutation of X, Y, Z and subtracting the third equation from the sum of the first two. Also using symmetry of $`H$ in the first two slots, we get

(3.10)

$2`H(X,Y,Z) = 2e^{\sigma} \sum_{i=3}^{n-1}\{(X\sigma)A_i(Y)A_i(Z) + (Y\sigma)A_i(Z)A_i(X) - (Z\sigma)A_i(X)A_i(Y)\} + (e^{\sigma} - 1)\sum_{i=3}^{n-1}\big[A_i(Z)\{(D_XA_i)(Y) + (D_YA_i)(X) - 2A_i(H(X,Y))\} + A_i(X)\{(D_YA_i)(Z) - (D_ZA_i)(Y)\} + A_i(Y)\{(D_XA_i)(Z) - (D_ZA_i)(X)\}\big]$

This implies

(3.11) $2H(X,Y) = 2e^{\sigma} \sum_{i=3}^{n-1}[(X\sigma)A_i(Y)T_i + (Y\sigma)A_i(X)\, T_i - ({}^{-1}G\nabla\sigma)A_i(\mathrm{X})A_i(\mathrm{Y})] + (e^{\sigma} - 1)\sum_{i=3}^{n-1}\big[\{(D_XA_i)(Y) + (D_YA_i)(X) - 2A_i(H(X,Y))\}\, T_i + A_i(X)(D_YT_i) + A_i(Y)(D_X\, T_i) - A_i(X)({}^{-1}G\nabla A_i)(\mathrm{Y}) - A_i(\mathrm{Y})({}^{-1}G\nabla A_i)(\mathrm{X})\big].$

(3.6) follows from (3.11) and (3.5).

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