# Labeling Techniques in Friendship Graph

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**Abstract**— This paper aims to focus on some labeling methods of Friendship Graphs. We investigate friendship graph with four types of labeling such that Harmonious, Cordial, distance antimagic labeling and sum labeling. The approach will be to summarize them in different flavor and possible different labeling. Also we will discuss about some important theorems and examples based on those theorems.

**Keywords**— Friendship graph, Harmonious labeling, cordial labeling, distance antimagic labeling, generalised friendship graph, sum labeling, optimal summable.

## Introduction

Lots of research work is been carried out in the labeling of graphs in past few work since the first initiated by A. Rosa [7]. Sum labeling of graphs was introduced by Harary [3] in 1990 and since that time the problem of finding an optimal labeling for a family of graphs has been shown to be difficult, even for fairly simple graphs.

### Definitions

1. The ***fan*** fn ($n \geq 2$) is obtained by joining all vertices of $P_n$ (Path of n vertices) to a further vertex called the center and contains n+ 1 vertex and 2n-1 edges. i.e. $f_n = P_n + K_1$.Fan f4 is shown in the following Figure I.

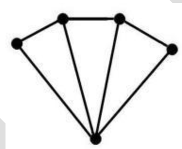


Figure I

2. A ***friendship graph*** $F_n$ is a graph which consists of n triangles with a common vertex. Friendship graph $F_4$ is shown in the following Figure II.

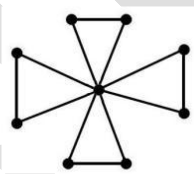


Figure. II

3. The ***wheel graph*** $W_n$ is defined to be the join of $K_1 + C_n$ i.e. the wheel graph consists of edges which join a vertex of $K_1$ to every vertex of $C_n$. Figure III shows a wheel $W_3$.

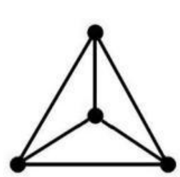


Figure III.

4. The ***helm*** $H_n$ is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the n – cycle as shown in the Fig. IV

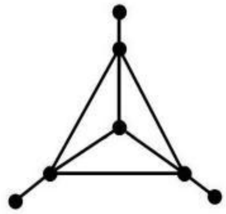


Figure IV

5. If the vertices of the graph are assigned values subject to certain conditions than it is known as ***graph labeling***. Most of the graph labeling problem will have following three common characteristics.

- A set of numbers from which vertex labels are chosen,
- A rule that assigns a value to each edge,
- A condition that these values must satisfy.

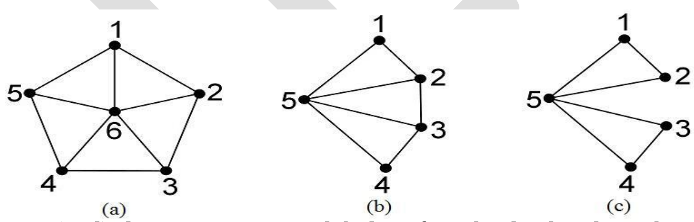
6. Let G be a graph with q edges. A function f is called ***harmonious labeling*** of graph G if f: V →{0, 1, 2,,,,q-1} is injective and the induced function f*: E → {0, 1, 2,,,,,,q} defined as f*(e = uv) = (f(u) +f(v))(mod q) is bijective. A Graph which admits harmonious labeling is called harmonious graph

7. A mapping $f : V(G) \rightarrow \{0,1\}$ is called ***binary vertex labeling*** of $G$ and $f(v)$ is called the *label* of vertex $v$ of $G$ under $f$. The induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of $G$ having labels 0 and 1 respectively under $f$ and let $e_f(0)$,$e_f(1)$ be the number of edges of $G$ having labels 0 and 1 respectively under $f^*$.

8. A binary vertex labeling of graph $G$ is called ***cordial labeling*** if $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is ***cordial*** if it admits cordial labeling.

9. Let G be a graph, x be a vertex in G, f be a bijection from V onto {1, 2,……… v}, and D ⊆ {0,1,…… diam(G)}.The bijection f is called ***distance antimagic labeling*** if all vertices have distinct vertex-weights. A graph is called ***distance antimagic*** if it admits a distance antimagic labeling.

10. The bijection f is called a ***D-distance antimagic labeling*** if the D-vertex-weights are all different. The bijection f is called an ***(a, d)-D-distance antimagic labeling*** if all D-vertex-weights constitute an arithmetic progression with difference d and starting value a, for a and d fixed integers with d≥0. A graph G is ***D-distance antimagic or (a, d)-D-distance antimagic*** if it admits a D-distance antimagic labeling or an (a, d)-D-distance antimagic labeling, respectively.




(a, d)-distance antimagic labeling for wheel-related graphs

11. A ***sum labeling*** is a mapping λ from the vertices of G into the positive integers such that, for any two vertices u, v belongs to V (G) with labels λ(u) and λ(v), respectively, (uv) is an edge if and only if λ(u) + λ(v) is the label of another vertex in V (G). Any graph supporting such a labeling is called a ***sum graph***.

12. The friendship graph $f_n$ is a collection of n triangles with a common vertex. It may be also pictured as a wheel with every alternate rim edge removed, see Figure II. The ***generalised friendship graph*** $f_{q,p}$ is a collection of p cycles (all of order q), meeting at a common vertex. In this section we will refer to the friendship graph $f_n$ as an instance of the generalised friendship graph and write it as $f_{3,n}$. The generalised friendship graph is, because of its shape, also referred to as a flower.

**Theorem : Friendship graph $F_n$ is harmonious except n≡ 2(mod 4)**

Proof: We consider the following three cases

Case (1): If n ≡ 2 (mod 4), then $F_2$ is not harmonious according to a theorem. Since number of vertices is 5 and number of edges are 6. Which is not divisible by 4 or 8. are 6. Which is not divisible by 4 or 8.

Case (2): If n ≡ 0 or 1 (mod 4), then the numbers {0,1,2,,,,2n} may be partitioned into n pairs ($a_r$, $b_r$) with $b_r - a_r = r$ for r = 1,2,,,,n. then a harmonious labeling is obtained by labeling the vertices of the triangle with (0,r, n+$a_r$) for r = 1,2,,,n.

Case (3): If n ≡ 3 (mod 4), then {1,2,3,,,,2n-6} may be partitioned into n-3 pairs ($a_r$, $b_r$) with $b_r - a_r = r+2$ for r = 1,2,,,,n-3. We label the triangles of Fn with (0,1, 3n-1), (0,2, 3n-6), (0,3n-2, 3n-3) and (0,r+2, n+ $a_r$) for r = 1,2,,,n-3. Thus, $F_n$ is harmonious except n ≡ 2 (mod 4).

**Theorem : fn is harmonious**.

Proof: Let m = [n/2] and label the centre with 0 and the vertices of path with m, n, m+1, n+1, m+2,,,. Then we get the harmonious labeling of fan.

In the following figure. V and VI the harmonious labeling of $F_4$ and $F_5$ is shown.

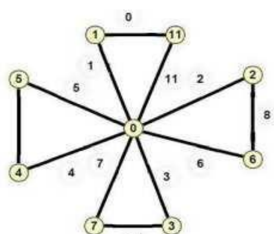


Figure V

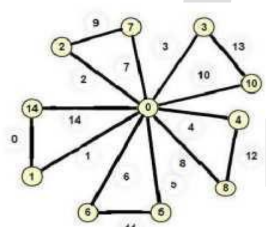


Figure VI

**Theorem : *$F_n$* is product cordial.**

*Proof :*

Let $F_n$ be the friendship graph with n copies of cycle $C_3$.

Let v′ be the apex vertex, $v_1, v_2, ..., v_{2n}$ be the other vertices and $e_1, e_2, ..., e_{3n}$ be the edges of $F_n$.

Define $f : V(F_n) \rightarrow \{0,1\}$, we consider following two cases.

Case 1: When *n* is even.

$f(v_i) = 0, \quad 1 \leq i \leq n$
$f(v_i) = 1$ otherwise
$f(v') = 1$

In view of the above labeling pattern we have,

$v_f(0) = v_f(1) - 1 = n$
$E_f(0) = e_f(1) = \frac{3n}{2}$

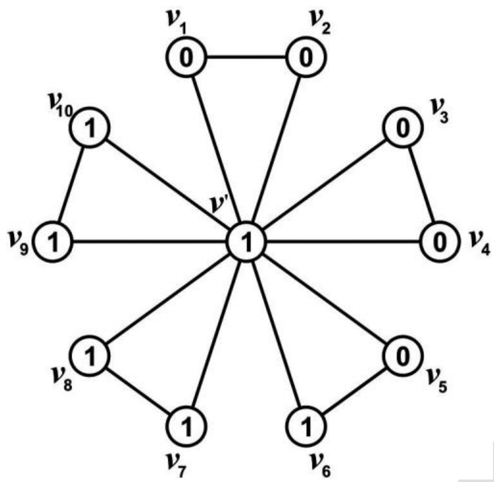
Case 1: When *n* is odd.

$f(v_i) = 0, \quad 1 \leq i \leq n$
$f(v_i) = 1$ otherwise
$f(v') = 1$

In view of the above labeling pattern we have,

$v_f(0) + 1 = v_f(1) = n + 1$
$E_f(0) = e_f(1) + 1 = \frac{3n}{2}$

Thus in each cases we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence $F_n$ is product cordial.




$F_5$ with product cordial labeling.

**Theorem : A friendship graph $f_n$ is (a, d)-distance antimagic if and only if n = 1 or n = 2**.

Proof:

For i = 1, 2,……. 2n, we have $3 \leq w(x_i) \leq 4n + 1$ and so d is at most $\frac{(4n+1)-3}{2n} = 2 - \frac{1}{n} \leq 2$.

On the other hand $n(2n+1) \leq w(x_0) \leq n(2n + 3)$.

Thus we have $w(x_0) - w(x_i) \geq n(2n+1) - (4n+1) = 2n^2 - 3n - 1$.

For $n \geq 3$, $w(x_0) - w(x_i) \geq 8$, a contradiction.

To complete the proof, we need to consider $f_1$ and $f_2$. Since $f_1 \simeq K_3$ then $f_1$ has a (3, 1)-distance antimagic labeling. Finally, a simple vertex labeling leads to the distance antimagicness of friendship graphs.

**Theorem : All friendship graphs are distance antimagic.**

Proof. We de ne a vertex labeling f of $f_n$ as follow $f(x_i) = \begin{cases} n+1, & \text{for } i=0, \\ i, & \text{for } i=1,2\ldots.n. \end{cases}$

and so we obtain the following vertex-weights $w(x_i) = \begin{cases} n(2n+1), & for \quad i = 0, \\ n+2+i, & for \quad i = 1,3 \ldots\ldots\ldots.2n-1, \\ n+i, & for \quad i = 2,4 \ldots\ldots\ldots\ldots..2n. \end{cases}$

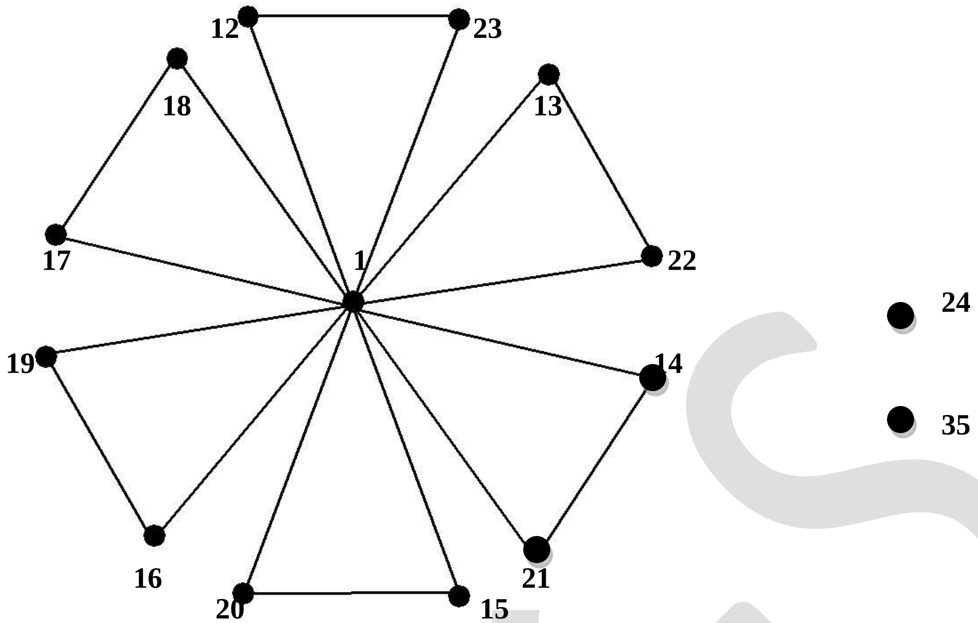
We can see that the weights are all distinct.

**Constructing a sum labelling for $f_{q,p}$**

**Theorem : $f_{3,p}$ is optimal summable**.

Proof: Begin a sum labelling for $f_{3,p}$ by labelling the centre vertex c = 1. Commencing from any triangle, distribute the labels a, a + 1, a + 2, . . . clockwise about the centre, one label for each triangle. The maximum label at this stage is a + (p − 1) on the $p^{th}$ triangle. Then continue the labelling sequence, distributing the labels a + p, a + (p + 1), . . . counterclockwise from the $p^{th}$ triangle. The maximum label is now a + (2p − 1) and is adjacent to the label a. Now all of the edges independent of the centre join vertices whose combined sum is 2a + (2p − 1) and the only edge not yet accounted for in the labelling is the edge between the maximum label and the centre (whose vertex sums add to a + 2p).

Since both 2a + (2p − 1) and a + 2p are larger than the maximum label in the graph, they must be (the only required) isolates. It only remains to ensure that no further edges are induced.Setting a large enough (say a = 2p) suffices.




The flower $f_{3,6}$ and its labeling

**Theorem: $f_{4,p}$ is optimal summable.**

Proof: Give the centre vertex the label 5 and label the outer vertices according to the scheme,

$$v_1^1 = 8,\ v_2^1 = 1,\ v_3^1 = 2^{p-3} \times 50 - 2^{p-1} - 1$$

$v_1^2 = 2^{p-3} \times 50 - 2^{p-1} - 10$,

$v_2^2 = 9$,

$v_3^2 = 2^{p-3} \times 50 - 2^{p-1} - 5$

$v_1^i = v_2^i - 5$,

$v_2^i = 2^{i-4} \times 50 - 2^{i-2}$,

$v_3^i = v_2^i - 10$.

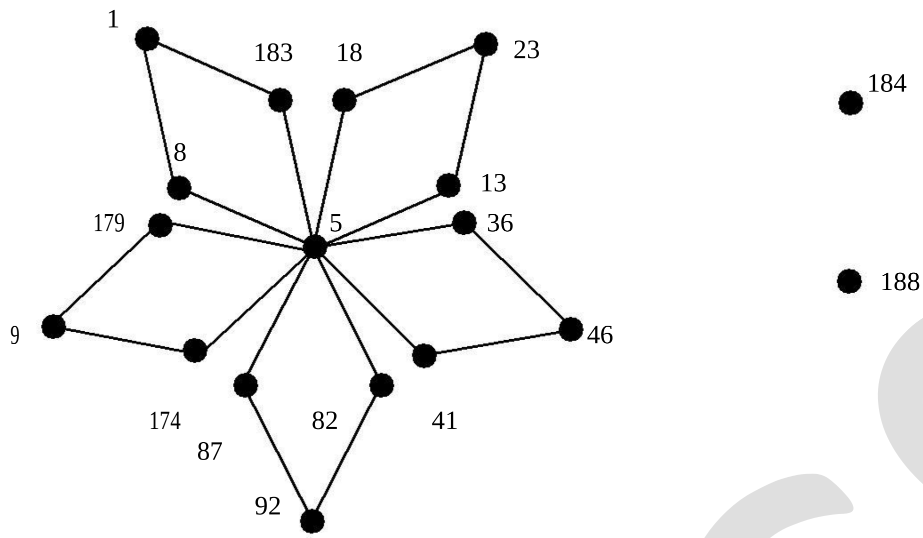
This labeling provides for 2 isolates (x, y) of the form

$$x = 2^{p-3} \times 50 - 2^{p-1},\ y = 2^{p-3} \times 50 - 2^{p-1} + 4.$$

Simple arithmetic verifies that the edges on the first two cycles are witnessed by the labelling. The following equations show that all edges of intermediate cycles are witnessed.

$$v_1^i = v_3^i + 5 = v_1^{i-1} + v_2^{i-1}$$
$$v_2^i = v_1^i + 5$$
$$v_3^i = v_3^{i-1} + v_2^{i-1}$$

The relationship $v_2^i = 2 \times v_2^{i-1}$ ensures that the labels on the individual cycles are sufficiently well spaced so as to prevent edges being induced between cycles.




The flower $f_{4,5}$ and its labeling

**Theorem : $f_{5,p}$ is optimal summable.**

Proof: Label the centre c = 1 and the vertices on the first petal by

$$v_1^1 = 3,$$
$$v_2^1 = 5,$$
$$v_3^1 = 8,$$
$$v_4^1 = 4$$

and similarly for subsequent petals (except for the last) following the scheme

$$\lambda(v_1^i) = \lambda(v_3^{i-1}) + \lambda(v_4^{i-1})$$
$$\lambda(v_2^i) = \lambda(v_4^i) + c$$
$$\lambda(v_3^i) = \lambda(v_1^i) + \lambda(v_2^i)$$
$$\lambda(v_4^i) = \lambda(v_1^i) + c.$$

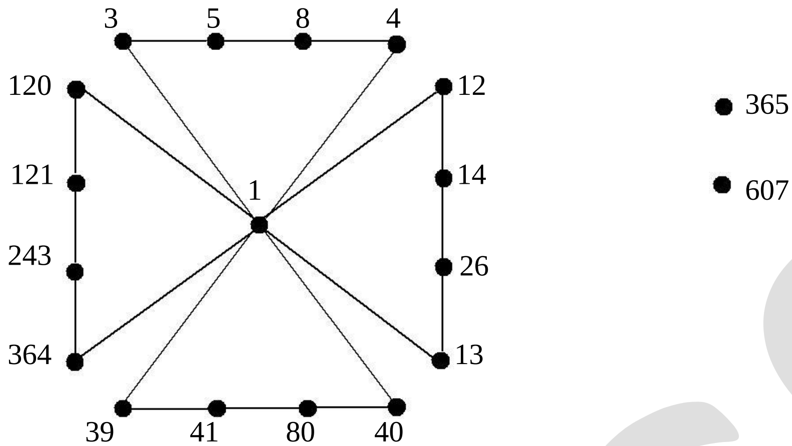
Note that every vertex is working except for the two smallest and that the multiplicity of $v_4^i$ is 2

$$\lambda(v_4^i) = \lambda(v_1^i) + c = \lambda(v_2^{i-1}) + \lambda(v_3^{i-1}).$$

The last cycle is labelled conventionally from $v_1^i$ to $v_4^i$ to avoid inducing any extra edges. More formally,

$$\lambda(v_1^p) = \lambda(v_3^{p-1}) + \lambda(v_4^{p-1})$$
$$\lambda(v_2^p) = \lambda(v_1^p) + c$$
$$\lambda(v_3^p) = \lambda(v_1^p) + \lambda(v_2^p)$$
$$\lambda(v_4^p) = \lambda(v_2^p) + \lambda(v_3^p)$$

leaving just two isolates to account for the edges adjacent to the vertex with the highest label($v_4^p$ ).




An optimal sum labelling of the graph $f_{5,4}$.

That no further edges are induced can best be shown using an explicit formula for the labelling. For the $i^{th}$ cycle, an alternative description of the labelling is as follows.

$$\lambda(v_1^i) = 2.3^{i-1} + \frac{1}{2}(3i-1)-1$$

$$\lambda(v_2^i) = \lambda(v_1^i) + 2$$

$$\lambda(v_3^i) = 2.\ \lambda(v_4^i)$$

$$\lambda(v_4^i) = \lambda(v_1^i) + 1$$

These formulae keep any "unwanted" edges being induced within a petal and the exponents of 3 forbid the inducing of any extra edges between the petals

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## CONCLUSION

It is very interesting to investigate the labeling types of friendship graphs. Here we investigate the four types of labeling techniques. In our next paper we discuss more about friendship graphs and its labeling.

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