Analysis of selected nonlinear electrical circuits based on lumped and distributed parameters

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Abstract. In the paper the equations of nonlinear electrical circuits derived from Hamilton-Ostrogradsky's action functional are analyzed with the use of Lagrangian function modified by two additional components. The components consider external and internal energy dissipation and energy of non-potential forces for the systems based on both lumped and distributed parameters. The results are presented as graphs.

Key words: electrical circuits, lumped and distributed parameters, Hamilton-Ostrograd–sky's action functional.

Introduction. Equations describing non-linear electrical circuits are basis for formulation of the mathematical models of electrical devices described with both lumped and distributed parameters [1,4]. These equations are used in order to formulate the mathematical models of transformers [1], all machines of direct and alternating current [1,4], choking coils [3,4], et al. The least action principle of Maupertuis is the fundamental law used in order to formulate the mathematical models of the abovementioned devices. The integral variational principle of Hamilton-Ostrogradsky [1,2] is the interpretation of Maupertuis' principle. In the paper the interdisciplinary method consisting in modification of Hamilton-Ostrogradsky's principle by the extension of available Lagrangian function based on two additional components is proposed. The components consider the external and internal energy dissipation as well as the energy of non-potential forces for the systems with both lumped and didtributed parameters [1]. The aforementioned modification allowed to use the variational approaches in order to solve any tasks of the applied physics.

Mathematical model of the system. The electrical circuits are given as the systems based on lumped parameters with the finite number of degrees of freedom. Generalized variables $\mathbf{q} \equiv (q_1, q_2, ..., q_n)^{\mathrm{T}}$ and their derivatives $\dot{\mathbf{q}} \equiv (\dot{q}_1, \dot{q}_2, ..., \dot{q}_n)^{\mathrm{T}}$ and time are the arguments of modified Lagrangian. The electromotive force (emf) $e_k(t) = e_k$, resistors $R_k(i_k) = R_k$, inductances $L_k(i_k) = L_k$ and capacitors $C_k(u_k) = C_k$ are the elements of analyzed circuit, where $k = 1, 2, ..., n$, n is number of generalized coordinates (circuits) in the electric system, $i(t)$, $u(t)$ are the respective currents and voltages. The charges of circuits $Q_k(t)$ are assumed to be the generalized coordinates whereas the derivatives of charges $i_k(t)$ are the currents in circuits.

The extended functional of electric system is given as follows [1,4]:

$$S = \int_{t_1}^{t_2} \left(\tilde{T} - P + \Phi - D\right) dt = \int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \int_0^{i_k} L_k(i_k) i_k di_k - \sum_{k=1}^{n} \int_0^{Q_k} C_k^{-1}(Q_k) Q_k dQ_k + \sum_{k=1}^{n} \int_0^{t} \int_0^{i_k} \left(R_k(i_k) i_k di_k\right)_{|t=\tau} d\tau - \sum_{k=1}^{n} \int_0^{t} \left(e_k i_k\right)_{|t=\tau} d\tau \right) dt, \quad (1)$$

where $\tilde{T}$ is kinetic coenergy of electric circuits.

At initial instant the currents are not flowing in the circuits. It means that the electrical energy dissipation is equal to zero: $\Phi|_{t=0} \equiv 0$. Variations of extended action functional of Hamilton-Ostrogradsky is expressed as follows:

$$\delta S = \int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \delta \int_0^{i_k} L_k(i_k) i_k di_k - \sum_{k=1}^{n} \delta \int_0^{Q_k} C_k^{-1}(Q_k) Q_k dQ_k + \sum_{k=1}^{n} \delta \int_0^{t} \int_0^{i_k} \left(R_k(i_k) i_k di_k\right)_{|t=\tau} d\tau - \sum_{k=1}^{n} \delta \int_0^{t} \left(e_k i_k\right)_{|t=\tau} d\tau \right) dt =$$

$$= \int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \frac{\partial}{\partial i_k} \int_0^{i_k} \left(L_k(i_k) i_k\right) di_k \delta i_k - \sum_{k=1}^{n} \frac{\partial}{\partial Q_k} \int_0^{Q_k} \left(C_k^{-1}(Q_k) Q_k\right) dQ_k \delta Q_k + \sum_{k=1}^{n} \frac{\partial}{\partial i_k} \int_0^{t} \int_0^{i_k} \left(R_k(i_k) i_k\right) di_k d\tau \delta i_{k i_k} - \sum_{k=1}^{n} \int_0^{t} \frac{\partial\left(e_k i_k\right)}{\partial i_k}_{|t=\tau} d\tau \delta i_k \right) dt \,. \quad (2)$$

The variational and integral procedures are independent and their sequence may be replaced. Grouping the similar components and taking into account the equality: $\delta i_k = \delta \frac{dQ_k}{dt} = \frac{d}{dt} \delta i_k$, it can be obtained:

$$\delta S = \int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \left[\left(L_k i_k + \int_0^{t} R_k i_{k|t=\tau} d\tau - \int_0^{t} e_{k|t=\tau} d\tau \right) \frac{d}{dt} \delta Q_k \right] - \sum_{k=1}^{n} \left[\left(C_k^{-1} Q_k \right) \delta Q_k \right] \right) dt = 0 \,. \quad (3)$$

For the first term of (3) the integration by parts is used, and the following dependency is obtained as a consequence:

$$\delta S = \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \sum_{k=1}^{n} \left[\left(L_k i_k + \int_0^{t} R_k di_{k|t=\tau} d\tau - \int_0^{t} e_{k|t=\tau} d\tau \right) \delta Q_k \right] - \sum_{k=1}^{n} \left[\frac{d}{dt} \left(L_k i_k + \int_0^{t} R_k i_{k|t=\tau} d\tau - \int_0^{t} e_{k|t=\tau} d\tau \right) \delta Q_k \right] - \sum_{k=1}^{n} \left[\left(C_k^{-1} Q_k \right) \delta Q_k \right] \right\} dt = 0 \,. \quad (4)$$

Hence, considering that variations of coordinates are equal to zero at instants t_1, t_2, it may be written:

$$\delta S=\int_{t_1}^{t_2}\sum_{k=1}^{n}\left[\frac{d}{dt}\left(L_k i_k+\int_0^t R_k i_{k|t=\tau}d\tau-\right.\right.$$

$$\left.\left.-\int_0^t e_{k|t=\tau}d\tau\right)+C_k^{-1}Q_k\right]\delta Q_k dt+0=0\,. \qquad (5)$$

The term δQ_k is never equal to zero, thus, the variation of action functional by Hamilton-Ostrogradsky is equal to zero only if the integrand in dependency (5) is equal to zero:

$$\frac{d}{dt}\sum_{k=1}^{n}L_k i_k+\sum_{k=1}^{n}R_k i_k-\sum_{k=1}^{n}e_k+\sum_{k=1}^{n}C_k^{-1}Q_k=0\,. \qquad (6)$$

where: $L_k i_k=\Psi_k$ is the full magnetic coupling (flux) of the k-th circuit,
$R_k i_k=u_{Rk}$ is voltage across the k-th circuit resistance,
e_k is EMF of the k-th circuit,
$C_k^{-1}Q_k=u_{Ck}$ is voltage across the capasitor of the k-th circuit.

Considering, the equation (6) has the following form [1,2]:

$$\sum_{k=1}^{n}\left(\frac{d\Psi_k}{dt}+u_{Rk}-e_k+u_{Ck}\right)=0\,. \qquad (7)$$

The dependency (7) is fundamental for the applied electrical engineering. It allows modelling the electrical processes in majority of electrical devices [1,2,4].

Simulation calculations. The analysis of transient state in nonlinear coils (inductance as function of current) was made using the above given equations. The simple nonlinear two-way electrical circuit consisted of series connections of EMF, nonlinear inductances and resistances, was analyzed. The extended functional based on (1) has the form:

$$S=\int_{t_1}^{t_2}\left(\tilde{T}-P+\Phi-D\right)dt=\int_{t_1}^{t_2}\left(\int_0^i L(i)idi+\int_0^t\int_0^i\left(R(i)idi\right)_{|t=\tau}d\tau-\right.$$

$$\left.-\int_0^t\left(ei\right)_{|t=\tau}d\tau\right)dt\,. \qquad (8)$$

Transforming the dependencies (2) to (11) it can be derived:

$$e=\frac{d}{dt}\left(L(i)i\right)+R(i)i=\frac{d\Psi(i)}{dt}+R(i)i=\frac{\partial\Psi(i)}{\partial i}\frac{di}{dt}+$$

$$+\frac{\partial\Psi(i)}{\partial t},\quad \frac{\partial\Psi(i)}{\partial t}\equiv 0\Rightarrow e=L^{\partial}(i)\frac{di}{dt}+R(i)i\,, \qquad (9)$$

where: i is current in coil winding, $L^{\partial}(i)$ is nonlinear differential circuit inductance.

$$\frac{di}{dt}=\frac{e-R(i)i}{L^{\partial}(i)}\,. \qquad (10)$$

The parameters of electrical circuit are as follows:
$e_1(t)=4500\exp\sin(30t+0{,}1)$ [V],
$e_2(t)=1000\sin(10t-2)$ [V],
$\Psi_1(i_1)=12{,}4\,\mathrm{arctg}\,0{,}066 i_1$ [Wb],
$\Psi_2(i_2)=10\,\mathrm{arctg}\,0{,}1 i_2$ [Wb],
$R_1=1{,}2+0{,}002 i_1$ [Ω], $R_2=0{,}7+0{,}003 i_2$ Ω.

The results of simulation are presented as graphs.

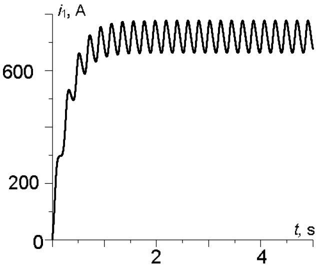

Fig. 1 Current of the first limb of analyzed circuit

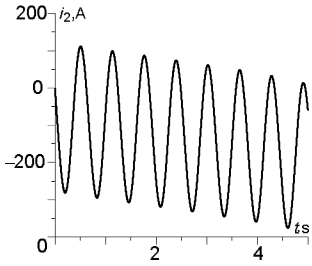

Fig. 2 Current of the second limb of analyzed circuit

The current of the first loop belonging to the nonlinear electrical circuit is shown in Fig. 1. The pattern of current depends on: (a) the nonlinear wave of feeding voltage of the first loop, (b) the dependency describing the magnetization curve of choking coil as well as (c) the dependency between voltage across the resistor and current. The frequency of current oscillation in steady state is equal to 5 Hz and depends on frequency of feeding voltage of the first loop.

The current of the second loop belonging to the nonlinear electrical circuit is shown in Fig. 2. There is not steady state for the assumed parameters of the circuit. The feeding voltage of the second loop is given as harmonic function, thus, all functional dependencies describing the loop in steady state should be harmonics for linear parameters of the loop. In the analyzed example the harmonic process is not possible.

Conclusions. The equations of nonlinear electrical circuits are presented in the paper. These are the most fundamental equations of electric theory based on interdisciplinary method proposed in [1]. The basic principles of variational calculus theory and the modified Hamilton-Ostrogradsky's principle were only used in order to derive the mathematical dependencies describing the analyzed electrical system. On the basis of carried out simulations it may be concluded that both methods are consistent, i.e. the classical method based on energy conservation low and the variational calculus method based on the least action principle.

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