Optimal Placement Problem in Printing Production

I. Grebennik, D. Grytsay and S. Shekhovtsov

***Abstract* — In the article we deal with the optimal placement problem in manufacturing printing products. A special set of two-dimensional geometric objects bounded by circular arcs and line segments are introduced as mathematical models of real-life printing objects. We derive phi-functions, as well as normalized and pseudo-normalized phi-functions to describe the relations (non-overlapping, containment and distance constraints) between the geometric objects. A mathematical model of the optimal placement of printing objects is constructed and a solution strategy is proposed.**

***Index Terms* – cutting, phi-functions, printing objects, mathematical model, solution strategy**

I. Introduction

Book-and-magazine editions are the main product type printing houses deal with. Although printing houses print not only books but also handouts, business cards, beer coasters, stickers, promo materials etc., the shape of those printing products types usually differs from the rectangle and generally their printrun is small. A variety of materials (cardboard of various types and density, coated paper of various density, different types of plastic and so on) is used for printing those printing products. Depending on the shape, the type of material, the printrun and other aspects of each printing product, the printing house chooses one of the two following variants of work: a) it can pick the cutting ticket (object layout on the template sheet) out of the existing set of object layouts or b) it can approximate the printing objects by rectangles and place them on a layout template sheet as ordinary rectangles. Hereinafter, we shall refer to parts of printing products which are placed on the printing sheet as printing objects.

Printing objects are always printed using the standard material sheet formats. Depending on the material type and the product binding it is possible to use different placement rules to place the objects on the printing sheet. For example, the book which consists of several printing objects collected from separate pages or which uses the spring as the binding can be printed without keeping to the folding rules.

The current problem in its statement relates to the cutting and packing problems [1]. The problem of placing such printing objects is relevant because printing objects are placed into the rectangular area (printing sheet) using the rectangular placing methods but the objects do not have a rectangular shape in most cases. The more the shape of the printing object differs from the rectangle the bigger blank spaces are and correspondently the more material is wasted. The next problem is to fill the remaining blank spaces of the existing layout using other printing objects (in other words it is necessary to place other printing objects into blank spaces of the existing layout sheet) to reduce material waste.

If printing objects are of the same material type, density, printrun, the printing house could place it in one combined printing sheet. It makes it possible to print products using fewer plates for printing, having cheaper printing cost and saving energy. It increases profit of the printing house and reduces environmental pollution due to saving material.

First, generally the printing objects mentioned above have the shape of a rectangular with “rounded corners”; second, a great part of the objects having complicated shapes can be approximated by simple objects. Figure 1 shows some examples of printing objects.

Fig.1. Shapes of printing objects and their approximation: printing objects (a); shape approximation (b)

It is obvious that the shape approximation by a “rectangle” with a “rounded corner” shown in Figure 1 is better than approximation by a regular rectangle. If we

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place these shapes instead of the rectangle we save material.

The problem is that prepress centers place the printing objects as ordinary rectangles even when the printing objects do not have a rectangular shape.

Lets us consider only those types of printing objects which have the shapes presented in Figure 2 (e.g. business cards, promotional handouts, stickers, brochures etc.).

The article [2] considers a multi-stock cutting problem of a collection of arbitrary-shaped two-dimensional objects in order to maximize usable space or, in other words, minimize waste of manufacturing material. The frontiers of the objects are formed by circular arcs and line segments.

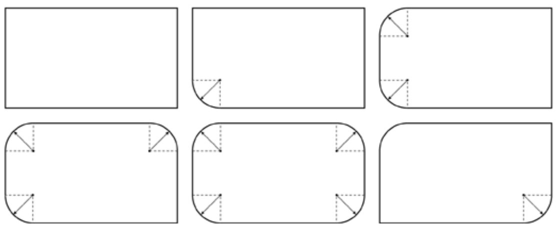
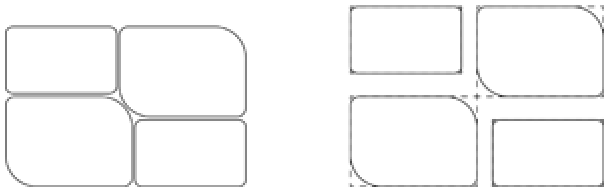


Fig. 2. Shapes used in printing

Today the printing products which have the proposed shapes are printed in small printruns and using the standard layout template sets. Typically each printing sheet collects the same type of the printing objects several times in the form of the array. After that if it is necessary to print several types of the printing objects in one time (using the same printrun and the same material for each printing object), different standard layout templates with array form should be used. It is useful if the printing objects have rectangular shapes. If it is necessary to round the corners for these objects then it has to be done in the end of the production cycle.

Generally when the curving radii are small enough it is useful to approximate these shapes by rectangles. At that the problem reduces to the standard problem of placing rectangles into a rectangular area [3-6]. In some cases if the rounded radii are compared to the edges length of the objects, there is a need for material saving (and what is the most important it is possible). Figure 3 shows the situation when the rectangular approximation can not be used.



(a) (b)

Fig. 3. Placement of printing objects: a) without approximations; b) with rectangular approximations

Figure 3 shows that the alternate version a) is more compact, therefore it is more material-saving. The placement problem for the proposed printing objects can be formulated as follows: 1) place all printing objects on the printing sheets of the standard format; 2) place as many objects as possible on printing sheets of a standard format; 3) place all printing objects using as few printing sheets of a standard format as possible; 4) determine the optimal printing sheet format to place the given printing objects set.

All mentioned alternatives can be formulated using sheet packing factor which also allows to economize by using fewer plates, saving the material, reducing the final product cost. Let us consider the shape of the objects in a more precise way.

As the mathematical models for the printing objects we consider a collection of phi-objects bounded by circular arcs and line segments [7]. We note the collection by $\mathfrak{R}$. The phi-function technique [7, 8] is used to describe the relations (non-intersection, containment, minimum allowable distances) between the geometric objects in the analytical form.

The aim of the paper is to construct a mathematical model and develop the solution strategy for the problem of determining the optimal printing sheet format to place the given printing objects.

II. Placement Objects

Let us consider a set of basic objects $\mathfrak{I} = \{C, \ R, \ D, \ K\}$, which are described in details in [7]. Figure 4 shows the basic objects.

Here C: is a circle of radius r, i.e. $m_C = (r)$; R: is the rectangle of a and half-height b, i.e. $m_R = (a, \ b)$; D: is a circular segment of radius r and height h, i.e. its metric characteristics is given by $(r, \ h)$, $D = T \cap C$, $O \notin D$, where T is a triangle which is constructed using two tangents and the chord that is drawn through the tangency points of the circle C; K: is a convex polygon with the vertices $v_1, \ \ldots, \ v_m$ which are given anticlockwise with the respect to the eigen coordinate system, i.e. $m_K = (v_1, \ \ldots, \ v_m)$.

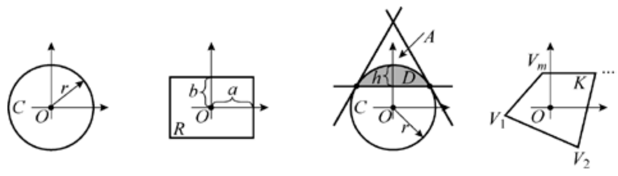


Fig.. 4. Basic objects $C, \ R, \ D, \ K$

Let us make the objects shape classification which comes to the phi-function technique. These shapes are composed phi-objects (hereinafter referred to as “the objects”).

1) If the rectangle corner curvings are equal then it is possible to construct the object using rectangle R determined by $(a, \ b)$ and circle C determined by radius r (Fig. 5,a). So the metric characteristics of type

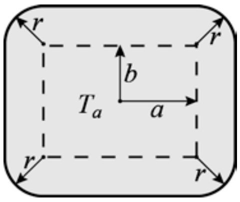
a object $m_a = (a,\ b,\ r)$. Therefore the first object type can be defined as

$$A_a = R(0) \oplus C(0), \tag{1}$$

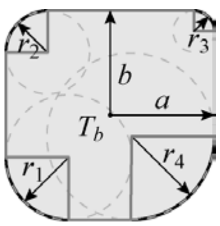
where $\oplus$ is the Minkovsky sum symbol.

2) If the rectangle corner curvings are different and all radii are less then the half-length of the rectangle after that it is possible to construct the object using polygon K with vertices v_i, $i = 1, \ldots, 8$, and four circles C_1, C_2, C_3, C_4 which are determined by different radii r_1, r_2, r_3, r_4 where the longest radius length is shorter than the half-height and half-width of the rectangle that circumscribes about the object A_b ($r_i \le \min\{a,\ b\}$, see Fig. 5,b). The metric characteristics of type b object are $m_b = (v_1, \ldots, v_8, r_i, \ldots, r_4)$. Therefore the second object type can be defined as

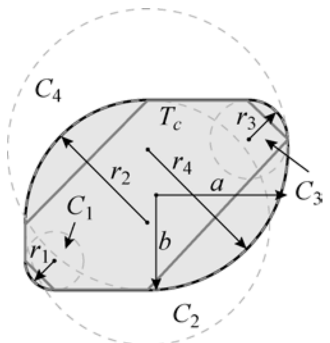
$$A_b = K \cup \left(\bigcup_{i=1}^{4} C_i \right). \tag{2}$$



(a)



(b)



(c)

Fig.5. Types of object shapes from collection $\Re$: the first type object (a) ; the second type object (b) ; the third type object (c)

3) If each corner of the rectangle has the arbitrary curving radius, it is possible to construct the objects as the union of the polygon K with the vertices v_i, $i = 1, \ldots, 8$, and four circular segments D_1, D_2, D_3, D_4, which are constructed using the tangents, the chord, the radii with unrestricted length whose the metric characteristics are $(r_j,\ h_j)$, $j = 1, \ldots, 4$ (see Fig. 5,c). The metric characteristics of type c object are $m_c = (v_1, \ldots, v_8, (r_1, h_1), \ldots, (r_4, h_4))$. Therefore the third object type can be defined as

$$A_c = K \cup (\bigcup_{i=1}^{4} D_i). \tag{3}$$

Let us note that the rectangle is degenerate case of object of type a, the type a object can be formulated like object of type b and all of the recommended objects can be formulated like a type c object. Depending on the problem requirements you can us all of this types without fail.

We provide a type c object that shows possible shapes depending on the metric characteristics.

The possible shapers of the type c object are given in the Figure 6.

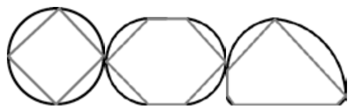


Fig.. 6. Possible shapers of the type c object

It has been proved in [7] that any phi-object bounded by circular arcs and line segments can be represented as a union of basic objects of four types.

Since A is a particular case which is described in [7] the following is true:

$$A = (A_1 \cup A_2 \cup \ldots \cup A_k),$$

where $A_i \in \Im$.

Let translation vectors $u_A = (x_A,\ \ y_A)$, $u_B = (x_B,\ \ y_B)$ be placement parameters of objects A and B respectively. We denote the vector of variables of both objects A and B by $u_{AB} = (x_A,\ \ y_A,\ \ x_B,\ \ y_B)$.

III. Placement Area

Taking into account the specific character of printing industry we note that the placement area is a printing sheet (or several printing sheets) which is always a rectangle $\Omega = R$. We set the pole $u_\Omega = (0, 0)$ of R in the bottom left corner (see Fig. 7) .

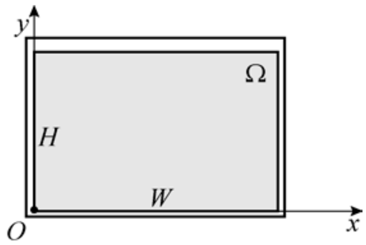


Fig.. 7. The placement area

According to the problem formulation defined in Section 1 each component of vector $P=(h,\ w)$, may be variable, where $h,\ w$ are the height and the width of R.

Now we introduce the possible additional constraints on $h,\ w$ for placement area R:

$$h_{\min} \le h \le h_{\max}\ ,\ w = \text{const}\ , \tag{3}$$

$$h = \text{const}\ ,\ w_{\min} \le w \le w_{\max}\ , \tag{4}$$

$$h = \text{const},\quad w = \text{const}\ , \tag{5}$$

$$h_{\min} \le h \le h_{\max}\ ,\ w_{\min} \le w \le w_{\max}\ , \tag{6}$$

where n is the number of the allowable printing sheet formats, $h_{\min}, h_{\max}, w_{\min}, w_{\max}$ are the minimal and maximal allowable values of $h,\ w$ respectively. For example, if we need to find the optimal printing sheet format to place the printing objects, the variables of the problem are h and w.

IV. Mathematical Modeling of The Relations of Printing Objects

The printing objects placement has its specific. It is clear that business cards, stickers, the handouts etc. can not be placed overlapping each other. It is also impossible to place the objects in such a way that any of the placed objects partially belong to the placement area (printing sheet) so any of the objects can not intersect the placement area boundary, all of the objects must be contained in the placement area completely. Sometimes, if it is necessary to place some specific kinds of printing objects, it is necessary to keep the minimal distance between the objects or between the frontier of placement area and the objects. For example, if we need to place stickers, the minimal allowed distance between the objects (depending on the equipment) can be 3 mm, which is due to engineering constrains of the knifes.

Taking into account all these object placement features the considered constraints can be formulated as follows.

1) The non-overlapping constraint, i.e. the objects do not overlap each other

$$\text{int}\, A \cap \text{int}\, B = \varnothing\,. \tag{7}$$

2) The containment constraint, i.e. each printing object has to be arranged within the placement area

$$A \subset \Omega \Leftrightarrow \Omega^* \cap A = \varnothing\ , \tag{8}$$

where $\Omega^* = R^2 \setminus \text{int}\,\Omega$.

3) The minimal allowed distances, i.e. the distance between the objects has to be greater than or equal to the minimal allowable distance:

$$\text{dist}(A,\ B) \ge \rho^-_{AB},\quad \text{dist}(\Omega^*,\ A) \ge \rho^-_{A}, \tag{9}$$

where $\text{dist}\left(A,\ \ B\right) = \min\limits_{a\in A,\ \ b\in B} \rho\left(a,\ b\right)$ is the Euclidian distance between objects A and B, $\rho(a,b)$ – distance between two points $a \in A$ and $b \in B$; $\text{dist}(\Omega^*,\ A)$ is the Euclidian distance between object A and object Ω^*.

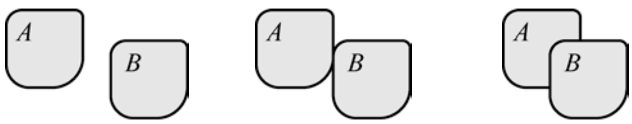
We apply the Stoyan phi-function technique [9, 10] in order to formalize constraints (8)-(10). As is known [11] within the field of Packing and Cutting the technique is the most powerful tool of mathematical modeling of relations between arbitrary shaped geometric objects in an analytical form. It should be noted that the algorithm of the constructing the No-Fit Polygon for the objects bounded by circular arcs proposed in [12] can be used only for heuristics solution methods.

V. Phi-Functions

According to the definition given in [9] continuous function $\Phi(u_A, u_B)$ defined everywhere is called a phi-function if it has the following characteristic properties:

$$\begin{aligned}
&\Phi\left(u_A,\ \ u_B\right) > 0,\ \text{if}\ \ A \bigcap B = \varnothing\\
&\Phi\left(u_A,\ \ u_B\right) = 0,\ \text{if}\ \ \text{int}\, A \bigcap \text{int}\, B = \varnothing,\\
&\qquad\qquad\qquad\qquad\ \ \text{fr}\ A \bigcap \text{fr}\ B \ne \varnothing\\
&\Phi\left(u_A,\ \ u_B\right) < 0,\ \text{if}\ \ \text{int}\, A \bigcap \text{int}\, B \ne \varnothing.
\end{aligned} \tag{11}$$

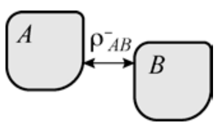
Fig. 8 shows arrangements of objects A and B.



(a) (b) (c)

Fig. 8. Arrangement of objects A and B non-overlapping (a), contact (b), overlapping (c)

When the distance between the objects or the distance between the objects and the frontier of placement area is important we use the normalized phi-function (Fig. 9,a) or pseudo-normalized phi-function (Fig. 9,b).



(a) (b)

Fig.9. The arrangement of A (a) and B (b) taking into account minimal allowable distance ρ^-_{AB}, $\Phi\left(u_A,\ \ u_B\right) = \rho^-_{AB}$, $\Phi\left(u_A,\ \ u_B\right) = 0$

The phi-function $\tilde{\Phi}(u_A,\ u_B)$ is said to be normalized [9] if its values are equal to the Euclidian distance $\rho(A,\ B)$ between two objects A and B, subject to $(u_A,\ u_B) \in D$,

$$D = \left\{ (u_A,\ u_B) \in R^6 :\ \operatorname{int} A(u_A) \cap \operatorname{int} B(u_B) = \varnothing \right\}.$$

The pseudo-normalized phi-function [8] is called the continuous, everywhere defined function $\hat{\Phi}(u_A,\ u_B)$ for which:

$$\begin{aligned}
&\hat{\Phi}(u_A,\ u_B) > 0, \text{ if } \operatorname{dist}(A,\ B) > \rho_{AB}^{-}\\
&\hat{\Phi}(u_A,\ u_B) = 0, \text{ if } \operatorname{dist}(A,\ B) = \rho_{AB}^{-}.\\
&\hat{\Phi}(u_A,\ u_B) < 0, \text{ if } \operatorname{dist}(A,\ B) < \rho_{AB}^{-}
\end{aligned} \tag{12}$$

Let $\hat{A} = A \oplus C(\rho)$, where $C(\rho)$ is the circle the radius of which is equal to the minimal allowed distance ρ_{ij}^{-} between two objects, the symbol $\oplus$ is the Minkovsky sum operation sign of object A and circle C, then $\hat{\Phi}(u_A,\ u_B) = \hat{\Phi}^{AB} = \Phi^{\hat{A}B}$, where $\Phi^{\hat{A}B}$ is the phi-function for a couple of objects $\hat{A}$ and B. Note that object $\hat{A}$ is an equidistant object, ρ_{ij}^{-} is radius of circle $C(\rho)$.

However, we note that the pseudo-normalized phi-function doesn't have radicals, it makes the function use in the modeling easier in the sense the effective local optimisation methods.

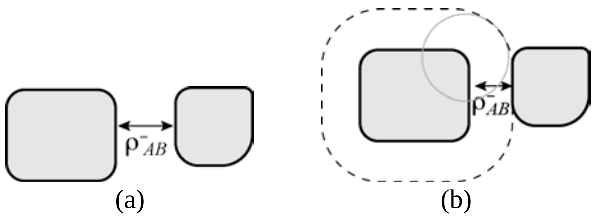


Fig..10. Minimal allowable distance ρ_{AB}^{-} between objects A and B: normalized phi-function $\tilde{\Phi} = \rho_{AB}^{-}$ (a), pseudo-normalized phi-function $\hat{\Phi}^{AB} = 0$ (b)

The problem constraints in terms of phi-functions

Constraints (9)–(11) described in Section IV can be formulated in the terms of phi-functions as follows:

1) Non-overlapping constraint:

$$\Phi^{AB} \ge 0, \tag{13}$$

where Φ^{AB} is a phi-function of objects A and B.

2) Containment constraint:

$$\Phi^{\Omega^* A} \ge 0, \tag{14}$$

where $\Phi^{\Omega^* A}$ is a phi-function for Ω^* and object A.

3) Distance constraints:

$$\begin{aligned}
&\tilde{\Phi}^{AB} \ge \rho_{AB}^{-} \Leftrightarrow \hat{\Phi}^{AB} \ge 0,\\
&\tilde{\Phi}^{\Omega^* A} \ge \rho_{A}^{-} \Leftrightarrow \hat{\Phi}^{\Omega^* A} \ge 0,
\end{aligned} \tag{15}$$

where $\tilde{\Phi}$, $\hat{\Phi}$ are the normalized and pseudo-normalized phi-functions respectively [7].

4) Based on additional constraints on metric characteristics of the placement area the vector can be given in the form:

$$u = (h,\ u_1,\ u_2,\ \ldots,\ u_m) \in R^{2m+1},\quad h_{\min} \le h \le h_{\max} \tag{16}$$

$$u = (w,\ u_1,\ u_2,\ \ldots,\ u_m) \in R^{2m+1},\quad w_{\min} \le w \le w_{\max} \tag{17}$$

$$u = (u_1,\ u_2,\ \ldots,\ u_m) \in R^{2m}, \tag{18}$$

$$u = (h,\ w,\ u_1,\ u_2,\ \ldots,\ u_m) \in R^{2m+2},\quad h_{\min} \le h \le h_{\max},\ w_{\min} \le w \le w_{\max} \tag{19}$$

where n is the number of the feasible printing sheet formats, m is the number of the objects which it is necessary to place on the printing sheet, h, w are the height and the width of the placement area respectively.

The phi-function constructing

For all object types which are mentioned in Fig. 5 we construct the phi-function that simulate the arrangement of a pair of objects.

Phi-functions for the arbitrary shaped and basic oriented objects bounded by circular arcs and line segments are proposed in works [7, 8]. Let us consider phi-functions for all combinations of pairs of objects from set A.

Non-overlapping constraints

Let there be objects A and B of type A_a given by (1) with placement parameters $u_A = (x_A,\ y_A)$, $u_B = (x_B,\ y_B)$ and metric characteristics $m_A = (a_A,\ b_A, r_A)$, $m_B = (a_B,\ b_B, r_B)$ respectively. Then, assuming $x = x_B - x_A$, $y = y_B - y_A$, the phi-function for the objects is defined as follows:

$$\Phi^{AB} = \max\left\{ \chi_i,\ \min\{\omega_i, \psi_i\},\ i = 1, \ldots, 4 \right\}, \tag{20}$$

where $\chi_1 = -x - A'$, $\chi_2 = y - B'$, $\chi_3 = x - A'$, $\chi_4 = -y - B'$,

$$\omega_1 = (x + A')^2 + (y + B')^2 - R^2,$$

$$\omega_2 = (x + A')^2 + (y - B')^2 - R^2,$$

$$\omega_3 = (x - A')^2 + (y - B')^2 - R^2,$$

$$\omega_4 = (x - A')^2 + (y + B')^2 - R^2,$$

$$\psi_1 = -x - y - A' - B' - R,\ \psi_2 = -x + y - A' - B' - R,$$

$$\psi_3 = x + y - A' - B' - R,\ \psi_4 = x - y - A' - B' - R.$$

Here $A' = a_A + a_B + R$; $B' = b_A + b_B + R$, $R = r_A + r_B$.

Let the objects $A = \bigcup_{i=1}^{5} A_i$, $A_i \in \{A_b\}$ and $B = \bigcup_{j=1}^{5} B_j$, $B_j \in \{A_b\}$ (2) with $u_A = (x_A, y_A)$ and $u_B = (x_B, y_B)$ as the placement parameters respectively then the phi-function for the pair of the objects is defined as follows:

$$\Phi^{AB} = \min\left\{\Phi^{A_i B_j},\quad i = 1, \ldots, 5,\quad j = 1, \ldots, 5\right\}, \quad (21)$$

where $\Phi^{A_i B_j}$ is the phi-function of the basic objects pair which form objects A_b and B_b.

A phi-function for the objects may be derived in the form (20), where $x = x_2 - x_1$, $y = y_2 - y_1$, $\chi_1 = -x - A'$, $\chi_2 = y - B'$, $\chi_3 = x - A'$, $\chi_4 = -y - B'$,

$$\omega_i = (x + A' - R_i)^2 + (y + B' - R_i)^2 - R_i{}^2,$$

$$\psi_i = x + y + A' + B' - R_i,\ i = 1, \ldots, 4,\ A' = a_A + a_B,$$

$$B' = b_A + b_B,\ a_A,\ a_B,\ b_A,\ b_B$$

are the length parameters which characterize the phi-objects A and B respectively.

Let us consider objects $A = K' \cup (\bigcup_{i=1}^{4} D_i')$, $A \in \{A_c\}$ and $B_c = K'' \cup (\bigcup_{i=1}^{4} D_i'')$, $B \in \{A_c\}$ (3) which have $u_A = (x_A, y_A)$ and $u_B = (x_B, y_B)$ as the placement parameters respectively. In this case the phi-function is defined as follows:

$$\Phi^{AB} = \min\{\Phi^{K'K''},\ \Phi^{K'D_i''},\ i = 1, \ldots, 4,$$

$$\Phi^{K''D_j'},\ j = 1, \ldots, 4\}, \quad (22)$$

where $\Phi^{K'K''}$ is a phi-function for $K' \subset A_c$ and $K'' \subset B_c$; $\Phi^{K'D_i''}$ is a phi-function for convex polygon $K' \subset A_c$ and circular object $D'' \subset B_c$; $\Phi^{K''D_j'}$ is a phi-function for convex polygon $K'' \subset B_c$ and circular object $D' \subset A_c$.

Containment constraints

According to the object classification given in Section II of this work let us construct the phi-functions describing relation of the containment the objects in the placement area for all mentioned objects.

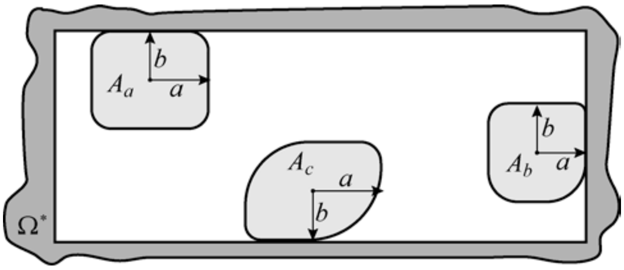


Fig. 11. The placement of the three types of printing objects

In view of the domain area specifics we note that the placement area is always a rectangle. As Figure 11 shows all of the objects have the half-width and the half-height as their parameter. Thus, based on the object of set A and placement area $\Omega^* = R^2 / \operatorname{int}\Omega$ have the placement parameters $u_A = (x_A, y_A)$ and $u_{\Omega^*} = (0, 0)$ then the following is true for all phi-object types of set A:

$$\Phi^{\Omega^* A} = \min\{\chi_i,\quad i = 1, \ldots, 4\}, \quad (23)$$

where H, W are the height and the width of the placement area respectively;

$\chi_1 = x - a$, $\chi_2 = w - b - x$, $\chi_3 = y - b$, $\chi_4 = h - b - y$, $x = x_A$, $y = y_A$.

Distance constraints

If the problems need the constraints (15), it should be taken using the normalized or pseudo-normalized phi-function (see Fig. 10 a, b).

The normalized phi-function for the type objects (1) can be defined by (20), assuming $x = x_B - x_A$, $y = y_B - y_A$, $\chi_1 = -x - A'$, $\chi_2 = y - B'$, $\chi_3 = x - A'$, $\chi_4 = -y - B'$, $\omega_1 = \sqrt{(x + A')^2 + (y + B')^2} - R$,

$$\omega_2 = \sqrt{(x + A')^2 + (y - B')^2} - R,$$

$$\omega_3 = \sqrt{(x - A')^2 + (y - B')^2} - R,$$

$$\omega_4 = \sqrt{(x - A')^2 + (y + B')^2} - R,$$

$$\psi_1 = -x - y - A' - B' - R,\ \psi_2 = -x + y - A' - B' - R,$$

$$\psi_3 = x + y - A' - B' - R,\ \psi_4 = x - y - A' - B' - R,$$

$$A' = a_A + a_B + R;\ B' = b_A + b_B + R.$$

Here a_A, a_B, b_A, b_B are the metric characteristics of objects A and B respectively, $R = r_A + r_B$; r_A, r_B are the radii of the circles which are characterized the corner curving for the objects A and B.

The normalized phi-function for the type objects (2) can be defined by (20), assuming $x = x_2 - x_1$, $y = y_2 - y_1$, $\chi_1 = -x - A'$, $\chi_2 = y - B'$, $\chi_3 = x - A'$, $\chi_4 = -y - B'$,

$$\omega_i = \sqrt{(x + A' - R_i)^2 + (y + B' - R_i)^2} - R_i,$$

$$\psi_i = x + y + A' + B' - R_i,\ i = 1, 2, 3, 4;\ A' = a_A + a_B$$

$B' = b_A + b_B$. Here a_A, a_B, b_A, b_B are the metric characteristics of objects A and B respectively.

The limitation of the determining the distances using the normalized phi-function is associated with the radicals that makes normalized phi-function use more complex in the local and the global optimization methods. Therefore it should be used the pseudo-normalized phi-function (12) to determine the distances between the objects.

V. Mathematical Model And Solution Strategy

Let us consider the printing objects placement problem of the given shape on the printing sheet in the following statement.

Let a rectangular region $P = \left\{ (x,\ y) \in R^2 \mid \ 0 \le x \le w,\ \ 0 \le y \le h \right\}$ of variable length w and height h, and objects $A_i \subset R^2$, $i \in I_m = \{1,\ \dots,\ m\}$ be given. Also the minimal distance constraints are given: between objects A_i and A_j, i.e. $\rho\left(A_i,\ \ A_j\right) \ge \rho_{ij}^-$, $i \ne j$, $i, j \in I_m$, as well as between object $A_i \subset R^2$ and the frontier of placement area Ω i.e. $\rho\left(\Omega^*,\ \ A_i\right) \ge \rho_i^-$, $i \in I_m$.

Placement problem. Place objects A_i, $i \in I_m$ within placement area Ω taking into account distance constraints so that

$$\operatorname{int} A_i\left(u_i\right) \cap \left(\Omega^* \oplus S_i\right) = \varnothing,$$

$$\operatorname{int} A_i \cap \left(A_j \oplus S_{ij}\right) = \varnothing,\ i \ne j,\ \ i, j \in I_m,$$

where S_i, $S_{ij} \subset R^2$ are circles of radii ρ_i^- and ρ_{ij}^- respectively.

In the phi-function terms the constraints can be described by the inequalities (13)-(15) and the placement problem can be formulated as the following optimisation problem:

$$\min F\left(u\right),\ \text{s.t. } u \in W \subset R^{2m+2} \qquad (24)$$

where $u = \left(h,\ w,\ u_1,\ u_2,\ \dots,\ u_m\right)$, $F\left(u\right) = h \cdot w$,

$$W = \begin{cases} u \in R^{2m+2} : \Phi_{ij}\left(u_i,\ u_j\right) \ge 0,\ i < j = 1, 2, \dots, m; \\ \Phi_i\left(0,\ u_i\right) \ge 0,\ i = 1, 2, .., m; \end{cases}$$

$h_{\min} \le h \le h_{\max}$, $w_{\min} \le w \le w_{\max}$.

The problem (24) is the nonlinear mathematical programming problem with the linear objective function, a solution space (a feasible region) W is described using $\frac{1}{n}(n-1)$ inequalities in the form $\Phi_{ij}(u_i, u_j) \ge 0$ and n inequalities in the form $\Phi_i(0, u_i) \ge 0$; phi-inequality $\Phi_{ij}(u_i, u_j) \ge 0$ provides non-overlapping of objects $A_i\left(u_i\right)$ and $A_j\left(u_j\right)$, $i < j = 2,\ \dots,\ n$; the condition $\Phi_i(0, u_i) \ge 0$ provides the containment of objects $A_i\left(u_i\right)$ to placement area P, $i = 1,\ 2,\ \dots,\ n$; m makes it possible to determine the number of objects which are placed in the area.

It is well known that feasible region W of problem (24) can be represented as a union of subregions W^k, $i = 1,\ 2,\ \dots, \eta$, because each phi-function is a superposition of the finite number of the minimum or the maximum functions.

Therefore to solve the problem (24) it is always possible to construct a solution tree. Each terminal node there corresponds to a non-linear inequality system which describe subregion W^k.

The problem (24) can be reduced to the following problem:

$$F\left(u^{*k}\right) = \min\left\{F\left(u^k\right), k = 1, 2, \dots, \eta\right\}, \qquad (25)$$

where

$$F\left(u^{*k}\right) = \min F\left(u\right),\ \text{s.t. } u \in W^k. \qquad (26)$$

Based on the characteristics of problem (24) we conclude that the problem is a multiextremal and NP-hard. We propose the solution strategy based on [13] which involves: construction of starting points from the feasible region; searching for a local minima of subproblems (26); searching for a good local optimal placement of problem (24).

This algorithm finds good solutions with reasonable computation times that do not increase significantly with the complexity of the objects. In order to obtain a good starting solution $u^0 \in W$ the algorithm employs a fast and the efficient heuristic given in [8]. The heuristic is based on searching for an approximate solution of problem (24) provided that the placement parameters of objects take discrete values. Then the algorithm applies IPOPT [14] search for local minima. Below we give a description of the algorithm.

Let us define function $\Lambda(u) = \min\{\Phi_{ij}(u_i, u_j), i < j = 1, \dots, n, \Phi_i(0, u_i), i = 1, \dots, n\}$. Our aim is to extract from $\Lambda(u^0) \ge 0$ an inequality system, which describes subregion $W_s \subset W$, such that $u^0 \in W_s$, using the solution tree strategy proposed in [13]. We form the subregion W_s as follows.

Each basic phi-function Φ_k may be given in the form:

$$\Phi_k = \max_{i=1,..,\eta_k} f_i^k = \max_{i=1,..,\eta_k} \min_{j=1,\dots,J_i^k} f_{ij}^k,$$

where f_{ij}^k are infinitely-differentiable functions. Since $\min_{j=1,\dots,J_i^k} f_{ij}^k \ge 0$ is equivalent to $f_{ij}^k \ge 0$ for all j, and $\max_{i=1,..,\eta_k} f_i^k \ge 0$ means at least one of the inequalities, say $f_{i_0}^k \ge 0$ has to be fulfilled, each of these terms can be considered as a system of (in general non-linear) inequalities. Then for each inequality $\Phi_k \ge 0$ we may construct a tree, called a *basic phi-tree* and noted by $\Im_k$, and η_k means the number of terminal nodes of the basic phi-tree. Each terminal node of $\Im_k$ corresponds to a system of inequalities $f_i^k \ge 0$, $i = 1, 2, \dots, \eta_k$.

The solution tree $\Im$ describes feasible region W of problem (24). We realise an exhaustive search of nodes $v_s^1, s = 1, \dots, \eta_1$, of the first level of the solution tree $\Im$ sequentially and search for the number s_1 such that

$$f_{s_1}^1(u^0) = f^1(u^0) = \max\{f_1^1(u^0),\ f_2^1(u^0), \dots,\ f_{\eta_1}^1(u^0)\}$$

Then we realize an exhaustive search of offsprings v_s^2, $s = 1, \dots, \eta_2$, of node $v_{s_1}^1$ and search for the number s_2 such that

$$f_{s_2}^2(u^0) = f^2(u^0) = \max\{f_1^2(u^0),\ f_2^2(u^0), \dots,\ f_{\eta_1}^2(u^0)\}$$

and so on.

On the *n*-th level of our solution tree $\Im$ we realise an exhaustive search of nodes $v_s^n, s = 1, \dots, \eta_n$ which are offsprings of node $v_{s_{n-1}}^{n-1}$ and search for the number s_n such that

$$f_{s_n}^n(u^0) = f^n(u^0) = \max\{f_1^n(u^0),\ f_2^n(u^0), \dots,\ f_{\eta_n}^n(u^0)\}$$

Then we form inequality system which corresponds to s-*th* terminal node of our solution tree $\Im$ in the form: $W_s = \{u \in R^\sigma : f_{s_1}^1 \ge 0,\ f_{s_2}^2 \ge 0, \dots,\ f_{s_n}^n \ge 0, \lambda \ge 0\}$. To each sequence of numbers $s_1, s_2, \dots, s_k, \dots, s_n$ there corresponds the number s.

Finally, we solve problem $\min_{u \in W_s} F(u)$ starting from point u^0.

VI. Conclusion

The proposed mathematical model and solution strategy allow us to employ the local and the global optimisation methods for solving the optimisation placement problem of printing objects.

The real placement problem of printing objects can be applied in the publishing-printing houses. Using the mathematical model allows us to get the layout templates for printing the different objects from several clients in one time reducing the time for processing the order and saving the material, which can be used for order group printing. Due to the involvement the proposed solution strategy to the production the publishing-printing houses can reduce the material waste by saving the material and this will influence to the environment pollution reduction.

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