Vector-Valued Function Application to Projectile Motion

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Abstract - *This research work study the motion of a projectile without air resistance using vector-valued function. In this work, we combined the factors that affect the path of a trajectory to determine how a pilot can jump off from an aircraft into a river which is located at a known distance without falling on the ground in case there is a failure in the parachute. Based on our study of the problem statement, we established a theorem which states that at every maximum point (time) of a projectile (ignoring air resistance), the tangential component of acceleration is equal to zero and the normal component of acceleration is equal to gravity.*

Keywords : *Acceleration, Velocity, Projectile motion, Tangential and Normal component of acceleration*

INTRODUCTION

In the Western world prior to the Sixteenth Century, it was generally assumed that the acceleration of a falling body would be proportional to its mass - that is, a 10 kg object was expected to accelerate ten times faster than a 1 kg object. It was an immensely popular work among academicians and over the centuries it had acquired a certain devotion verging on the religious.

During the Renaissance, the focus, especially in the arts, was on representing as accurately as possible the real world whether on a 2 dimensional surface or a solid such as marble or granite. This required two things. The first was new methods for drawing or painting, e.g. perspective. The second, relevant to this topic, was careful observation. With the spread of cannon in warfare, the study of projectile motion had taken on greater importance, and now, with more careful observation and more accurate representation, came the realization that the path of a projectile did not consist of two consecutive straight line components but was instead a smooth curve. It wasn't until the Italian scientist Galileo Galilei came along that anyone put Aristotle's theories to the test. Unlike everyone else up to that point, Galileo actually tried to verify his own theories through experimentation and careful observation. He then combined the results of these experiments with mathematical analysis in a method that was totally new at the time, but is now generally recognized as the way science gets done. For the invention of this method, Galileo is generally regarded as the world's first scientist. In a tale that may be apocryphal, Galileo (or an assistant, more likely) dropped two objects of unequal mass from the Leaning Tower of Pisa. Quite contrary to the teachings of Aristotle, the two objects struck the ground simultaneously (or very nearly so). Given the speed at which such a fall would occur, it is doubtful that Galileo could have extracted much information from this experiment.

Most of his observations of falling bodies were really of bodies rolling down ramps. This slowed things down enough to the point where he was able to measure the time intervals with water clocks and his own pulse (stopwatches and photo gates having not yet been invented). This he repeated "a full hundred times" until he had achieved "an accuracy such that the deviation between two observations never exceeded one-tenth of a pulse beat."This discovery led Galileo to an outstanding conclusion about Projectile Motion. He figured out that a projectile has two motions, instead of just one. He also said that "the motion that acts vertically is the force of gravity", which pulls the object back down to earth at 32 feet per second per second, and that "while gravity was pulling the object down to Earth, the projectile was also moving horizontally at the same time".

PROBLEM STATEMENT

Suppose an Air force jet is flying at a known altitude above sea level encounters a fault in the air and is about to crash and the pilot decides to jump off from the emergency exit with a known velocity and at a specified angle above the horizontal. (Assuming there is failure in the parachute and no air resistance). The problem therefore is how the pilot can jump into a river which is located at a known distance from where he jumped. Figure 1.1 illustrates the motion of the pilot from the aircraft.

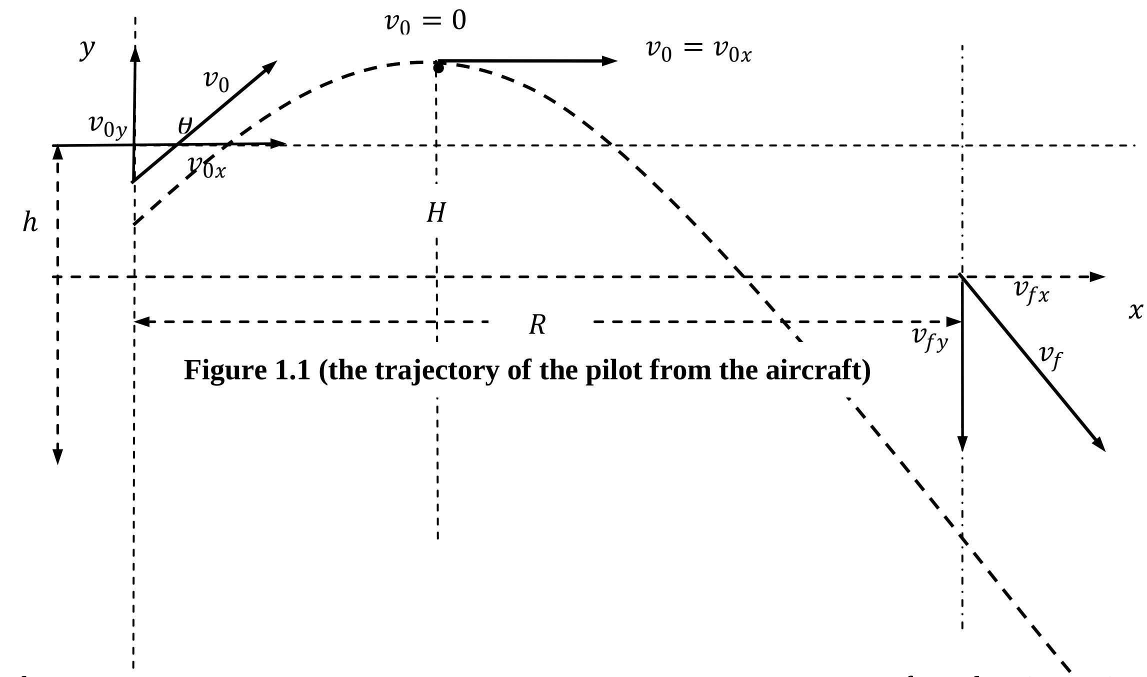

Figure 1.1 (the trajectory of the pilot from the aircraft)

where,
$v_0 = initial\ velocity$
$v_{0x} = initial\ velocity\ of\ the\ horizontal\ motion$
$v_{0y} = initial\ velocity\ of\ the\ vertical\ motion$
$h = initial\ height\ of\ the\ aircraft$
$H = maximum\ height\ reached\ by\ the\ pilot$
$R = horizontal\ distance$
$v_f = final\ velocity$
$v_{fx} = final\ velocity\ of\ the\ horizontal\ motion$
$v_{fy} = final\ velocity\ of\ the\ vertical\ motion$
$\theta = angle\ of\ elevation$

General objectives: The main aim of this paper is to investigates the projectile motion of the pilot in the problem statement. The objectives are therefore how we are going to use vector-valued function application in projectile motion to determine:

a) How long the pilot will be airborne (i.e. the pilot time of flight)?
b) Whether the pilot lands on the island or falls into the river which was located at an assume distance of 5 000 feet. (i.e. the horizontal distance travelled by the pilot).
c) The pilot final velocity at impact?
d) The vertical distance travelled (maximum height) by the pilot and the tangential and normal component of acceleration acting on the pilot at the time he reaches his maximum height.

EMPIRICAL LITERATURE REVIEW

The usual way of studying projectile motion is by using kinematics equations in physics. The following paragraphs will overview the various research works on the study of projectile motion.

Warburton and Wang (2004) provided an overview on calculating the range of a projectile experiencing air resistance in the asymptotic region of large velocities by introducing the Lambert function. From their exact solution for the range in terms of the Lambert function, they derive an approximation for the maximum range in the limit of large velocities. Analysis of the result confirmed an independent numerical result observed in an introductory physics class that the angle at which the maximum range occurs, goes rapidly to zero for increasing initial firing speeds.

Kantrowitz and Neumann, 2011, studied the motion of a projectile that was launched from the top of a tower and lands on a given surface in space. Their goal was to determine explicit and manageable formulas for the direction of launch in space that allows the projectile to travel as far as possible. They developed a deferent general approach that led to remarkably simple equations and solution formulas.

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Robinson and Robinson, 2013, developed differential equations which govern the motion of a spherical projectile rotating about an arbitrary axis in the presence of an arbitrary 'wind'. Three forces were assumed to act on the projectile: (i) gravity, (ii) a drag force proportional to the square of the projectile's velocity and in the opposite direction to this velocity and (iii) a lift or 'Magnus' force also assumed to be proportional to the square of the projectile's velocity and in a direction perpendicular to both this velocity and the angular velocity vector of the projectile. The problem was coded in Matlab and some illustrative model trajectories were presented for 'ball-games', specifically golf and cricket, although the equations could equally well be applied to other ball-games such as tennis, soccer or baseball. They found that the trajectories obtained were broadly in accord with those observed in practice.

Chudinov (2011) reviewed the classic problem of the motion of a point mass (projectile) thrown at an angle to the horizon. The air drag force was taken into account in the form of a quadratic function of velocity with the coefficient of resistance assumed to be constant. Analytical methods for the investigation were mainly used. With the help of simple approximate analytical formulas a full investigation of the problem was carried out. This study includes the determining of eight basic parameters of projectile motion (flight range, time of flight, maximum ascent height and others). The study also included the construction of the basic functional dependences of the motion, the determination of the optimum angle of throwing, providing the greatest range; constructing of the envelope of a family of trajectories of the projectile and finding the vertical asymptote of projectile motion. The motion of a baseball was presented as examples.

Warburton et al, (2010) studied the projectile motion with air resistance quadratic in speed. They considered three regimes of approximation: low-angle trajectory where the horizontal velocity; high-angle trajectory; and split-angle trajectory. The approximation was simple and accurate for low angle ballistics problems when compared to measured data. They also discovered that the range in this approximation is symmetric about, although the trajectories were asymmetric. They also gave simple and practical formulas for accurate evaluations of the Lambert W function.

Morales (2011) studied the motion of a projectile with linear drag shot from a nonzero height on an inclined plane that makes an angle $\emptyset$ with the horizontal, and obtain analytical expressions for the range, the time of flight, and the angle between the initial and final velocities as functions of the firing angle in terms of the Lambert *W* function. He observed that for $\emptyset = 0$, analytical expressions are also obtained for the maximum range, the optimum angle, and the optimum time of flight in terms of the Lambert *W* function. In the general case, he proved that when the projectile travels along the path of maximum range, the initial and final velocities are perpendicular.

METHODOLOGY

The path of a Projectile

We assume that gravity is the only force acting on the projectile after it is launched. So, the motion occurs in a vertical plane, which can be represented by the xy-coordinate system with the origin as a point on Earth's surface, as shown in Figure 1.2.

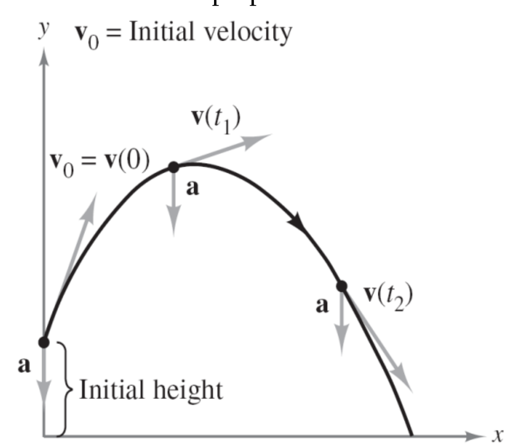

Figure 1.2

For a projectile of mass m, the force due to gravity is

$$\mathbf{F} = -mg\boldsymbol{j} \qquad \text{Force due to gravity}$$

where the acceleration due to gravity is $g = 32$ feet per second per second, or 9.81 meters per second per second.

By **Newton's Second Law of Motion,** this same force produces an acceleration $\mathbf{a} = \mathbf{a}(t)$, and satisfies the equation $\mathbf{F} = m\mathbf{a}$. Consequently, the acceleration of the projectile is given by $m\mathbf{a} = -mg\boldsymbol{j}$, which implies that

$$\mathbf{a} = -g\boldsymbol{j}. \qquad \text{Acceleration of projectile}$$

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We start by finding a position vector as a function of time(t). Beginning with the acceleration vector $\mathbf{a} = -g\boldsymbol{j}$ and integrating twice.

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = \int -g\boldsymbol{j}dt = -gt\boldsymbol{j} + \boldsymbol{C}_1$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \int (-gt\boldsymbol{j} + \boldsymbol{C_1})dt = -\frac{1}{2}gt^2\boldsymbol{j} + \boldsymbol{C}_1 t + \boldsymbol{C}_2$$

Solving for the constant vectors $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, we use the fact that $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$. Doing this produces $\boldsymbol{C}_1 = \mathbf{v}_0$ and $\boldsymbol{C}_2 = \mathbf{r}_0$. Therefore, the position vector is

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\boldsymbol{j} + t\boldsymbol{v_0} + \mathbf{r}_0 \qquad \text{position vector}$$

In many projectile problems, the constant vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not given explicitly. Often we are given the initial height h, the initial speed v_0 and the angle θ at which the projectile is launched, as shown in Figure 1.3.

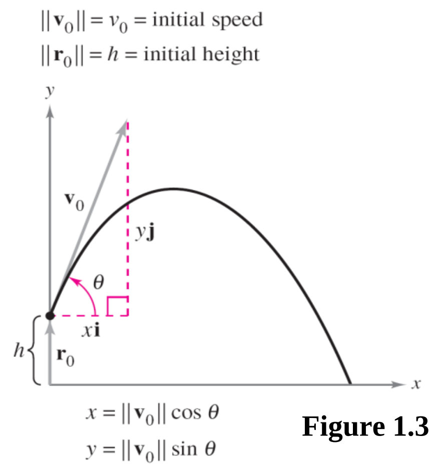

Figure 1.3

From the given height, we can deduce that $\mathbf{r}_0 = h\boldsymbol{j}$. Because the speed gives the magnitude of the initial velocity, it follows that $v_0 = \|\mathbf{v}_0\|$ and we can write

$$\mathbf{v}_0 = x\boldsymbol{i} + y\boldsymbol{j}$$
$$= (\|\mathbf{v}_0\| \cos\theta)\boldsymbol{i} + (\|\mathbf{v}_0\| \sin\theta)\boldsymbol{j}$$
$$= v_0 \cos\theta\boldsymbol{i} + v_0 \sin\theta\boldsymbol{j}$$

So, the position vector can be written in the form

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\boldsymbol{j} + t\boldsymbol{v_0} + \mathbf{r}_0 \qquad \text{Position vector}$$

$$= -\frac{1}{2}gt^2\boldsymbol{j} + tv_0 \cos\theta\boldsymbol{i} + tv_0 \sin\theta\boldsymbol{j} + h\boldsymbol{j}$$
$$= (v_0 \cos\theta)t\boldsymbol{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\boldsymbol{j}.$$

Theorem 1.1

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos\theta)t\boldsymbol{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\boldsymbol{j}$$

where g is the acceleration due to gravity.

The Normal and Binomial Vectors : At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $\|\mathbf{T}(t)\| = 1$ for all t, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that $\mathbf{T}'(t)$ is itself not a unit vector. But if $\mathbf{r}'$ is also smooth, we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

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The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the binomial vector. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector as shown in Figure 1.4

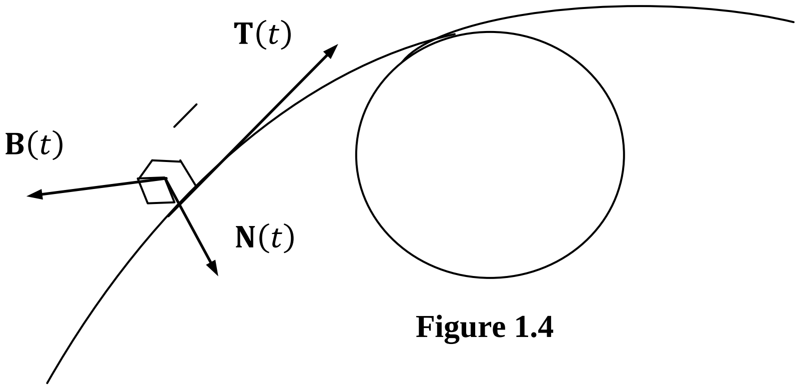

Figure 1.4

We can think of the normal vector as indicating the direction in which the curve is turning at each point

Tangential and Normal Component of Acceleration

Returning to the problem of describing the motion of an object along a curve. In the preceding section, we saw that for an object traveling at a *constant speed*, the velocity and acceleration vectors are perpendicular. This seems reasonable, because the speed would not be constant if any acceleration were acting in the direction of motion. We can verify this observation by noting that

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$$

if $\|\mathbf{r}'(t)\|$ is a constant.

However, for an object traveling at a variable speed, the velocity and acceleration vectors are not necessarily perpendicular. For instance, the acceleration vector for a projectile always points down, regardless of the direction of motion. In general, part of the acceleration (the tangential component acts in the line of motion, and part (the normal component) acts perpendicular to the line of motion. In order to determine these two components, we can use the unit vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ which serve in much the same way as do $\boldsymbol{i}$ and $\boldsymbol{j}$ in representing vectors in the plane.

The following theorem states that the acceleration vector lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

Theorem 1.2

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$

Proof

To simplify the notation, we write $\mathbf{T}$ for $\mathbf{T}(t)$, $\mathbf{T}'$ for $\mathbf{T}'(t)$, and so on. Because $\mathbf{T} = \mathbf{r}'/\|\mathbf{r}'\| = \mathbf{v}/\|\mathbf{v}\|$, it follows that

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{T}\ .$$

By differentiating, we obtain

$$\mathbf{a} = \mathbf{v}' = \frac{d}{dt}[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\mathbf{T}' \qquad \text{Product Rule}$$

$$= \frac{d}{dt}[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\mathbf{T}'\left(\frac{\|\mathbf{T}'\|}{\|\mathbf{T}'\|}\right)$$

$$= \frac{d}{dt}[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\|\mathbf{T}'\|\mathbf{N}. \qquad \text{since} \quad \mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$$

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Because **a** is written as a linear combination of **T** and **N**, it follows that **a** lies in the plane determined by **T** and **N** The coefficients of **T** and **N** in the proof of theorem 1.2 are called the **tangential and normal components of acceleration** and are denoted by $a_{\mathbf{T}} = \frac{d}{dt}[\|\mathbf{v}\|]$ and $a_{\mathbf{N}} = \|\mathbf{v}\|\|\mathbf{T}'\|$. So, we can write

$$\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$$

The following theorem at the next page gives some convenient formulas for $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$
Theorem 1.3

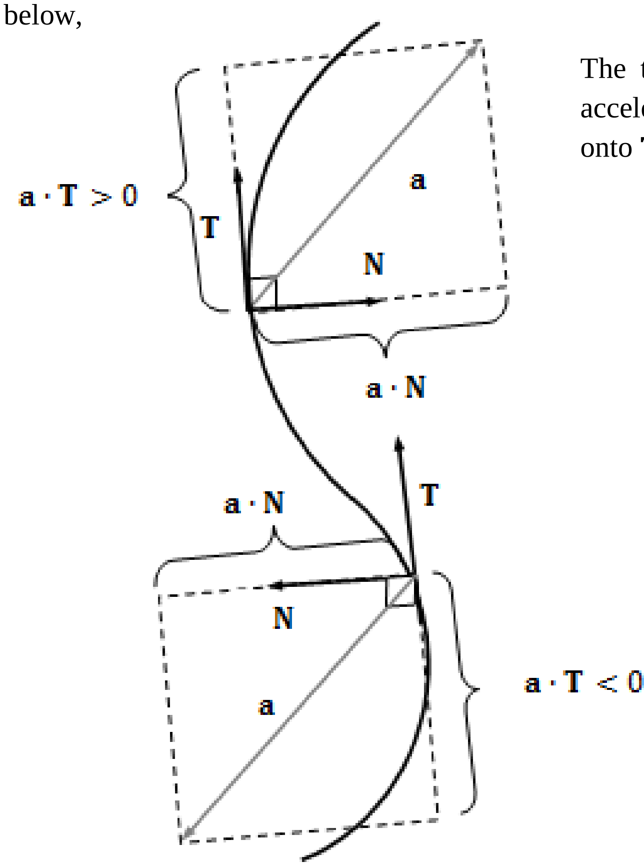
If $\mathbf{r}(t)$ is the position vector for a smooth curve C [for which $\mathbf{N}(t)$ exists], then the tangential and normal components of acceleration are as follows.

$$a_{\mathbf{T}} = \frac{d}{dt}\|\mathbf{v}\| = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}} = \|\mathbf{v}\|\|\mathbf{T}'\| = \mathbf{a}\cdot\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$$

Note that $a_{\mathbf{N}} \geq 0$. The normal component of acceleration is also called the

Proof of theorem 1.3
Consider the diagram below,



The tangential and normal components of acceleration are obtained by projecting **a** onto **T** and **N**

Figure 1.5

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Note that a lies in the plane of $\mathbf{T}$ and $\mathbf{N}$. So, we can use Figure 1.5 to conclude that, for any time t, the components of the projection of the acceleration vector onto $\mathbf{T}$ is given by $a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T}$, and onto $\mathbf{N}$ is given by $a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N}$. Moreover, because $\mathbf{a} = \mathbf{v}'$ and $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\|$, we have

$$\begin{aligned} a_{\mathbf{T}} &= \mathbf{a} \cdot \mathbf{T} \\ &= \mathbf{T} \cdot \mathbf{a} \\ &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{a} \\ &= \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} \end{aligned}$$

For the normal component of acceleration, using $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, $\mathbf{T} \times \mathbf{T} = 0$, and $\|\mathbf{T} \times \mathbf{N}\| = 1$, we have

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\|\mathbf{T} \times (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \\ &= \|\mathbf{v}\|a_{\mathbf{T}}(\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \\ &= \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\|a_{\mathbf{N}}\|\mathbf{T} \times \mathbf{N}\| \\ &= \|\mathbf{v}\|a_{\mathbf{N}} \end{aligned}$$

Thus,

$$a_{\mathbf{N}} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

Also, $\|\mathbf{a}\| = \mathbf{a} \cdot \mathbf{a} = (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \cdot (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})$

$$= a_{\mathbf{T}}^2\|\mathbf{T}\|^2 + 2a_{\mathbf{T}}a_{\mathbf{N}}\mathbf{T} \cdot \mathbf{N} + a_{\mathbf{N}}^2\|\mathbf{N}\|^2 = a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2$$

$$a_{\mathbf{N}}^2 = \|\mathbf{a}\| - a_{\mathbf{T}}^2$$

Since $a_{\mathbf{N}} > 0$, we have $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\| - a_{\mathbf{T}}^2}$

DATA ANALYSIS OF PROBLEM STATEMENT

From theorem 1.1 the position function for a projectile is given by

$$\mathbf{r}(t) = (v_0 \cos\theta)t\boldsymbol{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\boldsymbol{j} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

From the problem statement, we will assume that the known altitude, initial velocity and the angle of elevation are respectively.

$h = 30000\,feet$, $v_0 = 150\,feet\ per\ second$, and $\theta = 45^o$,

Since in air navigation the lowest safe altitude (LSALT) is an altitude that is at least 1,000 feet above any obstacle or terrain within a defined safety buffer region around a particular route that a pilot might fly. Also if someone fall's head down from a tower with their body straight, as if in a dive, it could be 102mph (149.6000034 feet per second). And one of the best known 'results' of the science of mechanics is that the optimum projection angle for achieving maximum horizontal range is 45°.

So using $g = 32$ feet per second per second, equation (1) becomes

$$\begin{aligned} \mathbf{r}(t) &= \left(150\cos\frac{\pi}{4}\right)t\boldsymbol{i} + \left[30000 + \left(150\sin\frac{\pi}{4}\right)t - \frac{1}{2} \cdot 32t^2\right]\boldsymbol{j} \\ &= \left(75\sqrt{2}\,t\right)\boldsymbol{i} + \left(30000 + 75\sqrt{2}\,t - 16t^2\right)\boldsymbol{j} \end{aligned}$$

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writing the trajectory in the form of parametric equation, the position vector $\mathbf{r}(t)$ becomes

$$x(t) = 75\sqrt{2}t \qquad\qquad y(t) = 30000 + 75\sqrt{2}t - 16t^2$$

a) For how long is the pilot airborne is the time taken by the pilot to complete its motion, which is represented by the curve in Figure 1.1

Impact occurs when $y(t) = 0$, thus

$$16t^2 - 75\sqrt{2}t - 30000 = 0$$

here $a = 16,\ \ b = -75\sqrt{2}$ and $c = -30000$ and so solving this quadratic equation yields

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{75\sqrt{2} \pm \sqrt{\left(75\sqrt{2}\right)^2 - 4(16)(-30000)}}{2(16)}$$

$$= \frac{75\sqrt{2} \pm 1389.6942211}{32} \approx 46.74 \quad \text{or} \quad -40.11$$

Since negative time has no meaning in this context we take the positive root. Therefore the pilot is in the air for about 46.74 seconds.

b) The horizontal distance travelled by the pilot is known as the range (R), which is represented by a straight horizontal line to where it cuts the plane below in Figure 1.1

The horizontal motion for the trajectory is

$$x(t) = 75\sqrt{2}t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (2)$$

applying the definition of limit gives

$$\lim_{t\to 46.74} x(t) = \lim_{t\to 46.74}\left(75\sqrt{2}\right)t = \left(75\sqrt{2}\right)(46.74) \approx 4957.52$$

Since the river is located 5 000 feet, the pilot falls on the island about 4 958 feet away from where he jumped.

a) The pilot final velocity upon landing on the ground is the speed at impact, which is represented by the diagonal arrow (v_f) in Figure 1.1

By the definition of velocity and speed in Chapter Three

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\boldsymbol{i} + y'(t)\boldsymbol{j} = \left(75\sqrt{2}\right)\boldsymbol{i} + \left(75\sqrt{2} - 32t\right)\boldsymbol{j}$$

and $\textbf{speed} = \|\mathbf{v}(\boldsymbol{t})\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$. So the pilot speed at impact is

$$\|\mathbf{v}(46.74)\| = \sqrt{\left(75\sqrt{2}\right)^2 + \left(75\sqrt{2} - 32 \cdot 46.74\right)^2} \approx 1394$$

Therefore the pilot final velocity (v_f) or speed at impact is 1 394 feet per second.

b) The maximum height is the maximum value of the vertical distance attain by the pilot above the horizontal plane to the point of projection. In Figure 1.1, the maximum height is represented by the middle vertical line (H).

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The maximum height occurs when

$$y'(t) = 75\sqrt{2} - 32t = 0$$

which implies that

$$t = \frac{75\sqrt{2}}{32} \approx 3.31 \ \ seconds$$

So the maximum height reached by the pilot is

$$y\left(\frac{75\sqrt{2}}{32}\right) = 30000 + 75\sqrt{2}\left(\frac{75\sqrt{2}}{32}\right) - 16\left(\frac{75\sqrt{2}}{32}\right)^2$$

$$\approx 30175.78 \ \text{feet}$$ maximum height when $t \approx 3.31\ seconds$

Our velocity vector, speed and acceleration vector are;

$$\mathbf{v}(t) = \left(75\sqrt{2}\right)\boldsymbol{i} + \left(75\sqrt{2} - 32t\right)\boldsymbol{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{\left(75\sqrt{2}\right)^2 + \left(75\sqrt{2} - 32t\right)^2} = 2\sqrt{5625 - 1200\sqrt{2}t + 256t^2}$$

and $\mathbf{a}(t) = \mathbf{v}'(t) = 0\boldsymbol{i} - 32\boldsymbol{j} = -32\boldsymbol{j}$

From theorem 1.3, the tangential component of acceleration is given by $a_{\mathbf{T}} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$

and so

$$a_{\mathbf{T}}(t) = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle 75\sqrt{2}\boldsymbol{i} + \left(75\sqrt{2} - 32t\right)\boldsymbol{j}\ \rangle\cdot\langle 0\boldsymbol{i} - 32\boldsymbol{j}\rangle}{2\sqrt{5625 - 1200\sqrt{2}t + 256t^2}}$$

$$a_{\mathbf{T}}(t) = \frac{-32\left(75\sqrt{2} - 32t\right)}{2\sqrt{5625 - 1200\sqrt{2}t + 256t^2}}$$

$$= \frac{-16\left(75\sqrt{2} - 32t\right)}{\sqrt{5625 - 1200\sqrt{2}t + 256t^2}}$$

and the normal component of acceleration is given by $a_{\mathbf{N}} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}$

note that $\mathbf{a}(t) = -32\boldsymbol{j} = 0\boldsymbol{i} - 32\boldsymbol{j} + 0\boldsymbol{k}$. Thus

$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 75\sqrt{2} & \left(75\sqrt{2} - 32t\right) & 0 \\ 0 & -32 & 0 \end{vmatrix} = 0\boldsymbol{i} + 0\boldsymbol{j} - 2400\sqrt{2}\,\boldsymbol{k}$$

Therefore,

$$a_{\mathbf{N}}(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|} = \frac{\left\|0\boldsymbol{i} + 0\boldsymbol{j} - 2400\sqrt{2}\boldsymbol{k}\right\|}{2\sqrt{5625 - 1200\sqrt{2}t + 256t^2}}$$

$$= \frac{\sqrt{0^2 + 0^2 + \left(-2400\sqrt{2}\right)^2}}{2\sqrt{5625 - 1200\sqrt{2}t + 256t^2}}$$

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$$= \frac{1200\sqrt{2}}{\sqrt{5625 - 1200\sqrt{2}t + 256t^2}}$$

at that particular time $t = \frac{75\sqrt{2}}{32}$, therefore we have

$$a_{\mathbf{T}}\left(\frac{75\sqrt{2}}{32}\right) = \frac{-16\left(75\sqrt{2} - 32\left(\frac{75\sqrt{2}}{32}\right)\right)}{\sqrt{5625 - 1200\sqrt{2}\left(\frac{75\sqrt{2}}{32}\right) + 256\left(\frac{75\sqrt{2}}{32}\right)^2}} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

$$a_{\mathbf{N}}\left(\frac{75\sqrt{2}}{32}\right) = \frac{1200\sqrt{2}}{\sqrt{5625 - 1200\sqrt{2}\left(\frac{75\sqrt{2}}{32}\right) + 256\left(\frac{75\sqrt{2}}{32}\right)^2}} = 32 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

From Figure 1.1, the tangential and the normal component of acceleration can be illustrated as

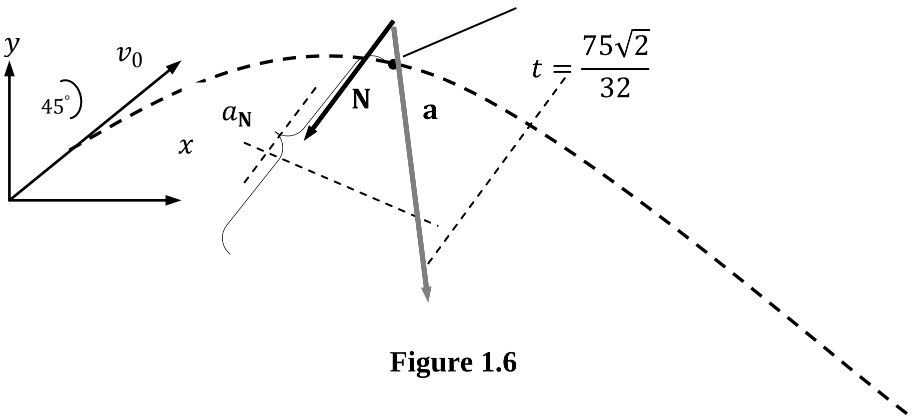

Figure 1.6

The results in equation (3) and (4) are always true for all situations under projectile motion with no air resistance. In order to prove this, we consider the model for the path of a projectile, given by the equation

$$\mathbf{r}(t) = (v_0\cos\theta)t\boldsymbol{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\boldsymbol{j}$$

Writing the trajectory in the form of parametric equation, the position vector $\mathbf{r}(t)$ becomes

$$x(t) = (v_0\cos\theta)t \qquad\qquad y(t) = h + (v_0\sin\theta)t - \frac{1}{2}gt^2$$

Now, at maximum, $y'(t) = 0$
which implies that

$$y'(t) = v_0\sin\theta - gt = 0$$
$$gt = v_0\sin\theta$$
$$t = \frac{v_0\sin\theta}{g} \qquad (maximum\ time\ for\ a\ projectile)$$

From theorem 1.3, the tangential component of acceleration is given by $a_{\mathbf{T}} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$

But,

$\mathbf{v}(t) = \mathbf{r}'(t) = (v_0\cos\theta)\boldsymbol{i} + (v_0\sin\theta - gt)\boldsymbol{j}$

$\|\mathbf{v}(t)\| = \sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}$ and

$\mathbf{a}(t) = \mathbf{v}'(t) = 0\boldsymbol{i} - g\boldsymbol{j} = -g\boldsymbol{j}$

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Therefore,

$$a_{\mathbf{T}}(t) = \frac{\langle (v_0\cos\theta)\boldsymbol{i} + (v_0\sin\theta - gt)\boldsymbol{j}\rangle \cdot \langle 0\boldsymbol{i} - g\boldsymbol{j}\rangle}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}}$$

$$= \frac{-g(v_0\sin\theta - gt)}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}}$$

Substituting $t = \frac{v_0 \sin\theta}{g}$ into the above equation, we obtain

$$a_{\mathbf{T}}\left(\frac{v_0\sin\theta}{g}\right) = \frac{-g\left[v_0\sin\theta - g\left(\frac{v_0\sin\theta}{g}\right)\right]}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}}$$

$$= \frac{-g(v_0\sin\theta - v_0\sin\theta)}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}} = 0 \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

And the normal component of acceleration is given by

$$a_{\mathbf{N}} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}$$

But,

$$\mathbf{v}(t)\times\mathbf{a}(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_0\cos\theta & v_0\sin\theta - gt & 0 \\ 0 & -g & 0 \end{vmatrix}$$

$$= 0\boldsymbol{i} + 0\boldsymbol{j} - g(v_0\cos\theta)\boldsymbol{k}$$

Therefore,

$$a_{\mathbf{N}}(t) = \frac{\|0\boldsymbol{i} + 0\boldsymbol{j} - g(v_0\cos\theta)\boldsymbol{k}\|}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}}$$

$$= \frac{\sqrt{0^2 + 0^2 + (-gv_0\cos\theta)^2}}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}} = \frac{\sqrt{(-gv_0\cos\theta)^2}}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}}$$

$$= \frac{\|-gv_0\cos\theta\|}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}} = \frac{\|-g\|\cdot\|v_0\cos\theta\|}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - gt)^2}}$$

Now substituting $t = \frac{v_0 \sin\theta}{g}$ into the above equation, we obtain

$$a_{\mathbf{N}}\left(\frac{v_0\sin\theta}{g}\right) = \frac{\|g\|\cdot\|v_0\cos\theta\|}{\sqrt{(v_0\cos\theta)^2 + \left(v_0\sin\theta - g\left(\frac{v_0\sin\theta}{g}\right)\right)^2}}$$

$$= \frac{\|g\|\cdot\|v_0\cos\theta\|}{\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - v_0\sin\theta)^2}}$$

$$= \frac{\|g\|\cdot\|v_0\cos\theta\|}{\sqrt{(v_0\cos\theta)^2}} = \frac{\|g\|\cdot\|v_0\cos\theta\|}{\|v_0\cos\theta\|} = \|g\| = g \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

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Hence, from equation (3) and (4), we establish the theorem given below.
Theorem 1.4

At every maximum point (time) of a projectile with no air resistance,

$$a_{\mathbf{T}}(t) = 0 \qquad \text{and} \qquad a_{\mathbf{N}}(t) = \|-g\| = g$$

where $g =$ gravity .

DISCUSSION OF RESULTS

Case 1

Keeping v_0 and θ constant and taking $h = 40000\ feet$ we have

$$y(t) = 40000 + 75\sqrt{2}t - 16t^2$$

Equating the above equation to zero and solving for t we obtain

$$t = \frac{75\sqrt{2} \pm 1603.511771}{32} \approx 53.42$$

From equation (2), applying the definition of limit, we have

$$\begin{aligned} \lim_{t\to 53.42} x(t) &= \lim_{t\to 53.42} \left(75\sqrt{2}\right)t \\ &= \left(75\sqrt{2}\right)(53.42\ ) \approx 5666.50 \end{aligned}$$

Thus the pilot falls into the river.

Case 2

Keeping h and θ constant and taking $v_0 = 100\ feet\ persecond$, equation (1) becomes

$$\mathbf{r}(t) = \left(50\sqrt{2}\,t\right)\boldsymbol{i} + \left(30000 + 50\sqrt{2}\,t - 16t^2\right)\boldsymbol{j}$$

writing the trajectory in the form of parametric equation, the position vector $\mathbf{r}(t)$ becomes

$$x(t) = 50\sqrt{2}t \qquad\qquad y(t) = 30000 + 50\sqrt{2}t - 16t^2$$

Equating $y(t)$ to zero and solving for t, we obtain

$$t = \frac{50\sqrt{2} \pm 50\sqrt{770}}{32} \approx 45.57$$

applying the definition of limit to the horizontal motion $x(t)$ in this case, we have

$$\begin{aligned} \lim_{t\to 45.57} x(t) &= \lim_{t\to 45.57} \left(50\sqrt{2}\right)t \\ \lim_{t\to 45.57} x(t) &= \left(50\sqrt{2}\right)(45.57) \approx 3222.29 \end{aligned}$$

Thus the pilot falls on the island.

We can also see from Figure 1.6 that at the maximum height, when $t = \frac{75\sqrt{2}}{32}$, the tangential component is 0. This is reasonable because the direction of motion is horizontal at the point and the tangential component of acceleration is equal to the horizontal component of the acceleration. Notice also that the normal component of acceleration is equal to the magnitude of the acceleration. In other words, because at that time the speed is constant, the normal component of acceleration is perpendicular to the velocity at that point in figure 1.1.

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CONCLUSIONS AND RECOMMENDATIONS

A projectile is an object in free fall: subject to gravity and air resistance. In this context, we did not account for air resistance acting on a projectile. Instead, we neglected it. Projectile motion consists of independent horizontal and vertical motions. The horizontal and vertical motions of a projectile take the same amount of time. Projectiles usually move horizontally at a constant velocity and undergo uniform acceleration in the vertical direction. This acceleration is due to gravity. Objects can be projected horizontally or at an angle to the horizontal. Projectile motion can begin and end at the same or at different heights.

Understanding how projectile motion works is very beneficial in determining how to best propel an object. In our discussion of results above, we were able to increase the horizontal distance travelled by the pilot to enable him cover a greater distance. The two variables which affected the horizontal distance were the initial velocity, and the height at which he jumped from the aircraft. One of these variables is not enough to ensure a good horizontal distance. In order to combine these two factors to increase the pilot horizontal distance, then the pilot has to gather much momentum before propelling himself off from the emergency exit to increase his initial velocity or increasing the altitude of the aircraft. Generally, when the pilot velocity and angle of projection are held constant, the higher the projection height, the longer the flight time. Hence, if flight time is longer the distance is greater.

Also in the data analysis of the problem statement, we were able to establish theorem (4.1) which states that at every maximum point (time) of a projectile (ignoring air resistance), the tangential component of acceleration is equal to zero and the normal (centripetal) component of acceleration is equal to gravity. This was due to the fact that, at that time the velocity of the projectile is horizontal and since the speed of the projectile remains constant throughout the flight, the tangential component of acceleration now becomes zero, causing the normal component of acceleration to be equal to the magnitude of the total acceleration (gravity).

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