A Finite Abelian Group of Two-Letter Inversions

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Abstract–*In abstract algebra, the study of concrete groups is fundamentally important to beginners. Most commonly used groups as examples are integer addition modulo n, real number addition and multiplication, permutation groups, and groups of symmetry. The last two examples are finite non-abelian groups and can be investigated with the aid of concrete representations. This study presents a finite abelian group of inversions of two letter symbols with vertical and horizontal axes of symmetry and whose binary operation is established through motions like alternation, rotation, reflection, and a combination of two or all motions.*

Keywords–*abstract algebra,alternation, rotation, reflection, commutative, concrete, teaching, learning*.

INTRODUCTION

A mathematical *group* is defined as a non-empty set G on which there is a binary operation * such that: for every element a,b in G, then $a*b$ is still in G (*closure*); for all a,b,c in G, $a*(b*c) = (a*b)*c$ (*associativity*); there is an element e in G such that $a*e = e*a = a$ for all a in G (*identity*); and if a is an element of G, then there is an element a^{-1} in G such that $a*a^{-1} = a^{-1}*a = e$ (*inverse*). Groups are one of the fundamental structures of algebra; they underlie most of the other objects (e.g., rings, fields, modules, and algebras) and are widely used in other branches of mathematics [1].

As part of the curriculum in teacher education, group theory is being studied in detail by Filipino preservice mathematics teachers whose mathematical background knowledge is developed more on foundations of arithmetic, basic and advanced algebras, geometry, trigonometry, and calculus than on purely abstract thinking. These students were accustomed to the concepts and principles of "familiar" mathematics in which they also exhibit learning difficulties [2] [3], and presenting to them an "unfamiliar" one would probably trigger aversion to the subject [4]. Furthermore, empirical evidence shows that undergraduate students have difficulties in understanding group theory [5] [6].

Hence, alternative methods of teaching that may be useful in overcoming these difficulties are deemed vital. In introducing abstract algebra to beginners, it is more effective to reduce the degree of abstractions [7] or to start with concrete models of groups and, eventually, to acquaint them with symbolic representations and abstract relationships [8]. Beginning lessons in abstract algebra with concrete and familiar situations will help the learner understand that there are things which are existing but unnoticed and see things which are not even visible [9]. The study of concrete, demonstrable or manipulative groups can potentially bridge the gap between concrete and abstract mathematical thinking.

Examples of groups which can be calculated manually or concretely include integer addition modulo n or $\mathbb{Z}_n$, real numbers under addition and multiplication $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \bullet)$, permutation groups (S_n), and symmetry groups (D_3 and D_4). The elements of these groups can be visualized and operated with ease by usual addition and multiplication of numbers, rearrangement of symbols, rotation of a plane, or flipping of a card. By way of these visualizations students tend to appreciate and understand the concepts that underlie the abstract symbols [10] [11].

Novel to the perceptions of pre-service mathematics teachers are the permutation and symmetry groups, as these were not studied previously. These groups are not commutative or non-abelian. A permutation group is a rearrangement of a set X in a one-to-one correspondence to itself [12]. On the other hand, a symmetry group contains elements which are cards operated by plane rotations and flipping on the plane's axes of symmetry. The manipulability of these groups enables the students to easily perform the binary operations and construct multiplication tables.

In search of supplemental groups which can be used in the study of group theory, this paper presents the inversions of two letter symbols. The elements of this group are two letters manipulated through alternation or change of places, 180° rotation, reflection or lateral inversion, and a combination of two or three of the first three motions.

OBJECTIVES OF THE STUDY

The general objective of this study is to determine the properties of two-letter inversions as potential group properties. The specific objectives are the following: a) Define the elements of the set of two-letter inversions; b) Construct the multiplication table for the inversions; c) Show that the binary operation is commutative; and d) Find the subgroups of the group-candidate.

METHODS AND PROCEDURES

Analysis of group-like properties, computations, visualizations, and relating the results to the following group theory definition [13] and theorems [14] were done to support the claim for a new group subject:

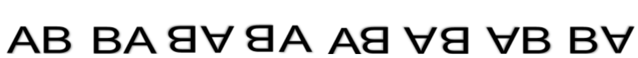
> [Groups] are sets with just one binary operation, and there are just three axioms that govern them: (i) the binary operation should be associative, (ii) there should be an identity for this operation, and (iii) every element should have an inverse with respect to this operation. (p. 157)

> Theorem 1. If G is a finite abelian group and the prime p divides the order of G, then G contains an element of order p and hence a subgroup of order p. (p. 85)

> Theorem 2. Lagrange's Theorem: If H is a subgroup of a finite group G, then the order of H is a divisor of the order of G. (p. 80).

RESULTS AND DISCUSSION

Two script letters A and B, assuming that A has a vertical axis of symmetry and B has a horizontal axis of symmetry, can be written in the following forms:



These forms are called inversion elements or *inversions*. The first inversion is the identity or *non-motion* element; the second is the *alternation* which switches the places of the two letters; the third is the result of 180° *rotation* of the plane on which the letters are written; the fourth is the *reflection* or the mirror image, which is laterally inverted; the fifth is the result of reflection and alternation; the sixth is the result of alternation and rotation; the seventh is the result of reflection and rotation; and the last is the product of alternation, rotation, and reflection in any order of motion. Furthermore, the cardinality of the set is eight; therefore, it is a finite set.

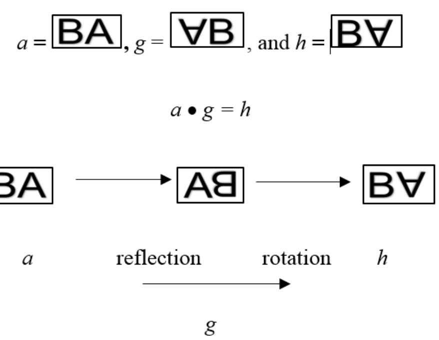


Figure 1. *Example of two-letter inversion computation*

For purposes of simplicity, we shall denote the identity element as e, alternation as a, rotation as b, reflection as c, the product $a\bullet c$ as d, the product $a\bullet b$ as f, the product $b\bullet c$ as g, and the product $a \bullet b\bullet c$ as h. We shall denote the set of inversions as B_2. Thus, in symbols,

$$B_2 = \{e, a, b, c, d, f, g, h\}.$$

Group Properties and Multiplication Table

Multiplying two elements in the set results to an element that belongs to the same set; hence, there is a closure property. There is an identity element e which, when multiplied by another element in the set, preserves the element. Furthermore, multiplying the element by itself produces the identity; therefore, the inverse of each element is itself. Finally, the binary operation in the set is associative as illustrated below. For $a, b, c, f, g, h \in B_2$,

$$a \bullet (b \bullet c) = (a \bullet b) \bullet c$$
$$a \bullet g = f \bullet c$$
$$h = h.$$

The products of each pair of elements in B_2 are shown in Table 1, with unique cell colour for each element.

Table 1. Two-letter inversion multiplication table with unique cell colour for each element

B_2	*e*	*a*	*b*	*c*	*d*	*f*	*g*	*h*
e	*e*	*a*	*b*	*c*	*d*	*f*	*g*	*h*
a	*a*	*e*	*f*	*d*	*c*	*b*	*h*	*g*
b	*b*	*f*	*e*	*g*	*h*	*a*	*c*	*d*
c	*c*	*d*	*g*	*e*	*a*	*h*	*b*	*f*
d	*d*	*c*	*h*	*a*	*e*	*g*	*f*	*b*
f	*f*	*b*	*a*	*h*	*g*	*e*	*d*	*c*
g	*g*	*h*	*c*	*b*	*f*	*d*	*e*	*a*
h	*h*	*g*	*d*	*f*	*b*	*c*	*a*	*e*

Subgroups

For ease of reference, Table 2 shows the multiplication table for the two-letter inversions with the elements of subgroups identified within broken lines or in cells/font of the same colour. (Intersection of two or more subgroup cells is coloured by mixing the colours of the subgroup cells or with grey).

Table 2. Multiplication table for the two-letter inversions and their subgroups

•	*e*	*a*	*b*	*c*	*d*	*f*	*g*	*h*
e	*e*	*a*	*b*	*c*	*d*	*f*	*g*	*h*
a	*a*	*e*	*f*	*d*	*c*	*b*	*h*	*g*
b	*b*	*f*	*e*	*g*	*h*	*a*	*c*	*d*
c	*c*	*d*	*g*	*e*	*a*	*h*	*b*	*f*
d	*d*	*c*	*h*	*a*	*e*	*g*	*f*	*b*
f	*f*	*b*	*a*	*h*	*g*	*e*	*d*	*c*
g	*g*	*h*	*c*	*b*	*f*	*d*	*e*	*a*
h	*h*	*g*	*d*	*f*	*b*	*c*	*a*	*e*

The binary operation is commutative as evidenced by the equality of the products of two elements regardless of their orders in the operation. For example, if $a, c, d \in B_2$,

$$a \bullet d = d \bullet a$$
$$c = c.$$

Hence, the given set with the binary operation characterizes the commutative property of a group, which is called abelian group.

The two-letter inversions B_2 contains subsets whose elements are closed under the same binary operation. These subsets are called subgroups. This study analysed the multiplication table above and found fifteen (15) proper subgroups in the two-letter inversions group-candidate as shown in Figure 2.

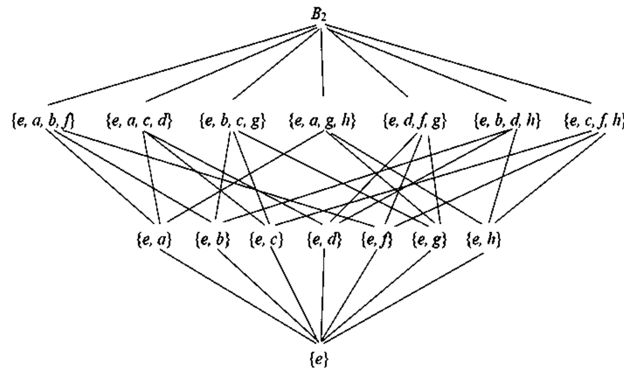


Figure 2. *Lattice diagram for the subgroups of the two-letter inversions*

The two-letter inversions contain seven four-membered, seven two-membered, and one singleton subgroups; hence, the subgroups have the orders 4, 2, and 1, respectively. Each subgroup has an order which divides the order of B_2 (order 8), which means that it is consistent with Lagrange's Theorem. Furthermore, the subgroups with order 4 have elements with order 1 or 2; order 2 is cyclic; and order 1 is isomorphic to {*e*} [14].

CONCLUSION AND RECOMMENDATION

The set of two-letter inversions B_2 and the binary operation (alternation, rotation, and/or reflection) constitute an finite abelian group, as the binary structure meets the group conditions or definitions for closure, identity, inverse, associativity, and commutativity. Hence, B_2 can be used as a supplemental subject in the study of group theory, which is suited to beginners as it can be learned through manipulation of concrete materials in the same manner as that of permutation and symmetry groups. However, teaching and learning group theory in this manner must be done with caution as they may jeopardize the ultimate aim of abstract algebra (from the word *abstract* itself) which is to develop advanced mathematical thinking.

The two-letter inversion group described in this study is deemed helpful in the introductory courses in abstract algebra. To find additional demonstrable groups, other inversion groups may be studied by using letter symbols with at most two axes of symmetry or without and by increasing the number of letters under operation (e.g., three-letter or four-letter inversions). Further, future investigations on the applications of inversion groups are encouraged.

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