STUDIA UNIVERSITATIS MOLDAVIAE, 2015, nr.2(82)
Seria "Ştiinţe exacte şi economice" ISSN 1857-2073 ISSN online 2345-1033 p.53-60

ON RECURSIVE DIFFERENTIABILITY OF BINARY QUASIGROUPS

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Recursively differentiable binary quasigroups and loops, are considered in the present paper. Invariants under the recursive differentiability of binary quasigroups are found. It is shown that the recursive derivative of order one of an LIP-loop $(Q,\cdot)$ (in particular of a left Bol loop) is an isostrophe of its core. The recursively 1-differentiable left Bol loops are studied. Some properties of recursive derivatives of order one of left Bol loops are established.

Keywords: *Recursively differentiable quasigroup, recursive derivative, isostrophe, core, LIP-loop, left Bol loop.*

CU PRIVIRE LA DERIVABILITATEA RECURSIVĂ A CVASIGRUPURILOR BINARE
În lucrare sunt studiate cvasigrupurile şi buclele recursiv derivabile. Sunt determinaţi invarianţi la derivarea recursivă a cvasigrupurilor binare. Se arată că derivatele recursive de ordinul unu ale LIP-buclelor (în particular, ale buclelor Bol la stânga) sunt izostrofi ai miezului lor. Sunt stabilite unele proprietăţi ale derivatelor recursive de ordinul unu ale buclelor Bol la stânga.

Cuvinte-cheie: *cvasigrup recursiv derivabil, derivate recursive, izostrof, miez, LIP-buclă, buclă Bol la stânga.*

Let (Q,A) be an n-ary groupoid and k a positive integer. The n-ary operation $A^{(k)}$ defined on Q as follows:

$$A^{(k)}=\begin{cases} A\left(x_{k+1},\dots,x_n,A^{(0)}\left(x_1,\dots x_n\right),\dots,A^{(k-1)}\left(x_1,\dots x_n\right)\right), \text{ if } k<n,\\ A\left(A^{(k-n)}\left(x_1,\dots x_n\right),\dots,A^{(k-1)}\left(x_1,\dots x_n\right)\right), \textit{if } k\ge n,\end{cases}$$

is called the recursive derivative of order k (or the k-recursive derivative) of the n-ary operation A. If the recursive derivatives $A^{(1)},A^{(2)},\dots A^{(s)}$ of a n-ary quasigroup (Q,A) are quasigroup operations then (Q,A) is called recursively s-differentiable. If $(Q,\cdot)$ is a binary groupoid then will denote its recursive derivative of order s by „ $\overset{s}{\cdot}$ ”, hence

$$x\overset{0}{\cdot}y=x\cdot y,\ \ x\overset{1}{\cdot}y=y\cdot xy,\ \ \dots,\ \ x\overset{s}{\cdot}y=(x\overset{s-2}{\cdot}y)\cdot(x\overset{s-1}{\cdot}y),$$

for every $x,y\in Q.$ The notions of recursive derivatives and recursively differentiable quasigroups raised in the algebraic coding theory ([6]). The general form of the recursive derivatives was announced in [4] for binary quasigroups and was generalized for n-ary quasigroups in [8]. Also, it is proved in [8] that an abelian group $(Q,\cdot)$ is recursively s-differentiable, where $s\ge 1,$ if and only if the mappings $x\to x^{b_i}$, where $(b_i)_{i\in N}$ is the Fibonacci sequence, are bijections, for every $i\in\{0,1,\dots,s+1\}$. Strongly recursively r-differentiable quasigroups are studied in [5].

Recursively differentiable binary quasigroups and loops, are considered in the present paper. Invariants under the recursive differentiability of binary quasigroups are found. Connections between the groups of automorphisms and semiautomorphisms of a quasigroup and of its recursive derivatives are established. It is shown that the recursive derivative of order 1 of an LIP-loop $(Q,\cdot)$ (in particular of a left Bol loop) is an isostrophe of its core. The recursively 1-differentiable left Bol loops are studied. Some properties of recursive derivatives of order one of left Bol loops are established.

Proposition 1. *If two binary groupoids* $(Q,\cdot)$ *and* $(Q',\circ)$ *are isomorphic, then* $(Q,\overset{i}{\cdot})\cong(Q',\overset{i}{\circ})$, *for every* $i\ge 1$.

Proof. Let φ be an isomorphism from $(Q,\cdot)$ to $(Q',\circ)$. Then $\varphi(x\cdot y)=\varphi(x)\circ\varphi(y)$, for $\forall x,y\in Q$, which implies $\varphi(x\overset{1}{\cdot}y)=\varphi(y\cdot xy)=\varphi(y)\circ[\varphi(x)\circ\varphi(y)]=\varphi(x)\overset{1}{\circ}\varphi(y)$, $\forall x,y\in Q$, i.e. $(Q,\overset{1}{\cdot})\cong(Q',\overset{1}{\circ})$.
Suppose that $\varphi(x\overset{i}{\cdot}y)=\varphi(x)\overset{i}{\circ}\varphi(y)$, for every $1\le i\le n-1$. Hence,

$$\varphi(x\overset{n}{\cdot}y)=\varphi\left[(x\overset{n-2}{\cdot}y)\cdot(x\overset{n-1}{\cdot}y)\right]=$$

$$\varphi(x\overset{n-2}{\cdot}y)\circ\varphi(x\overset{n-1}{\cdot}y)=\left[\varphi(x)\overset{n-2}{\circ}\varphi(y)\right]\circ\left[\varphi(x)\overset{n-1}{\circ}\varphi(y)\right]=\varphi(x)\overset{n}{\circ}\varphi(y),$$

for $\forall x,y\in Q$, i.e. $(Q,\overset{i}{\cdot})\cong(Q',\overset{i}{\circ})$, $\forall i\ge 1$. □

Lemma 1. *If $(Q,\cdot)$ is a binary groupoid then, for every $x,y\in Q$, the following equalities hold:*

1) $x\overset{n}{\cdot}y=y\overset{n-1}{\cdot}(x\overset{0}{\cdot}y),\quad \forall n\ge 1;\quad \forall x,y\in Q;$
2) $x\overset{n}{\cdot}y=(x\overset{j-1}{\cdot}y)\overset{n-j-1}{\cdot}(x\overset{j}{\cdot}y),\quad \forall j=\overline{1,n-1};\quad \forall n\ge 2.$

Proof. 1) If $(Q,\cdot)$ is a binary groupoid then $x\overset{1}{\cdot}y=y\overset{0}{\cdot}(x\overset{0}{\cdot}y)$. Suppose that $x\overset{i}{\cdot}y=y\overset{i-1}{\cdot}(x\overset{0}{\cdot}y)$, for every $1\le i\le n-1$, then

$$x\overset{n}{\cdot}y=(x\overset{n-2}{\cdot}y)\overset{0}{\cdot}(x\overset{n-1}{\cdot}y)=\left[y\overset{n-3}{\cdot}(x\overset{0}{\cdot}y)\right]\overset{0}{\cdot}\left[y\overset{n-2}{\cdot}(x\overset{0}{\cdot}y)\right]=y\overset{n-1}{\cdot}(x\overset{0}{\cdot}y).$$

2). Let $n=2$ and $j=1$, then:

$$x\overset{2}{\cdot}y=(x\overset{0}{\cdot}y)\overset{0}{\cdot}(x\overset{1}{\cdot}y).$$

Suppose that $x\overset{n}{\cdot}y=(x\overset{j-1}{\cdot}y)\overset{n-j-1}{\cdot}(x\overset{j}{\cdot}y)$, for every $1\le j\le n-1,\ \ n=k$. For $n=k+1$, we have

$$x\overset{k+1}{\cdot}y=(x\overset{k-1}{\cdot}y)\overset{0}{\cdot}(x\overset{k}{\cdot}y)=\left[(x\overset{j-1}{\cdot}y)\overset{(n-1)-(j+1)}{\cdot}(x\overset{j}{\cdot}y)\right]\overset{0}{\cdot}\left[(x\overset{j-1}{\cdot}y)\overset{n-(j+1)}{\cdot}(x\overset{j}{\cdot}y)\right]=(x\overset{j-1}{\cdot}y)\overset{n-(j+1)}{\cdot}(x\overset{j}{\cdot}y),$$

$\forall\ 1\le j\le n-1$. □

Recall that, if $(Q,\cdot)$ is a groupoid, then a bijection $\varphi:Q\to Q$ is called a (left) semi-automorphism of $(Q,\cdot)$ if $\varphi(x\cdot yx)=\varphi(x)\cdot\varphi(y)\varphi(x)$, $\forall x,y\in Q$. We will denote by $SAut(Q,\cdot)$ the set of all (left) semi-automorphisms of $(Q,\cdot)$. Note that $SAut(Q,\cdot)$ is a subgroup of the symmetric group S_Q.

Proposition 2. *If $(Q,\cdot)$ is a groupoid then the following statements hold:*

1) $Aut(Q,\cdot)\le Aut(Q,\overset{n}{\cdot}),\ \forall n\ge 1$;
2) $Aut(Q,\overset{i}{\cdot})\le SAut(Q,\overset{i}{\cdot}),\quad \forall i\ge 0;$
3) $SAut(Q,\cdot)=Aut(Q,\overset{1}{\cdot});$
4) $SAut(Q,\cdot)\le SAut(Q,\overset{1}{\cdot})$.

Proof. 1) If $\varphi \in Aut(Q,\cdot)$ then $\varphi(x\overset{1}{\cdot}y)=\varphi\left(y\cdot xy\right)=\varphi(y)\cdot\varphi(x)\varphi(y)=\varphi(x)\overset{1}{\cdot}\varphi(y),\ \forall x,y\in Q$, so $\varphi\in Aut(Q,\overset{1}{\cdot})$. Suppose that $\varphi\in Aut(Q,\overset{k}{\cdot})$, for every $1\le k\le n-1$. Then

$\varphi(x\overset{n}{\cdot}y)= =\varphi\left((x\overset{n-2}{\cdot}y)\cdot(x\overset{n-1}{\cdot}y)\right)=$

$\varphi(x\overset{n-2}{\cdot}y)\cdot\varphi(x\overset{n-1}{\cdot}y)=\left[\varphi(x)\overset{n-2}{\cdot}\varphi(y)\right]\cdot\left[\varphi(x)\overset{n-1}{\cdot}\varphi(y)\right]=\varphi(x)\overset{n}{\cdot}\varphi(y),$

for $\forall x,y\in Q$, i.e. $\varphi\in Aut(Q,\overset{n}{\cdot})$, so $Aut(Q,\cdot)\le Aut(Q,\overset{n}{\cdot}),\ \forall n\ge 1$.

2) If $\varphi\in Aut(Q,\overset{i}{\cdot})$ then $\varphi(x\overset{i}{\cdot}y)=\varphi(x)\overset{i}{\cdot}\varphi(y)$, hence $\varphi(x\overset{i}{\cdot}(y\overset{i}{\cdot}x))=\varphi(x)\overset{i}{\cdot}(\varphi(y)\overset{i}{\cdot}\varphi(x))$, $\forall x,y\in Q$, i.e. $\varphi\in SAut(Q,\overset{i}{\cdot}),\ \forall i\ge 0$.

3) If $\varphi\in SAut(Q,\cdot)$ then $\varphi(x\cdot yx)=\varphi(x)\cdot\varphi(y)\varphi(x)$, $\forall x,y\in Q$, so $\varphi(x\overset{1}{\cdot}y)=\varphi(y\cdot xy)=\varphi(y)\cdot\varphi(x)\varphi(y)=\varphi(x)\overset{1}{\cdot}\varphi(y)$, $\forall x,y\in Q$, i.e. $\varphi\in Aut(Q,\overset{1}{\cdot})$. Conversely, if $\varphi\in Aut(Q,\overset{1}{\cdot})$, then $\varphi(x\overset{1}{\cdot}y)=\varphi(x)\overset{1}{\cdot}\varphi(y)\Leftrightarrow\ \varphi(y\cdot xy)=\varphi(y)\cdot\varphi(x)\varphi(y),\ x,y\in Q$, i.e. $\varphi\in SAut(Q,\cdot)$.

4) Follows from 2. and 3. □

Let $(Q,\cdot)$ be a quasigroup and $a\in Q$, then the mapping $R_a:Q\to Q,\ R_a(x)=xa,\ \forall x\in Q$, is called the right translation with the element a and the mapping $L_a:Q\to Q,\ \mathrm{L}_a(x)=ax,\ \forall x\in Q$ is called the left translation with the element a. We will denote: $\mathrm{RM}(Q,\cdot)=\langle R_b\mid b\in Q\rangle$ (respectively $\mathrm{LM}(Q,\cdot)=\langle L_a\mid a\in Q\rangle$ and $\mathrm{M}(Q,\cdot)=\langle L_a,R_b\mid a,b\in Q\rangle$) – the right multiplication group (respectively, the left multiplication group and the multiplication group) of the quasigroup $(Q,\cdot)$. If $(G,\cdot)$ is a group and $h\in G$, then the set $G_h=\{x\in G\mid x\cdot h=h\cdot x\}$ is a subgroup of G and is called the centralizer of h in $(G,\cdot)$. If $(Q,\cdot)$ is a quasigroup and $h\in Q$, the group $I_h=\{\sigma\in M(Q,\cdot)\mid\sigma(h)=h\}$ is called the group of inner mappings, with respect to the element h.

Proposition 3. *If $(Q,\cdot)$ is a recursively 1-differentiable quasigroup then $RM(Q,\overset{1}{\cdot})\le M(Q,\cdot)$. More, if $(Q,\cdot)$ is recursively 2-differentiable and $(Q,\overset{1}{\cdot})$ is a commutative quasigroup then $RM(Q,\overset{2}{\cdot})\le M(Q,\cdot)$.*

Proof. If $(Q,\cdot)$ is a recursively 1-differentiable quasigroup then $R_y^{(\overset{1}{\cdot})}(x)=\ x\overset{1}{\cdot}y=\ y\cdot xy=\ L_y^{(\cdot)}R_y^{(\cdot)}(x)$, $\forall x\in Q$, which implies

$$R_y^{(\overset{1}{\cdot})}=L_y^{(\cdot)}R_y^{(\cdot)}, \tag{1}$$

for $\forall y\in Q$, so $RM(Q,\overset{1}{\cdot})\le M(Q,\cdot)$. If $(Q,\cdot)$ is recursively 2-differentiable and $(Q,\overset{1}{\cdot})$ is commutative, then $L_x^{(\overset{1}{\cdot})}=R_x^{(\overset{1}{\cdot})}$ and $R_x^{(\overset{2}{\cdot})}(y)=y\overset{2}{\cdot}x=x\overset{1}{\cdot}(y\cdot x)=L_x^{(\overset{1}{\cdot})}R_x^{(\cdot)}(y)=R_x^{(\overset{1}{\cdot})}R_x^{(\cdot)}(y)$, $\forall y\in Q$. So, using (1) we get $R_x^{(\overset{2}{\cdot})}=L_x^{(\cdot)}R_x^{(\cdot)}R_x^{(\cdot)}\in M(Q,\cdot)$, i.e. $RM(Q,\overset{2}{\cdot})\le M(Q,\cdot)$ □

Corollary. *If* $(Q,\cdot)$ *is a recursively 1-differentiable quasigroup, then* $RM(Q,\overset{1}{\cdot})_h \subseteq M(Q,\cdot)_h = I_h^{(\cdot)}$, *for every* $h \in Q$.

Proof. If $\varphi \in RM(Q,\overset{1}{\cdot})_h$ then $\varphi \in M(Q,\cdot)$ and $\varphi(h) = h$ i.e. $\varphi \in I_h^{(\cdot)}$, so $RM(Q,\overset{1}{\cdot}) \le I_h^{(\cdot)}$, $\forall h \in Q$. □

Remark 1. If $(Q,\cdot)$ is a quasigroup and $\alpha \in S_Q$, then α is a right regular mapping of $(Q,\cdot)$, i. e. $\alpha \in \Re_{(\cdot)} \quad \Leftrightarrow \quad \alpha(x\cdot y) = x\cdot\alpha(y) \quad \Leftrightarrow \quad \alpha L_x^{(\cdot)}(y) = L_x^{(\cdot)}\alpha(y),\ \forall x, y \in Q,$ which is equivalent to $\alpha \in C_{S_Q}(LM(Q,\cdot))$, so

$$\Re_{(\cdot)} = C_{S_Q}(LM(Q,\cdot)) \ . \tag{2}$$

Proposition 4. *If* $(Q,\cdot)$ *is a recursively 1-differentiable quasigroup then:*

1) $\Re_{(\cdot)} \cap \Re_{\overset{1}{(\cdot)}} = \{\varepsilon\}$;

2) $C_{S_Q}(LM(Q,\cdot)) \cap C_{S_Q}(LM(Q,\overset{1}{\cdot})) = \{\varepsilon\}$.

Proof. 1) If $\rho \in \Re_{(\cdot)} \cap \Re_{\overset{1}{(\cdot)}}$ then $\rho(x\overset{1}{\cdot}y) = x\overset{1}{\cdot}\rho(y) \ \Rightarrow \ \rho(y\cdot xy) = \rho(y)\cdot x\rho(y) \ \Rightarrow \ y\cdot x\rho(y) = \rho(y)\cdot x\rho(y)$, $\forall x, y \in Q \Rightarrow y = \rho(y),\ \forall y \in Q \Rightarrow \Re_{(\cdot)} \cap \Re_{\overset{1}{(\cdot)}} = \{\varepsilon\}$.

2) The proof follows from 1) and the equality (2). □

Proposition 5. *If* $(Q,\cdot)$ *is a recursively 1-differentiable loop then the following statements hold:*

1) $a\cdot a = a, \forall a \in N_r^{\overset{1}{(\cdot)}}$, *where* $N_r^{\overset{1}{(\cdot)}}$ *is the right nucleus of* $(Q,\overset{1}{\cdot})$;

2) $N_r^{\overset{1}{(\cdot)}} = \{e\}$, *where* e *is the unit of* $(Q,\cdot)$;

3) $\Re_{\overset{1}{(\cdot)}} = \{\varepsilon\}$.

Proof. 1) If $(Q,\cdot)$ is a recursively 1-differentiable loop with the unit e, then e is a right unit in $(Q,\overset{1}{\cdot})$. Let $a \in N_r^{\overset{1}{(\cdot)}}$, then $(x\overset{1}{\cdot}y)\overset{1}{\cdot}a = x\overset{1}{\cdot}(y\overset{1}{\cdot}a) \Leftrightarrow a\cdot(y\cdot xy)a = (a\cdot ya)\cdot x(a\cdot ya),\ \forall x, y \in Q$. Taking $x = y = e$ in the last equality, we get $a\cdot a = (a\cdot a)\cdot(a\cdot a)$, so $a\cdot a = e$.

2) Let $a \in N_r^{\overset{1}{(\cdot)}}$. Using 1), we have: $a\overset{1}{\cdot}a = a\cdot(a\cdot a) = a\cdot e = a = a\overset{1}{\cdot}e \Rightarrow a = e$, so $N_r^{\overset{1}{(\cdot)}} = \{e\}$.

3) According to 2), $N_r^{\overset{1}{(\cdot)}} = \{e\}$. So as $\Re_{\overset{1}{(\cdot)}} \cong N_r^{\overset{1}{(\cdot)}}$, we get $\Re_{\overset{1}{(\cdot)}} = \{\varepsilon\}$. □

Lemma 2. *If a quasigroup* $(Q,\cdot)$, *with the left unit, is recursively 1-differentiable, then the mapping* $x \to x^2$ *is a bijection.*

Proof. If the quaisgroup $(Q,\cdot)$ has the left unit f, then $f\overset{1}{\cdot}x = x\cdot fx = x\cdot x,\ \forall x \in Q$, so the mapping $x \to x\cdot x$ is a bijection on *Q*. □

Remark 2. 1. The converse of Lemma 2 is not true, i.e. if $(Q,\cdot)$ is a recursively 1-differentiable quasigroup and the mapping $Q \to Q,\ x \to x\cdot x$ is a bijection, than $(Q,\cdot)$ has not always a left unit. For example, the quasigroup ($Z_{7,\cdot}$), where $x\cdot y = 4x + 4y (\bmod 7),\ \forall x, y \in Q$, is recursively 1-differentiable and $x\cdot x = x$, for every x∊Z_7, so the mapping $x \to x\cdot x$ is a bijection, but ($Z_{7,\cdot}$) has not a left unit.

2. If $(Q,\cdot)$ is a recursively 1-differentiable loop then the mapping $x \to x^2$ is a bijection.

Proposition 6. *A diassociative loop* $(Q,\cdot)$ *is recursively 1-differentiable if and only if the mapping* $x \to x^2$ *is a bijection.*

Proof. Let $(Q,\cdot)$ be a diassociative loop with the unit e. If $(Q,\cdot)$ is recursively 1-differentiable then $(Q,\overset{1}{\cdot})$ is a quasigroup, so the equivalent equations $e\overset{1}{\cdot}x = a \Leftrightarrow x\cdot ex = a \Leftrightarrow x\cdot x = a$ have a unique solution, for each $a \in Q$, i.e. the mapping $x \to x\cdot x$ is a bijection. Conversely, if the mapping $x \to x\cdot x$ is a bijection on Q, then the equation $x^2 = a$ has a unique solution for each $a \in Q$. Let denote its solution by $a^{1/2}$ and let $a,b \in Q$. Then $x\overset{1}{\cdot}a = b \Leftrightarrow a\cdot xa = b \Leftrightarrow x = (a\backslash b)/a$ and, so as $(Q,\cdot)$ is a diassociative loop, $a\overset{1}{\cdot}y = b \Leftrightarrow y\cdot ay = b \Leftrightarrow ay\cdot ay = ab \Leftrightarrow (ay)^2 = ab \Leftrightarrow y = a\backslash(ab)^{1/2}$, we get that $(Q,\overset{1}{\cdot})$ is a quasigroup, i.e. $(Q,\cdot)$ is recursively 1-differentiable. □

Corollary 1. A Moufang loop $(Q,\cdot)$ is recursively 1-differentiable if and only if the mapping $x \to x^2$ is a bijection on Q.

Corollary 2. A group $(G,\cdot)$ is recursively 1-differentiable if and only if the mapping $x \to x^2$ is a bijection on G.

Corollary 3. Finite groups of even order are not recursively 1-differentiable.

Proof. It is known that such groups have elements of order 2. □

The notion of core was introduced by R.Bruck for Moufang loops ([3]) and by V. Belousov for Bol loops [1,2]. They proved, in particular, that the cores of Moufang or Bol loops are invariant under loops isotopy: the cores of isotopic Moufang (Bol) loops are isomorphic. I. Florja introduced and studied the notion of core for Bol quasigroups [7]. The core of Bol loops (left, right, middle) was also studied by Robinson, Vanzurova and others [1, 9, 10].

Let $(Q,\cdot)$ be an arbitrary loop. The groupoid $(Q,+)$, where

$$x + y = x(y\backslash x),$$

$\forall x,y \in Q,$ is called the core of the loop $(Q,\cdot)$. Note that if $(Q,\cdot)$ is an LIP-loop, then its core $(Q,+)$ is defined by the equality $x + y = x\cdot y^{-1}x,\ \ \forall x,y \in Q$, where $y\cdot y^{-1} = e$, and e is the unit of $(Q,\cdot)$.

Proposition 7. *If* $(Q,\cdot)$ *is an LIP-loop, then its core* $(Q,+)$ *and its recursive derivative* $(Q,\overset{1}{\cdot})$ *are isostrophic.*

Proof. Let $(Q,\cdot)$ be an LIP-loop and let $(Q,+)$ be its core and $(Q,\overset{1}{\cdot})$ its recursive derivative of order 1. Denoting $x\oplus y = y + x$, we have $x\oplus y = y + x = y\cdot x^{-1}y = x^{-1}\overset{1}{\cdot}y$, so $(\oplus) = {}^{(12)}(+)$ and $(\oplus) = (\overset{1}{\cdot})^{(I,\varepsilon,\varepsilon)}$, which imply ${}^{(12)}(+) = (\overset{1}{\cdot})^{(I,\varepsilon,\varepsilon)}$, i.e.

$$\left[{}^{(12)}(+)\right]^{(I,\varepsilon,\varepsilon)} = (\overset{1}{\cdot}), \tag{3}$$

hence $(Q,+)$ and $(Q,\overset{1}{\cdot})$ are isostrophic. □

Corollary 1. *An LIP- loop* $(Q,\cdot)$ *is recursively 1-differentiable if and only if its core is a quasigroup.*

STUDIA UNIVERSITATIS MOLDAVIAE, 2015, nr.2(82)
Seria "Ştiinţe exacte şi economice" ISSN 1857-2073 ISSN online 2345-1033 p.53-60

Remark 3. If $(Q,\cdot)$ is an LIP-loop, $(Q,+)$ is its core and $(Q,\overset{1}{\cdot})$ is its recursive derivative then the equality (3) gives (4) and (5):

$$x+y=y^{-1}\overset{1}{\cdot}x, \tag{4}$$

$$x\overset{1}{\cdot}y=y+x^{-1}, \tag{5}$$

for every $x,y\in Q$, where $x^{-1}\cdot x=y^{-1}\cdot y=e$, e is the unit of $(Q,\cdot)$.
It was proved in [7] that in the left Bol loops the cores satisfy the automorphic inverse property (AIP), i.e.

$$(x+y)^{-1}=x^{-1}+y^{-1}. \tag{6}$$

Proposition 8. *If $(Q,\cdot)$ is a left Bol loop, then its recursive derivative $(Q,\overset{1}{\cdot})$ satisfies the automorphic inverse property (AIP), i.e. for $\forall x,y\in Q$:*

$$(x\overset{1}{\cdot}y)^{-1}=x^{-1}\overset{1}{\cdot}y^{-1}, \tag{7}$$

where $x^{-1}\cdot x=y^{-1}\cdot y=e$, e is the unit of $(Q,\cdot)$.

Proof. According to (4), $x+y=y^{-1}\overset{1}{\cdot}x$ and then $x^{-1}+y^{-1}=y\overset{1}{\cdot}x^{-1}$, for $\forall x,y\in Q$. Now, using (6), we have $(y^{-1}\overset{1}{\cdot}x)^{-1}=y\overset{1}{\cdot}x^{-1}$, for $\forall x,y\in Q$. Denoting y^{-1} by x and x by y in the last equality we obtain $(x\overset{1}{\cdot}y)^{-1}=x^{-1}\overset{1}{\cdot}y^{-1}$, $\forall x,y\in Q$. □

It is known that the cores of left Bol loops are left distributive and satisfies the left keys law ([2]), i.e. if $(Q,\cdot)$ is a left Bol loop and $(Q,+)$ is its core, then for every $x,y,z\in Q$, the following equalities hold:

$$x+(y+z)=(x+y)+(x+z), \tag{8}$$

$$x+(x+y)=y. \tag{9}$$

Proposition 9. *If $(Q,\cdot)$ is a left Bol loop and $(Q,\overset{1}{\cdot})$ is its recursive derivative of order 1, then*

$$(x\overset{1}{\cdot}y)\overset{1}{\cdot}z=(x\overset{1}{\cdot}z^{-1})\overset{1}{\cdot}(y\overset{1}{\cdot}z),$$

$\forall x,y,z\in Q$. Moreover, if $(Q,\cdot)$ is recursively 1-differentiable then $(Q,\overset{1}{\cdot})$ is an RIP-quasigroup.
Proof. From equalities (4) and (7) follows :

$$x+(y+z)=(z^{-1}\overset{1}{\cdot}y)^{-1}\overset{1}{\cdot}x=\left[(z^{-1})^{-1}\overset{1}{\cdot}y^{-1}\right]\overset{1}{\cdot}x=(z\overset{1}{\cdot}y^{-1})\overset{1}{\cdot}x, \Rightarrow$$

$$x+(y+z)=(z\overset{1}{\cdot}y^{-1})\overset{1}{\cdot}x. \tag{10}$$

Analogously,

$$(x+y)+(x+z)=(y^{-1}\overset{1}{\cdot}x)+(z^{-1}\overset{1}{\cdot}x)=(z^{-1}\overset{1}{\cdot}x)^{-1}\overset{1}{\cdot}(y^{-1}\overset{1}{\cdot}x)=\left[(z^{-1})^{-1}\overset{1}{\cdot}x^{-1}\right]\overset{1}{\cdot}(y^{-1}\overset{1}{\cdot}x)=$$

$$=(z\overset{1}{\cdot}x^{-1})\overset{1}{\cdot}(y^{-1}\overset{1}{\cdot}x),$$

STUDIA UNIVERSITATIS MOLDAVIAE, 2015, nr.2(82)
Seria "Științe exacte și economice" ISSN 1857-2073 ISSN online 2345-1033 p.53-60

which implies

$$(x+y)+(x+z)=(z\overset{1}{\cdot}x^{-1})\overset{1}{\cdot}(y^{-1}\overset{1}{\cdot}x). \tag{11}$$

From (8), using equalities (10) and (11), we have

$$(z\overset{1}{\cdot}y^{-1})\overset{1}{\cdot}x=(z\overset{1}{\cdot}x^{-1})\overset{1}{\cdot}(y^{-1}\overset{1}{\cdot}x). \tag{12}$$

Denoting x by z, z by x and y^{-1} by y in (12), we obtain $(x\overset{1}{\cdot}y)\overset{1}{\cdot}z = (x\overset{1}{\cdot}z^{-1})\overset{1}{\cdot}(y\overset{1}{\cdot}z), \forall x,y,z\in Q$. If the left Bol loop $(Q,\cdot)$ is recursively 1-differentiable then, using equalities (4) and (7), from (9) we get $y=x+(x+y)=(x+y)^{-1}\overset{1}{\cdot}x=(y^{-1}\overset{1}{\cdot}x)^{-1}\overset{1}{\cdot}x=(y\overset{1}{\cdot}x^{-1})\overset{1}{\cdot}x$, so $(y\overset{1}{\cdot}x^{-1})\overset{1}{\cdot}x=y, \ \forall x,y\in Q$, i.e. $(Q,\overset{1}{\cdot})$ is an RIP-quasigroup. □

Proposition 10. *Let $(Q,\cdot)$ be a Moufang loop. If $x\cdot y^2\cdot x=y\cdot x^2\cdot y$, for every $x,y\in Q$, then $(Q,\overset{1}{\cdot})$ satisfies the identity $z\overset{1}{\cdot}(y\overset{1}{\cdot}x)=(z^{-1}\overset{1}{\cdot}y)\overset{1}{\cdot}(z\overset{1}{\cdot}x)$.*

Proof. It is known that the core $(Q,+)$ of a Moufang loop $(Q,\cdot)$ is right distributive (i.e. satisfies the identity $(x+y)+z=(x+z)+(y+z)$) if and only if $(Q,\cdot)$ satisfies the equality $x\cdot y^2\cdot x=y\cdot x^2\cdot y$, for every $x,y\in Q$ ([1]). Using (4), we have $(x+y)+z = (x\cdot y^{-1}x)\cdot z^{-1}(x\cdot y^{-1}x) = z^{-1}\overset{1}{\cdot}(y^{-1}\overset{1}{\cdot}x)$ and $\left(x+z\right)+\left(y+z\right) = (x\cdot z^{-1}x)+(y\cdot z^{-1}y) = (x\cdot z^{-1}x)\cdot(y\cdot z^{-1}y)^{-1}(x\cdot z^{-1}x) = (y\cdot z^{-1}y)^{-1}\overset{1}{\cdot}(x\cdot z^{-1}x) = (z^{-1}\overset{1}{\cdot}y)^{-1}\overset{1}{\cdot}(z^{-1}\overset{1}{\cdot}x) = (z\overset{1}{\cdot}y^{-1})\overset{1}{\cdot}(z^{-1}\overset{1}{\cdot}x)$, hence $z^{-1}\overset{1}{\cdot}(y^{-1}\overset{1}{\cdot}x)=(z\overset{1}{\cdot}y^{-1})\overset{1}{\cdot}(z^{-1}\overset{1}{\cdot}x)$. Denoting z^{-1} by z and y^{-1} by y, we obtain $z\overset{1}{\cdot}(y\overset{1}{\cdot}x)=(z^{-1}\overset{1}{\cdot}y)\overset{1}{\cdot}(z\overset{1}{\cdot}x)$, $\forall x,y,z\in Q$. □

Proposition 11. *If $(Q,\cdot)$ is a left Bol loop and the mapping $x\mapsto x^2$ is a bijection then the recursive derivative $(Q,\overset{1}{\cdot})$ satisfies the right Bol identity.*

Proof. It is known that the core $(Q,+)$ of a left Bol loop $(Q,\cdot)$ is a left Bol loop if and only if the mapping $x\mapsto x^2$ is a bijection on Q. Hence, the conjugate $(Q,\oplus)$ of $(Q,+)$, where $x\oplus y=y+x$, is a right Bol loop. Moreover, using the equality (4) we get

$$x\oplus y=y\cdot x^{-1}y=x^{-1}\overset{1}{\cdot}y. \tag{13}$$

So as $(Q,\oplus)$ satisfies the right Bol identity, we have

$$((c\oplus a)\oplus b)\oplus a=c\oplus((a\oplus b)\oplus a)$$

which implies (using (13) and (7)):

$$((c^{-1}\overset{1}{\cdot}a)^{-1}\overset{1}{\cdot}b)^{-1}\overset{1}{\cdot}a=c^{-1}\overset{1}{\cdot}((a^{-1}\overset{1}{\cdot}b)^{-1}\overset{1}{\cdot}a)\Leftrightarrow((c^{-1}\overset{1}{\cdot}a)\overset{1}{\cdot}b^{-1})\overset{1}{\cdot}a=c^{-1}\overset{1}{\cdot}((a\overset{1}{\cdot}b^{-1})\overset{1}{\cdot}a).$$

Denoting c^{-1} by c and b^{-1} by b, we obtain $((c\overset{1}{\cdot}a)\overset{1}{\cdot}b)\overset{1}{\cdot}a=c\overset{1}{\cdot}((a\overset{1}{\cdot}b)\overset{1}{\cdot}a)$, i.e. $(Q,\overset{1}{\cdot})$ satisfies the right Bol identity. □

Proposition 12. *If two left Bol loops $(Q,\cdot)$ and $(Q,\circ)$ are isotopic then their first recursive derivatives $(Q,\overset{1}{\cdot})$ and $(Q,\overset{1}{\circ})$ are isomorphic if and only if $\varphi(x^{-1})=\varphi(x)^{-1}$, for every $x\in Q$, where φ is the isomorphism between the cores of $(Q,\cdot)$ and $(Q,\circ)$ (the inverses are considered in the corresponding loops).*

Proof. Let $(Q,\cdot)$ and $(Q,\circ)$ be two isotopic left Bol loops, then their cores $(Q,+)$ and $(Q,\oplus)$ are isomorphic [7]. Let the mapping $\varphi: Q \to Q$ be an isomorphism i.e. $\varphi(x+y)=\varphi(x)\oplus\varphi(y)$, for $\forall x,y\in Q$. If $(Q,\overset{1}{\cdot})$ and $(Q,\overset{1}{\circ})$ are the first recursive derivatives of the left Bol loops $(Q,\cdot)$ and $(Q,\circ)$, respectively then, according to (4), $x+y=y^{-1}\overset{1}{\cdot}x$ and $x\oplus y=y^{-1}\overset{1}{\circ}x$, $\forall x,y\in Q$. If $\varphi(x^{-1})=\varphi(x)^{-1}$, for every $x\in Q$, then $\varphi(y^{-1}\overset{1}{\cdot}x)=\varphi(y^{-1})\overset{1}{\circ}\varphi(x)$, for $\forall x,y\in Q$. Denoting y^{-1} by y, we obtain $\varphi(y\overset{1}{\cdot}x)=\varphi(y)\overset{1}{\circ}\varphi(x)$, $\forall x,y\in Q$, i.e. $(Q,\overset{1}{\cdot})\overset{\varphi}{\cong}(Q,\overset{1}{\circ})$. □

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Acknowledgements: This work was partially supported by CSSDT ASM grant 15.817.02.26F

Prezentat la 12.03.2015